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KIMMERLE'S CONJECTURE FOR INTEGRAL GROUP RINGS OF SOME ALTERNATING GROUPS

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ABSTRACT. Using the Luthar–Passi method and results of Hertweck, we study the long-standing conjecture of Zassenhaus for integral group rings of alternating groups A_n, $n \leq 8$. As a consequence of our results, we confirm the Kimmerle's conjecture about prime graphs for those groups.

1. Introduction and main results

Let G be a finite group and let $\mathbb{Z}G$ be the integral group ring of G. By $V(\mathbb{Z}G)$ we denote the group

$$\left\{ \sum_{g\in G} \alpha_g g \in U(\mathbb{Z}G) : \sum_{g\in G} \alpha_g = 1 \right\}$$

of normalized units in $\mathbb{Z}G$. As the group of units $U(\mathbb{Z}G)$ is completely determined by $U(\mathbb{Z}G) = U(\mathbb{Z}) \times (\mathbb{Z}G)$, throughout this paper all units are normalized and distinct from the identity element of G.

A long-standing conjecture in the theory of integral group rings is the following conjecture of *H. Zassenhaus*(see [32]).

Conjecture (ZC). Every torsion unit u in $V(\mathbb{Z}G)$ is conjugate to an element in G within the rational group algebra $\mathbb{Q}G$; i.e. there exist a group element g in G and w in $\mathbb{Q}G$ for which $w^{-1}uw = g$.

A positive answer is given for nilpotent groups and for some other special types of groups. For finite simple groups, the main tool for investigating (ZC) is the *Luthar–Passi* method introduced in [26] to confirm the conjecture for A_5. Later, *M. Hertweck* applied the method to Brauer character tables in the investigation of the Zassenhaus conjecture for $PSL(2,7)$ (see [20]). Using the same technique of *Hertweck* on the group A_6 ($\approxeq PSL(2,9)$), almost a positive solution we achieved in [29] with one remaining case for units of order 6 which

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been completed by *M. Hertweck* in [23]. Also some related properties can be found in [1] and [27], while in the latter paper some weakened variations of the conjecture have been made. The *Zassenhaus* conjecture (ZC) is still open for all sporadic simple groups, and for some recent results on sporadic simple groups we refer to the papers [4]–[15].

In parallel to the (ZC) and as a useful technique that we have used is the conjecture of *W. Kimmerle* (KC), which involves the concept of prime graph (see [25]). For a group $\mathcal{H}$, let $pr(\mathcal{H})$ denotes the set of all prime divisors of the orders of torsion elements of $\mathcal{H}$. The *Gruenberg–Kegel* graph (or the prime graph) of $\mathcal{H}$ is a graph $\pi(\mathcal{H})$ with vertices labelled by primes in $pr(\mathcal{H})$, such that vertices p and q are adjacent if $\mathcal{H}$ contains an element of order pq. In [25], *W. Kimmerle* made the following conjecture.

Conjecture (KC). If G is a finite group then $\pi(G) = \pi(V(\mathbb{Z}G))$.

Obviously, the *Zassenhaus conjecture* (ZC) implies the *Kimmerle's conjecture* (KC). In [25], *W. Kimmerle* has shown that (KC) holds for finite Frobenius and solvable groups. However the (ZC) remains open for such groups. Although, (KC) still open for symmetric groups in general, it been confirmed for S_n, where $n \leq 7$ (see [27], [23], [29] and [30]).

In this paper, we continue these investigations for the alternating groups A_7 and A_8. Our main results, Theorems 1 and 2, give a reasonable amount of information on all possible torsion units in $V(\mathbb{Z}A_7)$ and $V(\mathbb{Z}A_8)$, respectively. For torsion units which we can not confirm (ZC), a strong restrictions on their partial augmentations been obtained. And hence, an immediate consequence of these results, we get a positive answer to (KC) for those two groups.

Let $u = \sum \alpha_g g$ be a normalized torsion unit of $\mathbb{Z}G$ of order k. For any character χ (of G) of degree n, we have that $\chi(u) = \sum_{i=2}^{m} \nu_i \chi(h_i)$, where $h_i's$ are representatives of the conjugacy classes $\mathcal{C}_i's$, $2 \leq i \leq m$, of G and $\nu_i = \varepsilon_{\mathcal{C}_i}(u)$ are the partial augmentations of u. In addition, from *Higman-Berman's* Theorem (see [1]), we know that $\nu_1 = 0$ and

$$\nu_2 + \nu_3 + \cdots + \nu_m = 1. \tag{1}$$

The main results of this paper are the following two theorems.

Theorem 1. *Let G be the alternating group A_7 and u be a torsion unit in $V(\mathbb{Z}G)$ of order $|u|$. Then the following statements hold.*

(i) *If $|u| \neq 12$, then $|u|$ coincides with the order of some element $g \in G$.*
(ii) *If $|u| \in \{2, 3, 5, 7\}$, then u is rationally conjugate to some element $g \in G$.*
(iii) *If $|u| = 4$, then the tuple of the partial augmentations of u belongs to*

$$\{(\nu_{2a}, \nu_{3a}, \nu_{3b}, \nu_{4a}, \nu_{5a}, \nu_{6a}, \nu_{7a}, \nu_{7b}) \in \mathbb{Z}^8 \mid (\nu_{2a}, \nu_{4a}) \in \{(0,1), (2,-1)\}, \nu_{nt} = 0, nt \notin \{2a, 4a\}\}.$$

