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ON THE ROOTS OF EXPANDING INTEGER POLYNOMIALS

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Abstract. Monic polynomials with integer coefficients all of whose roots lie outside the closed unit disk are studied. The results are applied to canonical number systems and Garsia numbers.

1. Introduction

In this short note we consider monic polynomials with integer coefficients all of whose roots lie outside the closed unit disk of the complex plane. Several occurrences of polynomials of this type are mentioned and some results on their roots are collected. To be more specific, in Section 2 we list some examples and study the moduli and the arguments of the roots of these polynomials. Our main motivation stems from so-called CNS polynomials which have been introduced by A. Pethő [21]. Therefore, Section 3 is devoted to these polynomials and some of their generalizations. Furthermore, we are interested in semi-CNS polynomials defined by P. Burcsi and A. Kovács [9] and we characterize irreducible semi-CNS polynomials with negative constant terms.

Clearly, minimal polynomials of algebraic integers with all conjugates larger than one in modulus belong to the class of polynomials which are dealt with here. One may think of the number of conjugates of an algebraic number ζ outside the closed unit disk as the complexity of ζ [15, p. 375]. Algebraic integers with many conjugates outside the unit disk were investigated in [10] aiming at generalizations of Pisot-Vijayaraghavan and Salem numbers.

The author is indebted to P. Varga for bringing the paper [24] to his attention.

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2. Moduli and arguments of the roots of expanding integer polynomials

Let us denote by $\mathcal{E}$ the collection of monic expanding polynomials of positive degrees with integer coefficients. We recall that a polynomial in $\mathbb{C}[X]$ is expanding if all its roots lie outside the closed unit disk.

Example 1. (i) Every monic nonconstant factor of a CNS polynomial (see [21] for the definition) belongs to $\mathcal{E}$. This was proved by W. J. Gilbert [14] for irreducible polynomials and by A. Pethő [21] (cf. the introduction of [3]) in an unrestricted form[1]. In these papers it was also shown that real roots of CNS polynomials are negative and that their constant terms are at least 2. To put it another way every CNS polynomial is Ω-stable where $\Omega = \mathbb{R}_{>0} \cup \{\zeta \in \mathbb{C} \ : \ |\zeta| \le 1\}$ (see [25, p. 81] for the definition). This fact is tacitly used in the sequel. The reader is referred to [9] and [11] for search methods for CNS polynomials and to [5] for the role of CNS polynomials in the theory of dynamical systems.

(ii) Let f be the minimal polynomial of a real algebraic integer larger than 1 all of whose conjugates have modulus larger than 1 and whose norm has modulus 2. These numbers were first considered by A. Garsia [13] (see [8], [7] for some examples). Clearly, f belongs to $\mathcal{E}$.

(iii) The reader is referred to [23] for extensive calculations of expanding integer polynomials.

In this section we collect some results on the moduli and the arguments of the roots of these polynomials in $\mathcal{E}$. Let us denote the multiset of zeros of the polynomial $f \in \mathbb{C}[X]$ by $\mathcal{Z}(f)$.

Proposition 2. *Let $f \in \mathcal{E}$ have degree $d \ge 2$.*

(i) For all $\zeta \in \mathcal{Z}(f)$ we have

$$1 + \frac{1}{2c-1} < |\zeta| < |f(0)| \left(1 - \frac{1}{2c}\right)^{d-1} \tag{1}$$

with

$$c = \left(|f(0)|\sqrt{d}\right)^d.$$

(ii)

$$\min\{|\zeta| \ : \ \zeta \in \mathcal{Z}(f)\} \le L_2(f)^{1/d}$$

and

$$\max\{|\zeta| \ : \ \zeta \in \mathcal{Z}(f)\} \ge |f(0)|^{1/d}$$

where $L_2(f)$ denotes the square root of the sum of the squares of the coefficients of f.

[1] W. J. Gilbert [14] showed $|\alpha| \ge 1$ for every root of an irreducible CNS polynomial. In the respective part of the proof of [21, Theorem 6.1] the assumption "without multiple roots" was not used.

