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ON COMPLETELY SIMPLE SEMIGROUPS

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Abstract. In this paper completely simple semigroups or generalized groups are considered. We characterize generalized groups which are normal generalized groups. Homomorphisms of generalized groups are considered. Equivalent conditions for the kernel of a homomorphism are deduced. We prove that the group components of a generalized group have the same cardinality. We also prove that if G is a finite normal generalized group, and $q||G|$, where q is a prime number, then G has a generalized subgroup of order q. We deduce a theorem such as second isomorphism theorem for generalized groups.

1. Introduction

The notion of generalized group has been studied in 1999 for constructing a geometric theory [7]. A generalized group is a semigroup $(G, .)$ with the following properties:

(i) For each $g \in G$ there is a unique $e(g) \in G$ such that $ge(g) = e(g)g = g$;
(ii) For each $g \in G$ there is a $g^{-1} \in G$ such that $gg^{-1} = g^{-1}g = e(g)$.

In 2002, Araujo and Konieczny proved that this structure is equivalent to the notion of completely simple semigroups [2]. We recall that a semigroup $(G, .)$ with an idempotent is called a completely simple semigroup [5] if for all $a \in G$, $GaG = G$, and if e and f are idempotents in G such that $ef = fe$ then $e = f$. Completely simple semigroups have characterized by Ree's matrix semigroups [14]. If I and Λ are two sets, D is a group and $P\colon \Lambda \times I \to D$ is a mapping, then $I \times D \times \Lambda$ with the operation $(i_1, d_1, \lambda_1)(i_2, d_2, \lambda_2) = (i_1, d_1P(\lambda_1, i_2)d_2, \lambda_2)$ is a completely simple semigroup and we denote it by $M(D, I, \Lambda, P)$. Each completely simple semigroup is isomorphic to a Ree's matrix semigroup. So in this paper *generalized groups*, *completely simple semigroups*, and *Ree's matrix semigroups* are the same. One must pay attention to this point that each generalized group is a completely regular semigroup [5], but the converse is

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not true. For example $S = \{1, 2\}$ with the binary operation $2.2 = 2,\ 2.1 = 1.2 = 2,\ 1.1 = 1$ is a semigroup. If we define $2^{-1} = 2,\ 1^{-1} = 1$, then S is a completely regular semigroup which is not a generalized group. Because the identity of 2 is not unique.

In the references [1, 4, 6, 7, 10] the notion of generalized group has been considered from the algebraic point of view. One can find the applications of generalized group in dynamical systems, genetic and geometry in the references [8, 9, 10, 11, 12, 13]. This structure has been considered from fuzzy point of view in [3].

In section two we find a condition which determines the normality of generalized groups. Two equivalent conditions for the kernel of a generalized groups homomorphism are determined. If G is a generalized group, and $a \in G$, then $G_a = e^{-1}(e(a))$ is a group with the binary operation of G, and it is called a group component of G. We will prove that the cardinality of all group components of G are the same. If G is a finite normal generalized group, and $q||G|$, where q is a prime number, then in section two we prove that G has a generalized subgroup of order q. Second isomorphism theorem for generalized groups is studied in section three.

2. Main Results

A generalized group G is called a normal generalized group if $e\colon G \to G$ is a homomorphism. In the next theorem we characterize normal generalized groups.

Theorem 2.1. *Let $M(D, I, \Lambda, P)$ be a Ree's matrix semigroup. Then e is a homomorphism if and only if P is a separable function i.e. there exists $h\colon I \to D$ and $g\colon \Lambda \to D$ such that $P(\lambda, i) = g(\lambda)h(i)$ for all $(\lambda, i) \in \Lambda \times I$.*

Proof. The direct calculations imply that:

$$\begin{aligned} e(i_1, a_1, \lambda_1) &= (i_1, h(i_1)^{-1}g(\lambda_1)^{-1}, \lambda_1), \\ e(i_2, a_2, \lambda_2) &= (i_2, h(i_2)^{-1}g(\lambda_2)^{-1}, \lambda_2). \end{aligned}$$

Now, we show that e is a homomorphism.

$$\begin{aligned} e((i_1, a_1, \lambda_1))e((i_2, a_2, \lambda_2)) &= (i_1, h(i_1)^{-1}g(\lambda_1)^{-1}, \lambda_1)(i_2, h(i_2)^{-1}g(\lambda_2)^{-1}, \lambda_2) \\ &= (i_1, h(i_1)^{-1}g(\lambda_2)^{-1}, \lambda_2) \\ &= e((i_1, a_1p(\lambda_1, i_2)a_2, \lambda_2)) \\ &= e((i_1, a_1, \lambda_1)(i_2, a_2, \lambda_2)). \end{aligned}$$