(iv) *If* $|u| = 6$*, then the tuple of the partial augmentations of* u *belongs to*

$$\{(\nu_{2a}, \nu_{3a}, \nu_{3b}, \nu_{4a}, \nu_{5a}, \nu_{6a}, \nu_{7a}, \nu_{7b}) \in \mathbb{Z}^8 \mid (\nu_{2a}, \nu_{3a}, \nu_{3b}, \nu_{6a}) \in \{ (-2, 1, 2, 0), (0, 0, 0, 1), (2, 0, 0, -1), (0, 1, -1, 1), (-2, 2, 1, 0), (2, -1, 1, -1) \}, \nu_{nt} = 0, nt \notin \{2a, 3a, 3b, 6a\}\}.$$

(v) *If* $|u| = 12$*, then the tuple of the partial augmentations of* u *does not belong to*

$$\mathbb{Z}^8 \setminus \{ (\nu_{2a}, \nu_{3a}, \nu_{3b}, \nu_{4a}, \nu_{5a}, \nu_{6a}, \nu_{7a}, \nu_{7b}) \in \mathbb{Z}^8 \mid (\nu_{2a}, \nu_{3a}, \nu_{3b}, \nu_{4a}, \nu_{6a}) \in \{ (2, 0, 0, -2, 1), (1, 0, 0, 1, -1), (0, 1, 2, -2, 0), (0, 0, 0, 2, -1), (1, 0, 0, -1, 1) \}, \ \nu_{nt} = 0, \ nt \notin \{2a, 3a, 3b, 4a, 6a\} \}.$$

Theorem 2. *Let* G *be the alternating group* A_8 *and* u *be a torsion unit in* $V(\mathbb{Z}G)$ *of order* $|u|$*. The following statements hold.*

(i) *If* $|u| \neq 12$*, then* $|u|$ *coincides with the order of some element* $g \in G$*.*
(ii) *If* $|u| \in \{5, 7, 15\}$*, then* u *is rationally conjugate to some* $g \in G$*.*
(iii) *If* $|u| = 2$*, then the tuple of the partial augmentations of* u *belongs to*

$$\{ (\nu_{2a}, \nu_{2b}, \nu_{3a}, \nu_{3b}, \nu_{4a}, \nu_{4b}, \nu_{5a}, \nu_{6a}, \nu_{6b}, \nu_{7a}, \nu_{7b}, \nu_{15a}, \nu_{15b}) \in \mathbb{Z}^{13} \mid (\nu_{2a}, \nu_{2b}) \in \{ (0, 1), (2, -1), (1, 0), (-1, 2) \}, \ \nu_{nt} = 0, \ nt \notin \{2a, 2b\} \}.$$

(iv) *If* $|u| = 3$*, then the tuple of the partial augmentations of* u *belongs to*

$$\{ (\nu_{2a}, \nu_{2b}, \nu_{3a}, \nu_{3b}, \nu_{4a}, \nu_{4b}, \nu_{5a}, \nu_{6a}, \nu_{6b}, \nu_{7a}, \nu_{7b}, \nu_{15a}, \nu_{15b}) \in \mathbb{Z}^{13} \mid (\nu_{3a}, \nu_{3b}) \in \{ (0, 1), (1, 0), (-1, 2) \}, \ \nu_{nt} = 0, \ nt \notin \{3a, 3b\} \}.$$

As an immediate consequence of above results, we have a positive answer for (KC).

Corollary. *If* $G = A_n$*, where* $n \leq 8$*, then* $\pi(G) = \pi(V(\mathbb{Z}G))$*.*

2. Preliminaries

Throughout the paper, we simply denote the p-Brauer character table of the group G will by $\mathfrak{BCT}(p)$. For a torsion unit u in $V(\mathbb{Z}G)$, the (ZC) provides that $\chi(u) = \chi(x)$ for some $x \in G$; and hence an equivalence statement for (ZC) was given by *Luthar–Passi* as follows.

Lemma 1. *(See* [26]*.) Let* $u \in V(\mathbb{Z}G)$ *be of order* k*. Then* u *is conjugate in* $\mathbb{Q}G$ *to an element* $g \in G$ *if and only if for each* d *dividing* k *there is precisely one conjugacy class* $\mathcal{C}$ *with partial augmentation* $\varepsilon_{\mathcal{C}}(u^d) \neq 0$*.*

In order to start our study, we consider the calculation (by GAP) of the indicated numbers $\mu_m(u, \chi, p)$ in what follow for each possible order k of a torsion unit u in $V(\mathbb{Z}G)$, taking in account the following five facts. These facts give the relations between the order k of u and the partial augmentations $\nu = \varepsilon_{\mathcal{C}_i}(u)$.

Lemma 2. *(See* [20] *and* [26]*) Let p be either* 0 *or a prime divisor of $|G|$ and let F be the associated prime field. Suppose that $u \in V(\mathbb{Z}G)$ has finite order k that is relatively prime to k (i.e. k and p are coprime) if $p \neq 0$. If z is a primitive $k-th$ root of unity and χ is either a classical character or a p-Brauer character of G, then, for every integer m, the number*

$$\mu_l(u, \chi, p) = \frac{1}{k} \sum_{d|k} Tr_{F(z^d)/F}\{\chi(u^d) z^{-dm}\} \tag{2}$$

is a non-negative integer.

Note that if $p = 0$, we use the notation $\mu_l(u, \chi, *)$ for $\mu_l(u, \chi, p)$.