Proof. (i) Applying [20, Th. 5] to the roots of the polynomial

$$\operatorname{sign}(f(0))X^d f(1/X)$$

we find

$$\frac{1}{|\zeta|} < 1 - \frac{1}{2c} = \frac{2c-1}{2c}$$

yielding the left inequality of (1) and then the right inequality of (1) by

$$|f(0)| = |\zeta| \prod_{\eta \in \mathcal{Z}(f)\setminus\{\zeta\}} |\eta| > |\zeta| \left(\frac{2c}{2c-1}\right)^{d-1}.$$

(ii) The left inequality is clear by [16, Theorem 28,4], and the right inequality is trivial. □

Theorem 3. *Let $f \in \mathcal{E}$ be irreducible of degree d and*

$$R(f) = \prod_{\zeta \in \mathcal{Z}(f)} (1 + |\zeta|).$$

(i) We have

$$R(f) \leq 2^{d-1}\bigl(1 + |f(0)|\bigr).$$

(ii) If all roots of f are real then

$$R(f) \geq \max\left\{G^{\frac{(5-\sqrt{5})d}{2}} |f(0)|^{\frac{\sqrt{5}-1}{2}},\ G^{\frac{3d}{2}} |f(0)|^{\frac{\sqrt{5}}{5}} |f(1)f(-1)|^{\frac{5-2\sqrt{5}}{10}}\right\}$$

with $G = (1+\sqrt{5})/2$.

Proof. (i) Clear by [12, Lemma 2.1].
(ii) Pick $c \in [0, \sqrt{5}-2]$ arbitrarily. By [12, proof of Lemma 4.2] we have

$$\begin{aligned} R(f) &\geq G^{\frac{(3-c)d}{2}} |f(0)|^{\frac{\sqrt{5}}{5} + \frac{(5+\sqrt{5})c}{10}} |f(1)f(-1)|^{\frac{5-2\sqrt{5}}{10} - \frac{\sqrt{5}c}{10}} \\ &= G^{\frac{3d}{2}} |f(0)|^{\frac{\sqrt{5}}{5}} |f(1)f(-1)|^{\frac{5-2\sqrt{5}}{10}} \left(|f(0)|^{\frac{5+\sqrt{5}}{10}}\right)^{c} \left(G^{\frac{d}{2}} |f(1)f(-1)|^{\frac{\sqrt{5}}{10}}\right)^{-c} \end{aligned}$$

and specializing $c = 0$ and $c = \sqrt{5}-2$, respectively, yields our assertion. □

Theorem 4. [22, Cor. 1.6] *Let $f \in \mathcal{E}$ and $d = \deg f \geq \max\{H(f), 55\}$ where $H(f)$ denotes the (naive) height*[2] *of f. If all roots of f are simple then we have*

$$\left| \frac{1}{d} \sum_{\zeta \in \mathcal{Z}(f)} \frac{1}{\zeta} \right| \leq 8\sqrt{\frac{\log d}{d}}.$$

Proof. Apply [22, Cor. 1.6] to the reciprocal polynomial of f. □

[2]See e.g. [18, p. 83] for the definition.

A famous result of M. Mignotte [17, p. 83] immediately yields the following statements where we use Catalan's constant

$$C = \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{2k+1} = 0.916\ldots$$

Theorem 5. *Let $f \in \mathcal{E}$ be irreducible, $d = \deg f$ and $\Theta \in [0,1]$. The number N of roots of f in the closed angular sector with opening $2\pi\Theta$ is bounded by*

$$N \leq 2\Theta d + \sqrt{\frac{2\pi}{C}} \sqrt{2d\Big(\frac{3}{2}\log(2d) + 2\log|f(0)|\Big)}.$$