Conversely if e is a homomorphism then

$$e((i_1, a_1, \lambda_1)(i_2, a_2, \lambda_2)) = e((i_1, a_1, \lambda_1))e((i_2, a_2, \lambda_2)).$$

So

$$e((i_1, a_1p(\lambda_1, i_2)a_2, \lambda_2)) = (i_1, p(\lambda_1, i_1)^{-1}, \lambda_1)(i_2, p(\lambda_2, i_2)^{-1}, \lambda_2).$$

Thus,

$$(i_1, p(\lambda_2, i_1)^{-1}, \lambda_2) = (i_1, p(\lambda_1, i_1)^{-1}p(\lambda_1, i_2)p(\lambda_2, i_2)^{-1}, \lambda_2).$$

Then

$$p(\lambda_2, i_1)^{-1} = p(\lambda_1, i_1)^{-1}p(\lambda_1, i_2)p(\lambda_2, i_2)^{-1}.$$

So

$$(*) \qquad p(\lambda_1, i_1) = p(\lambda_1, i_2)p(\lambda_2, i_2)^{-1}p(\lambda_2, i_1).$$

Let $i_0 \in I$ and $\lambda_0 \in \Lambda$ be fixed.

Then for $(\lambda, i) \in \Lambda \times I$ the equality $(*)$ implies

$$P(\lambda, i) = p(\lambda, i_0)p(\lambda_0, i_0)^{-1}p(\lambda_0, i).$$

If we define $g\colon \Lambda \to D$ and $h\colon I \to D$ by $g(\lambda) = p(\lambda, i_0)$ and $h(i) = p(\lambda_0, i_0)^{-1}p(\lambda_0, i)$, then $p(\lambda, i) = g(\lambda)h(i)$ so p is separable. □

If $f\colon G_1 \to G_2$ is a homomorphism then in [6] the kernel of f is defined as the set $\{x \in G_1 |\ \exists a \in G_1$ such that $f(x) = f(e(a))\}$.

Lemma 2.2. *If $f\colon G_1 \to G_2$ is a homomorphism then*

(i) $\ker f = \{x \in G_1 |\ f(x) = f(e(x))\}$;
(ii) $\ker f = \{x \in G_1 |\ f(x^2) = f(x)\}$.

Proof. (i) If $x \in \ker f$, then there is an $a \in G_1$ such that $f(x) = f(e(a))$. So $e(f(x)) = e(f(e(a)))$. Hence, $f(e(x)) = f(e(e(a)))$. So $f(e(x)) = f(e(a))$. Thus, $f(e(x)) = f(x)$. So $\ker f \subseteq \{x \in G_1 |\ f(x) = f(e(x))\}$.

Moreover, the definition of kernel implies $\{x \in G_1 |\ f(x) = f(e(x))\} \subseteq \ker f$. Thus,

$$\ker f = \{x \in G_1 |\ f(x) = f(e(x))\}.$$

(ii) By (i) $\ker f = \{x \in G_1 |\ f(x) = f(e(x))\} = \{x \in G_1 |\ f(x) = e(f(x))\}$. The condition $f(x) = e(f(x))$ implies

$$f(x)f(x) = f(x)e(f(x)) = f(x).$$

So $f(x^2) = f(x)$. Moreover, if $f(x^2) = f(x)$, then $f(x)f(x) = f(x)$. Thus, $e(f(x)) = f(x)$.

Hence $\ker f = \{x \in G_1 |\ f(x^2) = f(x)\}$. □

Remark 2.3. If $M(D, I, \Lambda, P)$ is a Ree's matrix semigroup, then $e(e(a)e(b)) = e(ab)$ for all $a, b \in M(D, I, \Lambda, P)$.

Since each generalized group is equivalent to a Ree's matrix semigroup, then $e(e(a)e(b)) = e(ab)$ for all a, b belong to a generalized group.

Theorem 2.4. *If $f\colon G_1 \to G_2$ is a homomorphism then* $\ker f$ *is a generalized subgroup of G_1 if and only if $Im(f)$ is a generalized subgroup of G_2, and e is a homomorphism on it.*

Proof. Let $\ker f$ be a generalized subgroup of G_1. Then

$$\ker f = \{x \in G_1 |\ f(x) = f(e(x))\}.$$

Since $e(x) \in \ker f$ for all $x \in G_1$, then $e(x)e(y) \in \ker f$ for all $x, y \in G_1$.

So $f(e(x)e(y)) = f(e(e(x)e(y)))$ by Lemma 2.2. The Remark 2.3, implies $f(e(x)e(y)) = f(e(xy))$. So

$$\begin{aligned} e(f(x))e(f(y)) &= f(e(x))f(e(y)) = f(e(x)e(y)) \\ &= f(e(xy)) = e(f(xy)) = e(f(x)f(y)). \end{aligned}$$

So e is a homomorphism on Im f. Moreover in [6] has been proved that: Im f is a generalized subgroup of G_2.