Lemma 3. *(See* [17]*.) The order of any torsion unit $u \in V(\mathbb{Z}G)$ divides the exponent of G.*

Lemma 4. *(See* [26]*.) Let u be a torsion unit in $V(\mathbb{Z}G)$ and $\mathcal{C}$ be a conjugacy class of G. If p is a prime divisor of the order of elements $a \in \mathcal{C}$ but not the order of u, then $\varepsilon_{\mathcal{C}}(u) = 0$.*

Lemma 5. *(See* [21] *and* [22]*.) Let G be a finite group and u be a torsion unit in $V(\mathbb{Z}G)$. If x is an element of G whose p-part, for some prime p, has order strictly greater than the order of the p-part of u, then $\varepsilon_x(u) = 0$.*

Lemma 6. *(See* [17]*.) Let p be a prime and u be a torsion unit in $V(ZG)$ of order $p^n, n \geq 1$. Then for any $m (\neq n)$ the sum of all partial augmentations of u with respect to conjugacy classes of elements of order p^m is divisible by p.*

3. Proof of Theorem 1

Within this section G denote the alternating group A_7. It is well-known that $exp(G) = 420 = 2^2 \cdot 3 \cdot 5 \cdot 7$. The character table of G, as well as the Brauer character tables $\mathfrak{BCT}(p)$, where $p \in \{2, 3, 5, 7\}$, can be found using the computational algebra system GAP (see [6] and [19]).

Since the group G possesses elements of orders 2, 3, 4, 5, 6 and 7, we shall study units in $V(\mathbb{Z}G)$ of these orders. Moreover, by Lemma 3, the order of each torsion unit divides $exp(G)$, thus it remains to consider units of orders 10, 12, 14, 15, 21 and 35. But among these numbers, except 12, we prove that there is no unit of such orders are exist in $V(\mathbb{Z}G)$.

Obviously, G has exactly 8 distinct non-singleton conjugacy classes $2a$, $3a$, $3b$, $4a$, $5a$, $6a$, $7a$ and $7b$. We consider each possible unit order separately, where we use Lemma 4 and Equation (1) to determine the non-zero augmentations ν_{nt} for which $\sum_{\mathcal{C}_{nt}} \nu_{nt} = 1$. Then we use the Formula (2) of Lemma 2 to write the appropriate system of inequalities for such order.

• If $|u| \in \{2, 5\}$, then, by Lemmas 1 and 5, u has only one non-zero partial augmentation.

• Let $|u| = 3$. Then we have $\nu_{3a} + \nu_{3b} = 1$, and hence get the system

$$\mu_0(u, \chi_2, *) = \frac{1}{3}(6\nu_{3a} + 6) \geq 0; \mu_1(u, \chi_2, *) = \frac{1}{3}(-3\nu_{3a} + 6) \geq 0;$$
$$\mu_0(u, \chi_2, 2) = \frac{1}{3}(-4\nu_{3a} + 2\nu_{3b} + 4) \geq 0; \mu_0(u, \chi_2, 7) = \frac{1}{3}(4\nu_{3a} - 2\nu_{3b} + 5) \geq 0,$$

which has only the trivial solutions $(\nu_{3a}, \nu_{3b}) \in \{(1, 0), (0, 1)\}$ satisfying that all $\mu_i(u, \chi_j, p)$ are non-negative integers.

• Let $|u| = 7$. Then we have $\nu_{7a} + \nu_{7b} = 1$, and hence get the system

$$\mu_1(u, \chi_3, *) = \frac{1}{7}(t_1 + 10) \geq 0; \mu_3(u, \chi_3, *) = \frac{1}{7}(-t_2 + 10) \geq 0;$$
$$\mu_1(u, \chi_2, 2) = \frac{1}{7}(-t_1 + 4) \geq 0; \mu_3(u, \chi_2, 2) = \frac{1}{7}(3t_2 + 4) \geq 0,$$

where $t_1 = 4\nu_{7a} - 3\nu_{7b}$ and $t_2 = 3\nu_{7a} - 4\nu_{7b}$.This system has only the trivial solutions $(\nu_{7a}, \nu_{7b}) \in \{(1, 0), (0, 1)\}$ satisfying that all $\mu_i(u, \chi_j, p)$ are non-negative integers.

Thus, for units of orders $2, 3, 5$ and 7 there is precisely one conjugacy class with non-zero partial augmentation. Hence, by Lemma 1, part (ii) of Theorem 1 is proved.

• Let $|u| = 4$. Then we have $\nu_{2a} + \nu_{4a} = 1$, and hence get the system

$$\mu_0(u, \chi_2, *) = \frac{1}{4}(4\nu_{2a} + 8) \geq 0; \mu_2(u, \chi_2, *) = \frac{1}{4}(-4\nu_{2a} + 8);$$
$$\mu_0(u, \chi_2, 7) = \frac{1}{4}(2\nu_{2a} - 2\nu_{4a} + 6) \geq 0,$$

which has only two solutions $(\nu_{2a}, \nu_{4a}) \in \{(0, 1), (2, -1)\}$ satisfying Lemma 6 and the requirement for $\mu_i(u, \chi_j, p)$ to be non-negative integers.

• Let $|u| = 6$. Then we have $\nu_{2a}+\nu_{3a}+\nu_{3b}+\nu_{6a} = 1$, and either $\chi(u^2) = \chi(3a)$ or $\chi(u^2) = \chi(3b)$ since u^2 is of order 3.

Put $t_1 = \nu_{3a} + \nu_{3b} - 3\nu_{6a}$, $t_2 = \nu_{2a} - 3\nu_{3a} + \nu_{6a}$, $t_3 = \nu_{2a} + 2\nu_{3a} - \nu_{3b} - 2\nu_{6a}$ and $t_4 = \nu_{2a} - 2\nu_{3a} + \nu_{3b} - 2\nu_{6a}$.