Corollary 6. *The number of real positive roots of the irreducible polynomial $f \in \mathcal{E}$ of degree d does not exceed*

$$\sqrt{\frac{2\pi}{C}} \sqrt{3d\log(2d) + 4d\log|f(0)|}.$$

Proof. Let Θ tend to 0 in Theorem 5. □

Corollary 7. *Let $f \in \mathcal{E}$ of degree d be irreducible and assume that all roots of f are real and of equal sign. Then we have*

$$\log|f(0)| \geq \frac{Cd}{8\pi} - \frac{3}{4}\log(2d).$$

Proof. Let $\mathcal{Z}(f) \subset \mathbb{R}_{>0}$, otherwise consider the polynomial $(-1)^d f(-X)$. By Corollary 6 we have

$$d \leq \sqrt{\frac{2\pi}{C}} \sqrt{d(3\log(2d) + 4\log|f(0)|)}$$

which immediately yields the result. □

Finally we state a result on the sum of the roots of an irreducible polynomial $f \in \mathcal{E}$ in the case that all roots of f are real and positive.

Theorem 8. *Let $\sum_{i=0}^{d} a_i X^i \in \mathcal{E}$ be irreducible of degree $d \geq 3$ with only real positive roots. Then*

$$a_{d-1} < -\Big(2.66 + \frac{1}{2\big(|a_0|\sqrt{d}\big)^d - 1}\Big) \cdot d.$$

Proof. Let $f = \sum_{i=0}^{d} a_i X^i$ and $\zeta \in \mathcal{Z}(f)$. Clearly, the trace of ζ satisfies

$$Tr(\zeta) = -a_{d-1}, \tag{2}$$

and we can write

$$f(0) = (-1)^d m \tag{3}$$

with $m \in \mathbb{Z}_{\geq 2}$. Set

$$\begin{aligned} A = \{&X^3 - 5X^2 + 6X - 1,\ X^3 - 6X^2 + 9X - 3,\ X^3 - 7X^2 + 14X - 7, \\ &X^4 - 7X^3 + 13X^2 - 7X + 1,\ X^4 - 7X^3 + 14X^2 - 8X + 1, \\ &X^4 - 9X^3 + 27X^2 - 31X + 11\}. \end{aligned}$$

We distinguish two cases.

Case 1. For all $g \in A$, $\beta \in \mathcal{Z}(g)$ and $k \in \mathbb{N}$ we have $\zeta \neq \beta + k$.

By [1] we then have

$$\frac{1}{d} Tr(\zeta) > 1.66 + \min \mathcal{Z}(f),$$

and Proposition 2 yields

$$\min \mathcal{Z}(f) > 1 + \frac{1}{2\big(|a_0|\sqrt{d}\big)^d - 1}.$$

Now, (2) implies our assertion.

Case 2. There exist $g \in A$, $\beta \in \mathcal{Z}(g)$ and $k \in \mathbb{N}$ with $\zeta = \beta + k$.

Then we have $f(X) = g(X - k)$ for some $g \in A$. First, let

$$g = X^3 - aX^2 + bX - c,$$

hence

$$f = X^3 - (3k + a)X^2 + (3k^2 + 2ak + b)X - (k^3 + ak^2 + bk + c),$$

and

$$3k + a \geq 8:$$

This is clear for $k > 0$, and $k = 0$ is excluded by (3).

Second, let $g = X^4 - aX^3 + bX^2 - cX + e$, hence

$$f = X^4 - (4k + a)X^3 + h,$$

with some $h \in \mathbb{Z}[X]$ of degree less than 3, and

$$4k + a \geq 11:$$

Again, this is clear for $k > 0$, and $k = 0$ is excluded by (3) or by the fact that f does not have a root in $[-1, 1]$. $\square$

Example 9. (1) The only Garsia number with all conjugates real and positive is 2: There is only one quadratic Garsia number, namely $\sqrt{2}$ (see [8, Corollary 3.3])), and this number has a negative conjugate. Let us now assume that $\zeta_1, \ldots, \zeta_d$ are the real positive conjugates of a Garsia number of degree $d > 2$. Then Theorem 8 yields the contradiction

$$2d \geq \zeta_1 + \cdots + \zeta_d > 2.66d.$$

(2) Let $f = \sum_{i=0}^{d} a_i X^i$ be a monic irreducible factor of a CNS polynomial of degree $d \geq 3$. If

$$a_{d-1} \leq 2.66d \quad \text{or} \quad a_0 \leq 1.66d + 2 \quad \text{or} \quad a_0 < \exp\left\{\frac{Cd}{8\pi} - \frac{3}{4}\log(2d)\right\}$$

then f has at least one pair of complex conjugate roots: This can easily be deduced from Theorem 8, [9, Statement 2.1] and Corollary 7.