Conversely if Im (F) is a normal generalized subgroup of G_2.

If $a, b \in \ker f$, then

$$\begin{aligned} f(e(ab^{-1})) &= e(f(ab^{-1})) = e(f(a)f(b)^{-1}) = e(f(a))e(f(b)^{-1}) \\ &= f(e(a))f(e(b^{-1})) = f(a)f(b^{-1}) = f(ab^{-1}). \end{aligned}$$

So Lemma 2.2 implies $ab^{-1} \in \ker f$.

By Theorem 2.1 of [6] $\ker f$ is a generalized subgroup of G_1. □

We define the center of a generalized group G by

$$Z(G) = \{x \in G |\ xy = yx \text{ for all } y \in G\}.$$

Theorem 2.5. *$Z(G) \neq \emptyset$ if and only if G is a group.*

Proof. If G is a group then its identity is element of $Z(G)$. So $Z(G) \neq \emptyset$.

Conversely, if $Z(G) \neq \emptyset$, then there is $x \in G$ such that $xy = yx$ for all $y \in G$. Hence, $xy = xye(y)$ and $e(y)xy = e(y)yx = yx = xy$. So $e(xy) = e(y)$. Similarly $e(xy) = e(x)$. So $e(y) = e(x)$ for all $y \in G$. Thus, G is a group. □

Theorem 2.6. *Let G be a generalized group with the Ree's matrix representation $M(D, I, \Lambda, P)$ then $|G_a| = |D|$ for all $a \in G$.*

Proof. If $a = (i_1, a_1, \lambda_1)$, then for $d \in D$.

$e((i_1, d, \lambda_1)) = (i_1, p(\lambda_1, i_1)^{-1}, \lambda_1) = e(a)$. So $\{(i_1, d, \lambda_1) |\ d \in D\} \subseteq G_a$. If $(i_2, d, \lambda_2) \in G_a$, then $e((i_2, d, \lambda_2)) = e(a)$.

So $(i_2, p(\lambda_2, i_2)^{-1}, \lambda_2) = (i_1, p(\lambda_1, i_1)^{-1}, \lambda_1)$. Thus, $i_2 = i_1$, $\lambda_2 = \lambda_1$. Then $G_a \subseteq \{(i_1, d, \lambda_1) |\ d \in D\}$. Hence, $G_a = \{(i_1, d, \lambda_1) |\ d \in D\}$. So $|G_a| = |D|$. □

Theorem 2.7. *Let G be a finite generalized group with non constant identity. Moreover, let the cardinality of G be a prime number. Then $e(x) = x$ for all $x \in G$.*

Proof. Let E be a subset of G such that $E \cap G_a$ be a singleton for all $a \in G$. Then $G = \bigcup_{a \in E} G_a$. So $|G| = |E||G_a|$ for $a \in G$. Since, $|E| \neq 1$, then $|G_a| = 1$ for all $a \in G$. Thus $e(a) = a$ for all $a \in G$. □

Corollary 2.8. *If G is a finite generalized group and its cardinality is prime, then G is a normal generalized group.*

Lemma 2.9. *If G is a finite generalized group and $e(x) = x$ for all $x \in G$, then for all prime number q which $q||G|$, G has a generalized subgroup of order q.*

Proof. We use of the Ree's matrix representation for G, i.e. $G = M(D, I, \Lambda, P)$.
Since $e(x) = x$, then $|D| = |G_x| = 1$. So $q||G| = |I| \times |\Lambda|$. Hence, we have the following two Cases:

1) If $q||I|$, then for a fixed $\lambda_0 \in \Lambda$ we define G_1 by
$$\{(i_r, e, \lambda_0)|\ i_r \in I, r \in \{1, 2, \dots, q\}\}.$$
G_1 with the product of G is a generalized subgroup of G of order q.
2) If $q||\Lambda|$, then as the previous case we can fined a generalized subgroup of order q.

□

Theorem 2.10. *If G is a finite normal generalized group, and $q||G|$, where q is a prime number, then G has a generalized subgroup of order q.*

Proof. $G = \bigcup_{a \in A} G_a$ where $A = \{e(x)|\ x \in G\}$. Then $|G| = |A||G_a|$, and $q||A||G_a|$.
Case 1. If $q||G_a|$, then G_a has a subgroup of order q.
Case 2. If $q||A|$, then the previous lemma implies A has a generalized subgroup of order q. □

3. Second Isomorphism Theorem

In the group theory if G is a group and H is a normal subgroup of it, then any subgroup of G/H has the form K/H where K is a subgroup of G and $H \subseteq K$. This fact is not true in generalized group, for example if $G = \{x_1, x_2, \dots, x_{10}\}$, then G with the operation $x_i x_j = x_i$ is a normal generalized group, then $N = \{x_1, \dots, x_8\}$ is a generalized normal subgroup of G. So $G/N = \bigcup_{i=1}^{8} G_{x_i}/N_{x_i} = \{\{x_1\}, \{x_2\}, \dots, \{x_8\}\}$, and its operation is $\{x_i\}\{x_j\} = \{x_i\}$. G/N is a normal generalized group. If $K = \{\{x_1\}, \{x_2\}, \dots, \{x_5\}\}$, then K is a generalized normal subgroup of G/N, but the next theorem implies there is no any generalized normal subgroup M of G such that $K = M/N$ and $N \subseteq M$.