For the first case $\chi(u^2) = \chi(3a)$, we get the system

$$\begin{aligned}
&\mu_1(u, \chi_3, 5) = \frac{1}{6}(-t_1 + 9) \geq 0;\ \mu_2(u, \chi_3, 5) = \frac{1}{6}(t_1 + 9) \geq 0;\\
&\mu_0(u, \chi_8, *) = \frac{1}{6}(2t_2 + 16) \geq 0;\ \mu_3(u, \chi_8, *) = \frac{1}{6}(-2t_2 + 14) \geq 0;\\
&\mu_0(u, \chi_6, 5) = \frac{1}{6}(2t_3 + 10) \geq 0;\ \mu_3(u, \chi_6, 5) = \frac{1}{6}(-2t_3 + 8) \geq 0;\\
&\mu_1(u, \chi_2, *) = \frac{1}{12}(-2t_2 + t_3 + t_4 + 1) \geq 0;\ \mu_0(u, \chi_3, 5) = \frac{1}{6}(-2t_1 + 6) \geq 0,
\end{aligned}$$

that leads to

$$t_1 \in \{-9, -3, 3, 9\},\ t_2 \in \{-8, -5, -2, 1, 4, 7\} \text{ and } t_3 \in \{-5, -2, 1, 4\}.$$

Solving this system, we have the solutions

$$(-2, 1, 2, 0),\ (0, 0, 0, 1) \text{ and } (2, 0, 0, -1).$$

For the second case $\chi(u^2) = \chi(3b)$, we get the system

$$\begin{aligned}
&\mu_1(u, \chi_3, 5) = \frac{1}{6}(-t_1 + 9) \geq 0;\ \mu_2(u, \chi_3, 5) = \frac{1}{6}(t_1 + 9) \geq 0;\\
&\mu_3(u, \chi_7, *) = \frac{1}{6}(2t_2 + 16) \geq 0;\ \mu_3(u, \chi_8, *) = \frac{1}{6}(-2t_2 + 20) \geq 0;\\
&\mu_0(u, \chi_6, 5) = \frac{1}{6}(2t_4 + 16) \geq 0;\ \mu_3(u, \chi_6, 5) = \frac{1}{6}(-2t_4 + 14) \geq 0;\\
&\mu_1(u, \chi_2, *) = \frac{1}{12}(-2t_2 + t_3 - t_4 + 4) \geq 0;\\
&\mu_0(u, \chi_3, *) = \frac{2}{6}(-t_2 - 2t_3 - t_4 + 10) \geq 0;\\
&\mu_0(u, \chi_2, 7) = \frac{1}{6}(2t_3 + 4) \geq 0;\ \mu_3(u, \chi_2, 7) = \frac{1}{6}(-2t_3 + 2) \geq 0.
\end{aligned}$$

Similarly, we have that $t_1 \in \{-9, -3, 3, 9\}$, $t_2 \in \{-8, -5, -2, 1, 4, 7, 10\}$ and $t_3 \in \{-8, -5, -2, 1, 4, 7\}$. Which give the solutions $(0, 1, -1, 1)$, $(-2, 2, 1, 0)$ and $(2, -1, 1, -1)$. Union of solutions for both cases ends the proof of part (iv) of Theorem 1.

• Let $|u| = 12$. Then we have $\nu_{2a} + \nu_{3a} + \nu_{3b} + \nu_{4a} + \nu_{6a} = 1.$. Since u^4, u^3 and u^2 have orders $3, 4$ and 6, respectively, we need to consider $2 \cdot 2 \cdot 6 = 24$ cases those given by parts (ii), (iii) and (iv) of Theorem 1, i.e. we have either $\chi(u^4) = \chi(3a)$ or $\chi(u^4) = \chi(3b)$.

Let $s_1 = 2\nu_{2a}+3\nu_{3a}-\nu_{6a}$, $s_2 = 2\nu_{2a}-\nu_{3a}+2\nu_{3b}-\nu_{6a}$, $s_3 = \nu_{2a}-3\nu_{3a}+\nu_{4a}+\nu_{6a}$ and $s_4 = \nu_{3a} + \nu_{3b} - 3\nu_{6a}$. So, we may consider the following two cases.

Case 1. If $\chi(u^4) = \chi(3a)$, $\chi(u^3) = \chi(4a)$ and $\chi(u^2) = \chi(6a)$. we get the system

$$\mu_2(u,\chi_2,*) = \frac{1}{12}(2s_1+6) \geq 0; \mu_4(u,\chi_2,*) = \frac{1}{12}(-2s_1+6) \geq 0;$$
$$\mu_0(u,\chi_3,*) = \frac{1}{12}(-8\nu_{2a}+4\nu_{3a}+4\nu_{3b}+4\nu_{6a}+12) \geq 0;$$
$$\mu_4(u,\chi_3,*) = \frac{1}{12}(4\nu_{2a}-2\nu_{3a}-2\nu_{3b}-2\nu_{6a}+6) \geq 0;$$
$$\mu_0(u,\chi_6,*) = \frac{1}{12}(4s_2+12) \geq 0; \mu_6(u,\chi_6,*) = \frac{1}{12}(-4s_2+12) \geq 0;$$
$$\mu_0(u,\chi_7,*) = \frac{1}{12}(-4s_3+16) \geq 0; \mu_2(u,\chi_7,*) = \frac{1}{12}(-2s_3+14) \geq 0;$$
$$\mu_4(u,\chi_7,*) = \frac{1}{12}(2s_3+10) \geq 0;$$
$$\mu_0(u,\chi_8,*) = \frac{1}{12}(4\nu_{2a}-12\nu_{3a}-4\nu_{4a}+4\nu_{6a}+16) \geq 0;$$
$$\mu_0(u,\chi_9,*) = \frac{1}{12}(-4\nu_{2a}-4\nu_{3a}-4\nu_{3b}+4\nu_{4a}-4\nu_{6a}+32) \geq 0;$$
$$\mu_2(u,\chi_3,5) = \frac{1}{12}(-2s_4+6) \geq 0; \mu_4(u,\chi_3,5) = \frac{1}{12}(2s_4+6) \geq 0;$$
$$\mu_0(u,\chi_6,5) = \frac{1}{12}(4\nu_{2a}-8\nu_{3a}+4\nu_{3b}-4\nu_{4a}-8\nu_{6a}+4) \geq 0.$$