3. On the roots of semi-CNS polynomials

In this section we deal with semi-CNS polynomial which were introduced by P. Burcsi and A. Kovács [9]. Further, we mention algebraic integers with positive finiteness which were defined and studied by S. Akiyama [2]. For the convenience of the reader we recall the definitions in a form slightly adapted to our purposes here.

Definition 10. (1) [9, Definition 3.2] The polynomial $f \in \mathbb{Z}[X]$ is called a semi-CNS polynomial if f is monic with $f(0) \neq 0$ and if for all $p, q \in \mathcal{D}[X]$ there exists some $r \in \mathcal{D}[X]$ such that $p + q \equiv r \pmod{f}$. Here we set $\mathcal{D} = [0, |f(0)| - 1] \cap \mathbb{Z}$.

(2) [2, p. 4] Let ζ be a nonzero algebraic integer. We say that ζ has positive finiteness if for all $q \in \mathbb{N}[X]$ there is some $r \in \mathcal{D}[X]$ such that $q(\zeta) = r(\zeta)$. Here we set $\mathcal{D} = [0, |N(\zeta)| - 1] \cap \mathbb{Z}$ where $N(\zeta)$ denotes the absolute norm of ζ.

In the following theorem we exhibit the relations between CNS and semi-CNS polynomials and algebraic integers with finiteness property. The assertion that semi-CNS polynomials belong to $\mathcal{E}$ was stated in [24], and for the sake of completeness we include a proof here. Further, Theorem 11 (ii) essentially expresses [2, Theorem 2] in our language relating semi-CNS polynomials to the notion of positive finiteness. In the last part of the theorem we characterize irreducible semi-CNS polynomials with negative constant terms. We point out that P. Varga [24] achieved an analogous result for arbitrary cubic semi-CNS polynomials.

Theorem 11. *Let $f \in \mathbb{Z}[X]$.*

(i) If f is a semi-CNS polynomial with

$$|f(0)| > 1 \tag{4}$$

then f belongs to $\mathcal{E}$. Furthermore, for every $q \in \mathbb{N}[X]$ there is some polynomial

$$r \in \big([0, |f(0)| - 1] \cap \mathbb{Z}\big)[X]$$

such that $q \equiv r \pmod{f}$.

(ii) Let f be the minimal polynomial of an algebraic integer $\zeta \neq 0$. Then ζ has positive finiteness if and only if f is a semi-CNS polynomial with (4).

(iii) f *is a CNS polynomial if and only if* f *is a semi-CNS polynomial with* $f(0) > 1$ *and* $-1 \equiv r \pmod{f}$ *for some* $r \in \{0, 1, \dots, f(0) - 1\}[X]$.

(iv) Let f *be irreducible with* $f(0) < -1$. *Then* f *is a semi-CNS polynomial if and only if* f *satisfies the following two conditions.*

(a) f *is monic and* $f(1) < 0$.

(b) Apart from the constant term all coefficients of f *are nonnegative.*

Proof. Let $m = |f(0)|$ and $\mathcal{D} = \{0, 1, \dots, m-1\}$.
(i) Observing $1 \in \mathcal{D}$ we see that for every $k \in \mathbb{N}$ there is some $r \in \mathcal{D}[X]$ such that $k \equiv r \pmod{f}$. Let $q \in \mathbb{N}[X]$. By the semi-CNS property and induction on the degree of q we easily find $r \in \mathcal{D}[X]$ such that $q \equiv r \pmod{f}$. Thus we have shown our second assertion.