Theorem 3.1. *If N is a generalized normal subgroup of G and M is a generalized subgroup of G such that $N \subseteq M$, then N is a generalized normal subgroup of M.*

Proof. Since, N is a generalized normal subgroup of G, then there is a generalized group E and a homomorphism $f\colon G \to E$ such that $N \cap G_a = \emptyset$ or $N \cap G_a = \ker f_a$ for all $a \in G$. Let g be the restriction of f on M. Then $g\colon M \to E$ is a homomorphism and $M_a = M \cap G_a$, and $N \cap M_a = \emptyset$ or $N \cap M_a = \ker g_a$. So N is a generalized normal subgroup of M. □

Theorem 3.2. *Let G be a normal general group and M and N be to generalized normal subgroup of G and $N \subseteq M$. Then M/N is a generalized normal subgroup of G/N.*

Proof. The previous theorem implies that N is a generalized normal subgroup of M. There is a generalized group E and a homomorphism $f \colon G \to E$ such that $M_a = \emptyset$ or $M_a = \ker f_a$ for all $a \in G$.

We define $\overline{f} \colon G/N \to E$ by $\overline{f}(xN_a) = f(x)$. $\overline{f}$ is a well defined mapping, because if $xN_a = yN_b$, then there is $r \in N_b$ such that $x = yr$. Since $N \subseteq M$, then $N_b \subseteq M_b$, and

$$f(x) = f(yr) = f(y)f(r) = f(y)f(e(b)) = f(y)f(e(y)) = f(y).$$

So $\overline{f}(xN_a) = \overline{f}(yN_b)$. $\overline{f}$ is also a homomorphism. Because

$$\overline{f}(xN_a yN_b) = \overline{f}(xyN_{ab}) = f(xy) = f(x)f(y) = \overline{f}(xN_a)\overline{f}(yN_b).$$

Moreover

$$\begin{aligned}
\ker \overline{f} &= \{xN_a |\ \overline{f}(xN_a) = \overline{f}(e(xN_a))\} \\
&= \{xN_a |\ f(x) = \overline{f}(e(x)N_a)\} \\
&= \{xN_a |\ f(x) = f(e(x))\} \\
&= \{xN_a |\ f(x) = f(e(a))\} \\
&= \{xN_a |\ x \in \ker f_a = M_a\} = \bigcup_{a \in G} M_a/N_a = M/N.
\end{aligned}$$

So M/N is a generalized normal subgroup of G/N. □

Theorem 3.3. *If G is a normal generalized group and M and N are generalized normal subgroups of G and $N \subseteq M$, then $\frac{G/N}{M/N} \cong G/M$.*

Proof. We have

$$\begin{aligned}
(G/N)_{xN_a} &= \{yN_b |\ e(yN_b) = e(xN_a)\} \\
&= \{yN_b |\ e(y)N_b = e(x)N_a\} \\
&= \{yN_b |\ N_b = N_a \text{ and } e(y) = e(x) = e(a)\} \\
&= \{yN_b |\ e(b) = e(a) \text{ and } e(y) = e(a)\} \\
&= \{yN_b |\ e(y) = e(b)\} = \{yN_b |\ y \in G_b\} = G_b/N_b.
\end{aligned}$$

Since $e(a) = e(b)$, then $G_b/N_b = G_a/N_a$. Similarly, $(M/N)_{xN_a} = M_a/N_a$. Thus,

$$\frac{G/N}{M/N} = \bigcup_{xN_a \in G/N} \frac{(G/N)_{xN_a}}{(M/N)_{xN_a}} = \bigcup_{a \in G} \frac{G_a/N_a}{M_a/N_a} \cong \bigcup_{a \in G} G_a/M_a = G/M.$$

□

4. Conclusion

In this paper we deduce new results on generalized groups. The reader must pay attention to this point that if H and K are two generalized normal subgroups of a generalized group G, then it is not necessary that $HK = KH$.

In generalized groups we have no any theorem such as Lagrange theorem. For example if $G = \{x_1, x_2, x_3, x_4\}$ with the operation $x_i x_j = x_j$ for all $i, j \in \{1, 2, 3, 4\}$. Then $\{x_1, x_2, x_3\}$ is a generalized subgroup of G and $3 \nmid 4$.

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