Which leads to $t_1, t_4 \in \{-3, 3\}$, $t_2 \in \{-3, 0, 3\}$ and $t_3 \in \{-5, 1, 7\}$. Then we have the solutions $(2, 0, 0, -2, 1)$, $(1, 0, 0, 1, -1)$ and $(0, 1, 2, -2, 0)$.

Case 2. If $\chi(u^4) = \chi(3a)$, $\chi(u^3) = 2\chi(2a) - \chi(4a)$ and $\chi(u^2) = \chi(6a)$.

$$\mu_2(u,\chi_2,*) = \frac{1}{12}(2s_1-2) \geq 0; \mu_4(u,\chi_2,*) = \frac{1}{12}(-2s_1+14) \geq 0;$$
$$\mu_6(u,\chi_2,*) = \frac{1}{12}(-8\nu_{2a}-12\nu_{3a}+4\nu_{6a}+4) \geq 0;$$
$$\mu_4(u,\chi_3,*) = \frac{1}{12}(4\nu_{2a}-2\nu_{3a}-2\nu_{3b}-2\nu_{6a}-2) \geq 0;$$
$$\mu_0(u,\chi_6,*) = \frac{1}{12}(4s_2+20) \geq 0; \mu_6(u,\chi_6,*) = \frac{1}{12}(-4s_2+4) \geq 0;$$
$$\mu_2(u,\chi_7,*) = \frac{1}{12}(-2s_3+14) \geq 0;$$
$$\mu_4(u,\chi_7,*) = \frac{1}{12}(2s_3+10) \geq 0;$$
$$\mu_2(u,\chi_3,5) = \frac{1}{12}(-2s_4+6) \geq 0; \mu_4(u,\chi_3,5) = \frac{1}{12}(2s_4+6) \geq 0.$$

Which leads to $s_1 \in \{1, 7\}$, $s_2 \in \{-5, -2, 1\}$, $s_3 \in \{-5, 1\}$ and $s_4 \in \{-3, 3\}$. Then we have the solutions $(0, 0, 0, 2, -1)$ and $(1, 0, 0, -1, 1)$.

In all remaining 22 cases we get that $\mu_1(u,\chi_3,*) = 3/2 \notin \mathbb{Z}$ whenever $\chi(u^4) = \chi(3a)$ and $\chi(u^2) = -2\chi(2a) + \chi(3a) + 2\chi(3b)$, a contradiction. In all other cases we get similar contradictions which are $\mu_1(u,\chi_2,*) \in \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$.

Thus, parts (ii)-(v) of the Theorem 1 are proved. Now, to prove part (i) it remains to consider units of order 10, 14, 15, 21 and 35.

• Let $|u| = 10$. Then we have $\nu_{2a} + \nu_{5a} = 1$, and hence get

$$\begin{aligned}
&\mu_0(u,\chi_3,*) = \frac{8}{10}(-\nu_{2a}+1) \geq 0; \mu_5(u,\chi_3,*) = \frac{4}{10}(2\nu_{2a}+3) \geq 0;\\
&\mu_1(u,\chi_7,*) = \frac{1}{10}(-\nu_{2a}+16) \geq 0,
\end{aligned}$$

which has no integral solutions.

• Let $|u| = 14$. Then we have $\nu_{2a} + \nu_{7a} + \nu_{7b} = 1$, and since $\chi(u^2) \in \{\chi(7a), \chi(7b)\}$, we may calculate that

$$\mu_0(u,\chi_2,*) = -\mu_7(u,\chi_2,*) = \frac{1}{14}(6(2\nu_{2a} - \nu_{7a} - \nu_{7b}) + 2) = 0,$$

that leads to a contradiction.

• Let $|u| = 15$. Then we have $\nu_{3a} + \nu_{3b} + \nu_{5a} = 1$, and hence get

$$\begin{aligned}
\mu_0(u,\chi_7,*) &= \frac{1}{15}(24\nu_{3a}+\alpha_1) \geq 0;\ \mu_5(u,\chi_7,*) = \frac{1}{15}(-12\nu_{3a}+\alpha_2) \geq 0;\\
\mu_1(u,\chi_2,*) &= \frac{1}{15}(3\nu_{3a}+\nu_{5a}+\alpha_3) \geq 0; \mu_3(u,\chi_2,*)\\
&= \frac{1}{15}(-6\nu_{3a}-2\nu_{5a}+\alpha_4) \geq 0;\\
\mu_0(u,\chi_3,*) &= \frac{1}{15}(8\nu_{3a}+8\nu_{3b}+12) \geq 0,
\end{aligned}$$

where $(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \begin{cases} (21,12,2,11), & \text{if } \chi(u^3) = \chi(5a);\\ (15,15,5,5), & \text{if } \chi(u^3) = \chi(5b), \end{cases}$ that has no integral solution.