As $m > 1$ there must be some $\beta \in \mathcal{Z}(f)$ with $|\beta| > 1$, and for all $k \in \mathbb{N}_{>0}$ we find a polynomial $D_k \in \mathcal{D}[X] \setminus \{0\}$ of degree n_k such that

$$km \equiv D_k \pmod{f}. \tag{5}$$

We observe that the sequence (n_k) is unbounded: Suppose to the contrary that $n_k \leq N$ for all k. Then the finite set

$$\left\{ \sum_{i=0}^{N} d_i X^i \ : \ d_0, \dots, d_N \in \mathcal{D} \right\}$$

contains all polynomials D_k. Hence there exist $k \neq \ell$ such that $D_k = D_\ell$. But then (5) yields

$$km = D_k(\beta) = D_\ell(\beta) = \ell m$$

which is absurd.

Now, let us assume that there is some $\zeta \in \mathcal{Z}(f)$ with $|\zeta| \leq 1$. Write

$$D_k = \sum_{i=0}^{n_k} d_{k,i} X^i,$$

hence by (5) we have

$$\sum_{i=0}^{n_k} d_{k,i} \bigl(\beta^i - \zeta^i\bigr) = 0$$

and further

$$0 = \left| \sum_{i=0}^{n_k} \frac{d_{k,i}}{d_{k,n_k}} \bigl(\beta^i - \zeta^i\bigr) \right| \geq |\beta^{n_k} - \zeta^{n_k}| - \sum_{i=0}^{n_k - 1} \frac{d_{k,i}}{d_{k,n_k}} \left|\beta^i - \zeta^i\right|.$$

Thus we have

$$\begin{aligned}
|\beta|^{n_k} - 1 &\leq |\beta|^{n_k} - |\zeta|^{n_k} = \bigl||\beta|^{n_k} - |\zeta|^{n_k}\bigr| \leq |\beta^{n_k} - \zeta^{n_k}| \\
&\leq \sum_{i=0}^{n_k - 1} \frac{d_{k,i}}{d_{k,n_k}} \left|\beta^i - \zeta^i\right| \leq (m-1) \sum_{i=0}^{n_k - 1} \bigl(|\beta|^i + 1\bigr) \\
&= (m-1)\Bigl(n_k + \frac{|\beta|^{n_k} - 1}{|\beta| - 1}\Bigr)
\end{aligned}$$

which leads to

$$\big(|\beta|^{n_k}-1\big)(|\beta|-1) \le (m-1)\big(n_k(|\beta|-1)+|\beta|^{n_k}-1\big)$$

which is impossible for large n_k.
(ii) If ζ has positive finiteness then by [2, proof of Theorem 2] we have $f \in \mathcal{E}$, hence (4). Let $p, q \in \mathcal{D}[X]$. As $\mathcal{D}[X] \subset \mathbb{N}[X]$ and f is irreducible there is some $r \in \mathcal{D}[X]$ such that $p + q \equiv r \pmod{f}$. Therefore, f is a semi-CNS polynomial. The converse implication is clear by (i).
(iii) As pointed out in Example 1 (i) every CNS polynomial satisfies the properties mentioned above. To show the converse implication we proceed similarly as in the proof of the second assertion of (i).
(iv) Let f is a semi-CNS polynomial, hence f is monic by definition. Clearly, f cannot be a CNS polynomial by (iii). Let $\zeta \in \mathcal{Z}(f)$, thus by (ii) ζ has positive finiteness and by [2, Theorem 2] we find

$$f = \sum_{i=0}^{d} a_i X^i$$

with $a_1, \ldots, a_d \ge 0$ and $a_1 + \cdots + a_d < -a_0$ which implies $f(1) < 0$.

Now we turn to the converse. By [3, Lemma 1] we know $f \in \mathcal{E}$, and an application of [9, Theorem 3.4] completes the proof. □

A result of M. Mignotte and M. Waldschmidt [19] immediately yields:

Theorem 12. *Let ζ be a root of an irreducible semi-CNS polynomial f of degree $d \ge 2$ with $|f(0)| > 1$. Then we have*

$$|\zeta - 1| \ge \frac{1}{|f(0)|^{3\sqrt{d \log d}}}.$$

Proof. Denoting by $M(\zeta)$ the Mahler measure of ζ we have

$$|\zeta - 1| \ge \max\{2, M(\zeta)\}^{-3\sqrt{d \log d}}$$

by [19], and Theorem 11 (i) yields $M(\zeta) = |f(0)|$. □

We recall that by a result of E. Meissner – A. Durand [6, Théorème 2] for every polynomial $f \in \mathbb{R}[X]$ without nonnegative roots there exists a polynomial $g \in \mathbb{R}[X] \setminus \{0\}$ such that the product fg admits only nonnegative coefficients. The following theorem is an immediate consequence of a result of J.-P. Borel [6, Corollaire, p. 101]. It shows that the roots of CNS polynomials are relatively far away from the positive real axis.

Theorem 13. *Let ζ be a root of the CNS polynomial f. Then we have*

$$|\arg \zeta| \ge \frac{\pi}{deg(f) + \delta_0(f)}$$

where $\delta_0(f)$ denotes the minimal degree of a real polynomial $g \ne 0$ such that the product fg admits only nonnegative coefficients.

Proof. Let $g \in \mathbb{R}[X] \setminus \{0\}$ have minimal degree such that fg admits only nonnegative coefficients and assume

$$|\arg \zeta| < \frac{\pi}{deg(f) + \delta_0(f)} = \frac{\pi}{deg(f) + \deg(g)} = \frac{\pi}{deg(fg)}.$$

Then $(fg)(\zeta) \neq 0$ by [6, Theorem 4] which contradicts our hypothesis. □

Corollary 14. *Let f be a CNS polynomial with only nonnegative coefficients. For each root ζ of f we have*

$$|\arg(\zeta)| \geq \frac{\pi}{deg(f)}.$$

We conclude with an illustration of our results by a simple example.

Example 15. Let $f = \sum_{i=0}^{d} a_i X^i \in \mathbb{Z}[X]$ with $d \geq 3,\ a_d = 1,\ a_1 < 0,\ a_2 \geq -a_1,\ a_3, \ldots, a_{d-1} \geq 0$ and

$$a_0 > \sum_{i=2}^{d} a_i - a_1.$$

By [3, Lemma 1] we know $f \in \mathcal{E}$, and in view of

$$\sum_{i=1}^{d} a_i \geq \sum_{i=3}^{d} a_i > 0$$

f is a CNS polynomial by [4, Theorem 3.2]. Observing that the polynomial $(X+1) \cdot f$ has only nonnegative coefficients we have $\delta_0(f) = 1$, thus by Theorem 13 we have

$$|\arg \zeta| \geq \frac{\pi}{d+1}$$

for each $\zeta \in \mathcal{Z}(f)$. If a_0 is prime then f is irreducible by [18, Proposition 2.6.1], hence f has at most

$$\min\left\{ v,\ \sqrt{\frac{2\pi}{C}} \sqrt{3d \log(2d) + 4d \log a_0} \right\}$$

real roots by [21, Theorem 6.1], Corollary 6 and Descartes's rule of signs (see e. g. [26]); here v denotes the number of variations in sign in the sequence $-a_1, a_2, -a_3, \ldots, (-1)^{d-1} a_{d-1}, (-1)^d$.

References

[1] J. Aguirre, M. Bilbao, and J. C. Peral. The trace of totally positive algebraic integers. *Math. Comp.*, 75(253):385–393 (electronic), 2006.

[2] S. Akiyama. Positive finiteness of number systems. In *Number theory*, volume 15 of *Dev. Math.*, pages 1–10. Springer, New York, 2006.

[3] S. Akiyama and A. Pethő. On canonical number systems. *Theoret. Comput. Sci.*, 270(1-2):921–933, 2002.