• Let $|u| = 21$. Then we have $\nu_{3a} + \nu_{3b} + \nu_{7a} + \nu_{7b} = 1$, consider two cases.

If $\chi(u^7) = \chi(3a)$ and $\chi(u^3) \in \{\chi(7a), \chi(7b)\}$, we get the system

$$\mu_0(u,\chi_8,*) = \frac{1}{21}(-36\nu_{3a}+15) \geq 0; \quad \mu_7(u,\chi_8,*) = \frac{1}{21}(18\nu_{3a}+24) \geq 0.$$

Secondly, if $\chi(u^7) = \chi(3b)$ and $\chi(u^3) \in \{\chi(7a), \chi(7b)\}$ we get the system

$$\mu_0(u,\chi_8,*) = \frac{1}{21}(-36\nu_{3a}+21) \geq 0; \quad \mu_7(u,\chi_8,*) = \frac{1}{21}(18\nu_{3a}+21) \geq 0.$$

Both systems have no integral solution.

• Let $|u| = 35$. Then we have $\nu_{5a} + \nu_{7a} + \nu_{7b} = 1$, and hence for $\chi(u^5) \in \{\chi(7a), \chi(7b)\}$ we get the system

$$\mu_0(u,\chi_5,*) = \frac{1}{35}(-24\nu_{5a}+10) \geq 0; \quad \mu_0(u,\chi_8,*) = \frac{1}{35}(24\nu_{5a}+25) \geq 0,$$

which has no integral solution, too.

4. Proof of Theorem 2

Within this section $G = A_8$. It is well-known that $exp(G) = 420 = 2^2 \cdot 3 \cdot 5 \cdot 7$. The character table of G, as well as the Brauer character tables $\mathfrak{BCT}(p)$, where $p \in \{2, 3, 5, 7\}$, can be found using the computational algebra system GAP (see [6] and [19]).

Since the group G possesses elements of orders 2, 3, 4, 5, 6, 7 and 15, we shall study units in $V(\mathbb{Z}G)$ of these orders. Moreover, by Lemma 3, the order of such torsion unit divides $exp(G)$, so it remains to consider units of orders 10, 12, 14, 21 and 35. But among these numbers, except 12, we prove that there is no unit of such orders are exist in $V(\mathbb{Z}G)$.

Obviously, G has exactly 13 distinct non-singleton conjugacy classes $2a$, $2b$, $3a$, $3b$, $4a$, $4b$, $5a$, $6a$, $6b$, $7a$, $7b$, $15a$ and $15b$. We consider each possible unit order separately, where we use Lemma 4 and Equation (1) to determine the non-zero augmentations ν_{nt} for which $\sum_{\mathcal{C}_{nt}} \nu_{nt} = 1$, and hence use formula (2) of Lemma 2 to write the appropriate system of inequalities for such order.

- If $|u| = 5$, then by Lemma 1 and 5, there is only one non-zero partial augmentation of u.
- Let $|u| = 7$. Then we have $\nu_{7a} + \nu_{7b} = 1$, and hence get the system

$$\begin{aligned}
\mu_1(u, \chi_{10}, *) &= \frac{1}{7}(4\nu_{7a} - 3\nu_{7b} + 45) \geq 0; \mu_3(u, \chi_{10}, *) \\
&= \frac{1}{7}(-3\nu_{7a} + 4\nu_{7b} + 45) \geq 0; \\
\mu_1(u, \chi_2, 2) &= \frac{1}{7}(-4\nu_{7a} + 3\nu_{7b} + 4) \geq 0; \mu_3(u, \chi_2, 2) = \frac{1}{7}(3\nu_{7a} - 4\nu_{7b} + 4) \geq 0,
\end{aligned}$$

which has only two trivial solutions $(\nu_{7a}, \nu_{7b}) \in \{(1, 0), (0, 1)\}$ satisfying that all $\mu_i(u, \chi_j, p)$ are non-negative integers.

Thus, for units of orders 5 and 7, there is precisely one conjugacy class with non-zero partial augmentation. So, by Lemma 1, this proves part (ii) of Theorem 2, except for the unit of order 15, that will be considered later after studying the units of order 3.

- Let $|u| = 2$. Then we have $\nu_{2a} + \nu_{2b} = 1$, and hence get the system

$$\begin{aligned}
\mu_0(u, \chi_{12}, *) &= \frac{1}{2}(8\nu_{2a} + 56) \geq 0; \quad \mu_1(u, \chi_{12}, *) = \frac{1}{2}(-8\nu_{2a} + 56) \geq 0; \\
\mu_0(u, \chi_2, *) &= \frac{1}{2}(-\nu_{2a} + 3\nu_{2b} + 7) \geq 0; \mu_1(u, \chi_2, *) = \frac{1}{2}(\nu_{2a} - 3\nu_{2b} + 7) \geq 0,
\end{aligned}$$

which has only four solutions $(\nu_{2a}, \nu_{2b}) \in \{(0, 1), (1, 0), (-1, 2), (2, -1)\}$ satisfying that all $\mu_i(u, \chi_j, p)$ are non-negative integers.