[4] S. Akiyama and H. Rao. New criteria for canonical number systems. *Acta Arith.*, 111(1):5–25, 2004.

[5] G. Barat, V. Berthé, P. Liardet, and J. Thuswaldner. Dynamical directions in numeration. *Ann. Inst. Fourier (Grenoble)*, 56(7):1987–2092, 2006. Numération, pavages, substitutions.

[6] J.-P. Borel. Polynômes à coefficients positifs multiples d'un polynôme donné. In *Cinquante ans de polynômes (Paris, 1988)*, volume 1415 of *Lecture Notes in Math.*, pages 97–115. Springer, Berlin, 1990.

[7] H. Brunotte. A class of quadrinomial Garsia numbers. *Integers (to appear).*

[8] H. Brunotte. On Garcia numbers. *Acta Math. Acad. Paedagog. Nyházi. (N.S.)*, 25(1):9–16, 2009.

[9] P. Burcsi and A. Kovács. Exhaustive search methods for CNS polynomials. *Monatsh. Math.*, 155(3-4):421–430, 2008.

[10] D. G. Cantor. On sets of algebraic integers whose remaining conjugates lie in the unit circle. *Trans. Amer. Math. Soc.*, 105:391–406, 1962.

[11] A. Chen. On the reducible quintic complete base polynomials. *J. Number Theory*, 129(1):220–230, 2009.

[12] V. Flammang. Comparaison de deux mesures de polynômes. *Canad. Math. Bull.*, 38(4):438–444, 1995.

[13] A. M. Garsia. Arithmetic properties of Bernoulli convolutions. *Trans. Amer. Math. Soc.*, 102:409–432, 1962.

[14] W. J. Gilbert. Radix representations of quadratic fields. *J. Math. Anal. Appl.*, 83(1):264–274, 1981.

[15] E. Hironaka. What is ... Lehmer's number? *Notices Amer. Math. Soc.*, 56(3):374–375, 2009.

[16] M. Marden. *Geometry of polynomials.* Second edition. Mathematical Surveys, No. 3. American Mathematical Society, Providence, R.I., 1966.

[17] M. Mignotte. Sur un théorème de M. Langevin. *Acta Arith.*, 54(1):81–86, 1989.

[18] M. Mignotte and D. Ştefănescu. *Polynomials.* Springer Series in Discrete Mathematics and Theoretical Computer Science. Springer-Verlag Singapore, Singapore, 1999. An algorithmic approach.

[19] M. Mignotte and M. Waldschmidt. On algebraic numbers of small height: linear forms in one logarithm. *J. Number Theory*, 47(1):43–62, 1994.

[20] J.-L. Nicolas and A. Schinzel. Localisation des zéros de polynômes intervenant en théorie du signal. In *Cinquante ans de polynômes (Paris, 1988)*, volume 1415 of *Lecture Notes in Math.*, pages 167–179. Springer, Berlin, 1990.

[21] A. Pethö. On a polynomial transformation and its application to the construction of a public key cryptosystem. In *Computational number theory (Debrecen, 1989)*, pages 31–43. de Gruyter, Berlin, 1991.

[22] I. E. Pritsker. Means of algebraic numbers in the unit disk. *C. R. Math. Acad. Sci. Paris*, 347(3-4):119–122, 2009.

[23] C. van de Woestijne. Enumerating expanding and Pisot polynomials. *http://www.opt.math.tugraz.at/˜cvdwoest/maths/expanding/*, 2010.

[24] P. Varga. Characterization of semi-cns polynomials. *8th Joint Conf. on Math. and Comp. Sci., Komárno, Slovakia*, 2010.

[25] D. G. Wagner. Multivariate stable polynomials: theory and applications. *Bull. Amer. Math. Soc. (N.S.)*, 48(1):53–84, 2011.

[26] X. Wang. A simple proof of descartes's rule of signs. *Am. Math. Monthly*, 111(6):525–526, 2004.

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