• Let $|u| = 3$. Then we have $\nu_{3a} + \nu_{3b} = 1$, and hence get the system

$$\mu_0(u, \chi_5, *) = \frac{1}{3}(12\nu_{3a} + 21) \geq 0; \quad \mu_1(u, \chi_4, 2) = \frac{1}{3}(-3\nu_{3a} + 6) \geq 0;$$
$$\mu_0(u, \chi_2, 2) = \frac{1}{3}(-4\nu_{3a} + 2\nu_{3b} + 4) \geq 0;$$

which has the three solutions $(\nu_{2a}, \nu_{2b}) \in \{(0, 1), (1, 0), (-1, 2)\}$ satisfying that all $\mu_i(u, \chi_j, p)$ are non-negative integers.
• Let $|u| = 15$. Then we have $\nu_{3a} + \nu_{3b} + \nu_{5a} + \nu_{15a} + \nu_{15b} = 1$. Since $\chi(u^5)$ is of order 3, then by part (iv) of Theorem 2, we need to consider 3 cases: $\chi(u^5) = \chi(3a)$, $\chi(u^5) = \chi(3b)$ and $\chi(u^5) = -\chi(3a) + 2\chi(3b)$.

First, using (2) for the case $\chi(u^5) = \chi(3a)$, we get the system

$$\mu_1(u, \chi_4, *) = \frac{1}{15}(5\nu_{3a} - \nu_{3b} + 15) \geq 0; \mu_3(u, \chi_4, *) = \frac{1}{15}(-10\nu_{3a} + 2\nu_{3b} + 30) \geq 0,$$

which implies that $t_1 = 5\nu_{3a} - \nu_{3b} \in \{-15, 0, 15\}$. If $t_1 = 15$, then

$$\mu_5(u, \chi_4, *) = \frac{1}{15}(-20\nu_{3a} + 4\nu_{3b} + 15) = -45,$$

a contradiction. If $t_1 = -15$, then

$$\mu_0(u, \chi_4, *) = \frac{1}{15}(40\nu_{3a} - 8\nu_{3b} + 30) = -90,$$

a contradiction. Therefore, $\nu_{3b} = 5\nu_{3a}$. Let

$$t_2 = 9\nu_{3a} + 2\nu_{5a} - \nu_{15a} - \nu_{15b}; \; t_3 = 9\nu_{3a} - \nu_{5a} - \nu_{15a} - \nu_{15b};$$
$$t_4 = 6\nu_{3a} + \nu_{5a} + \nu_{15a} + \nu_{15b}; \; t_5 = 3\nu_{3a} - \nu_{5a} - 7\nu_{15a} + 8\nu_{15b}.$$

Using (2), we get the system

$$\mu_0(u, \chi_2, *) = \frac{1}{15}(8t_2 + 23) \geq 0; \mu_3(u, \chi_2, *) = \frac{1}{15}(-2t_2 + 13) \geq 0;$$
$$\mu_0(u, \chi_3, *) = \frac{1}{15}(8t_3 + 8) \geq 0; \mu_3(u, \chi_3, *) = \frac{1}{15}(-2t_3 + 13) \geq 0;$$
$$\mu_0(u, \chi_5, *) = \frac{1}{15}(8t_4 + 37) \geq 0; \mu_3(u, \chi_5, *) = \frac{1}{15}(-2t_4 + 32) \geq 0;$$
$$\mu_1(u, \chi_6, *) = \frac{1}{15}(-t_5 + 23) \geq 0; \mu_1(u, \chi_2, 2) = \frac{1}{15}(t_5 + 7) \geq 0.$$

From which we have $t_2 = t_3 = -1$, $t_4 \in \{1, 16\}$ and $t_5 \in \{-7, 8, 23\}$, that gives the solutions $(0, 0, 0, 0, 1)$, $(0, 0, 0, 1, 0)$ and $(0, 0, 0, -1, 2)$, where the last solution may excluded as it does not satisfy the inequality

$$\mu_7(u, \chi_2, 2) = \frac{1}{15}(-2\nu_{3a} + \nu_{3b} - \nu_{5a} + 8\nu_{15a} - 7\nu_{15b} + 7) \geq 0.$$

Using (2) for the other two cases $\chi(u^5) = \chi(3b)$ and $\chi(u^5) = -\chi(3a) + 2\chi(3b)$, we get the system

$$\mu_0(u, \chi_4, *) = \frac{1}{15}(40\nu_{3a} - 8\nu_{3b} + \alpha_1) \geq 0;$$
$$\mu_5(u, \chi_4, *) = \frac{1}{15}(-20\nu_{3a} + 4\nu_{3b} + \alpha_2) \geq 0,$$

where $\quad (\alpha_1, \alpha_2) = \begin{cases} (18, 21), & \text{if} \quad \chi(u^5) = \chi(3b); \\ (6, 27), & \text{if} \quad \chi(u^5) = -\chi(3a) + 2\chi(3b), \end{cases}$,

which has no integral solution satisfying that all $\mu_i(u, \chi_j, *)$ are non-negative integers.

Thus, we prove parts (ii)-(iv) of the Theorem 2. To justify part (i) , we need to consider possible units of orders 10, 14, 21 and 35.

• Let u be a unit of order 10. We consider the four pairs in part (iii) of the Theorem 2 and the fact that u^5 is of order 2. then , using (2), we get the system

$$\mu_0(u, \chi_4, *) = \frac{1}{10}(16t + 24) \geq 0; \mu_5(u, \chi_4, *) = \frac{1}{10}(-16t + 16) \geq 0;$$
$$\mu_1(u, \chi_{10}, *) = \frac{1}{10}(-3t + 48) \geq 0;$$

where $t = \nu_{2a} + \nu_{2b}$. This system has no integral solutions satisfying that all $\mu_i(u, \chi_j, *)$ are non-negative integers.

• Let $|u| = 14$. Since $\chi(u^2) \in \{\chi(7a), \chi(7b)\}$ and u^7 is of order 2, we consider four cases defined by part (ii) of the Theorem 2.

Case 1. If $\chi(u^7) = \chi(2a), \chi(u^2) \in \{\chi(7a), \chi(7b)\}$. then we get

$$\mu_0(u, \chi_{12}, *) = \frac{1}{14}(48\nu_{2a} + 64) \geq 0; \mu_7(u, \chi_{12}, *) = \frac{1}{14}(-48\nu_{2a} + 48) \geq 0;$$
$$\mu_0(u, \chi_2, *) = \frac{1}{14}(-6\nu_{2a} + 18\nu_{2b} + 6) \geq 0; \mu_7(u, \chi_2, *)$$
$$= \frac{1}{14}(6\nu_{2a} - 18\nu_{2b} + 8) \geq 0;$$
$$\mu_2(u, \chi_2, *) = \frac{1}{14}(\nu_{2a} - 3\nu_{2b} + 6) \geq 0.$$

Case 2. If $\chi(u^7) = \chi(2b), \chi(u^2) \in \{\chi(7a), \chi(7b)\}$, then we get

$$\mu_0(u, \chi_{12}, *) = \frac{1}{14}(48\nu_{2a} + 56) \geq 0; \;\; \mu_7(u, \chi_{12}, *) = \frac{1}{14}(-48\nu_{2a} + 56) \geq 0;$$
$$\mu_0(u, \chi_2, *) = \frac{1}{14}(-6\nu_{2a} + 18\nu_{2b} + 10) \geq 0; \mu_7(u, \chi_2, *)$$
$$= \frac{1}{14}(6\nu_{2a} - 18\nu_{2b} + 4) \geq 0.$$

Case 3. If $\chi(u^7) = 2\chi(2a) - \chi(2b), \chi(u^2) \in \{\chi(7a), \chi(7b)\}$, then we get

$$\mu_0(u, \chi_{12}, *) = \frac{1}{14}(48\nu_{2a} + 72) \geq 0; \quad \mu_7(u, \chi_{12}, *) = \frac{1}{14}(-48\nu_{2a} + 40) \geq 0.$$

Case 4. If $\chi(u^7) = -\chi(2a) + 2\chi(2b), \chi(u^2) \in \{\chi(7a), \chi(7b)\}$, then we get

$$\mu_0(u, \chi_{12}, *) = \frac{1}{14}(48\nu_{2a} + 48) \geq 0; \quad \mu_7(u, \chi_{12}, *) = \frac{1}{14}(-48\nu_{2a} + 64) \geq 0;$$

$$\begin{aligned} \mu_0(u, \chi_2, *) &= \frac{1}{14}(-6\nu_{2a} + 18\nu_{2b} + 14) \geq 0; \mu_7(u, \chi_2, *) \\ &= \frac{1}{14}(6\nu_{2a} - 18\nu_{2b}) \geq 0. \end{aligned}$$

All these above four systems have no integral solutions satisfying that all $\mu_i(u, \chi_j, *)$ are non-negative integers.

• Let $|u| = 21$. Since $\chi(u^3) \in \{\chi(7a), \chi(7b)\}$ and u^7 has order 3, we consider six cases defined by part (iv) of the Theorem 2.

If $(\chi(u^7), \chi(u^3))$ belongs to

$$\{(\chi(3a), \chi(7a)), (\chi(3a), \chi(7b)), \\ (-\chi(3a) + 2\chi(3b), \chi(7a)), (-\chi(3a) + 2\chi(3b), \chi(7b))\},$$

then we get the system

$$\mu_0(u, \chi_5, *) = \frac{1}{21}(72\nu_{3a} + \alpha_1) \geq 0; \quad \mu_7(u, \chi_5, *) = \frac{1}{21}(-36\nu_{3a} + \alpha_2) \geq 0,$$

where

$$(\alpha_1, \alpha_2) = \begin{cases} (33, 15), & \text{if} \quad \chi(u^7) = \chi(3a); \\ (9, 27), & \text{if} \quad \chi(u^7) = -\chi(3a) + 2\chi(3b), \end{cases}$$

which has no integral solutions satisfying that all $\mu_i(u, \chi_j, *)$ are non-negative integers.

If $(\chi(u^7), \chi(u^3))$ belongs to $\{(\chi(3b), \chi(7a)), (\chi(3b), \chi(7b))\}$, then we get the system

$$\begin{aligned} \mu_0(u, \chi_2, *) &= \frac{1}{21}(48\nu_{3a} + 12\nu_{3b} + 9) \geq 0; \mu_1(u, \chi_2, *) \\ &= \frac{1}{21}(4\nu_{3a} + \nu_{3b} + 6) \geq 0; \\ \mu_7(u, \chi_2, *) &= \frac{1}{21}(-24\nu_{3a} - 6\nu_{3b} + 6) \geq 0; \end{aligned}$$

which has no integral solution satisfying that all $\mu_i(u, \chi_j, *)$ are non-negative integers.

• Let $|u| = 35$. In both cases, determined by $\chi(u^5) \in \{\chi(7a), \chi(7b)\}$, using (2), we get the system

$$\mu_0(u, \chi_2, *) = \frac{1}{35}(48\nu_{5a} + 15) \geq 0; \; \mu_0(u, \chi_8, *) = \frac{1}{35}(-48\nu_{5a} + 20) \geq 0,$$

which has no integral solution such that all $\mu_i(u, \chi_j, *)$ are non-negative integers.

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