Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
28 (2012), 177–187
www.emis.de/journals
ISSN 1786-0091

ESTIMATES FOR THE INTEGRAL OF MAXIMAL FUNCTIONS OF FEJÉR KERNEL

NACIMA MEMIĆ

Abstract. In the first part of this paper we prove an estimate for the integral of the maximal function of Fejér kernel related to Walsh-Paley system. In the second part we study local integrability of partial maximal functions obtained from Walsh-Kaczmarz system.

1. Introduction

G. Gát [1] estimated the maximal function of Fejér kernel with respect to Walsh-Paley system, where a specific form of the Fejér kernel K_n has been used. Here we express K_n in a new form which enables us to obtain a sharper estimate.

Many authors have compared the behaviors of Walsh-Paley and Walsh-Kaczmarz systems. Parallel studies resulted in some analogies ([5], [8]) and many contrasts ([7], [2]).

As G. Gát proved in [2] the maximal function of Fejér kernel with respect to Walsh-Kaczmarz system is not integrable on any interval of the dyadic group. However, U. Goginava [3] obtained that the same function is an element from L^p for every $p \in (0,1)$. Besides, K. Nagy [4] established necessary and sufficient conditions for the integrability of weight maximal functions related to the same system.

Our attempt is to study local integrability for partial maximal functions, namely those related to some subsequences of the set of natural numbers.

Let $\mathbb{Z}_2$ denote the discrete cyclic group $\mathbb{Z}_2 = \{0,1\}$, where the group operation is addition modulo 2. If $|E|$ denotes the measure of the subset $E \subset \mathbb{Z}_2$, then $|\{0\}| = |\{1\}| = \frac{1}{2}$.

The dyadic group G is obtained from $G = \prod_{i=0}^{\infty} \mathbb{Z}_2$, where topology and measure are obtained from the product.

2010 *Mathematics Subject Classification.* 42C10.
Key words and phrases. Walsh-Kaczmarz system, Fejér kernels.

Let $x = (x_n)_{n\geq 0} \in G$. The sets $I_n(x) := \{y \in G : y_0 = x_0, \dots, y_{n-1} = x_{n-1}\}$, $n \geq 1$ and $I_0(x) := G$ are dyadic intervals of G. Set $e_n := (\delta_{in})_i$. The Walsh-Paley system is defined as the set of Walsh-Paley functions:

$$\chi_n(x) = \prod_{k=0}^{\infty} (r_k(x))^{n_k},\ n \in \mathbb{N},\ x \in G,$$

where $n = \sum_{k=0}^{\infty} n_k 2^k$ and $r_k(x) = (-1)^{x_k}$. The n-th Walsh-Kaczmarz function is

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} (r_{|n|-1-k}(x))^{n_k},$$

for $n \geq 1$, $k_0(x) := 1$ and $|n| = \max_{n_k \neq 0} k$, so that $n \sim 2^{|n|}$.

We denote the Dirichlet and the Fejér kernel functions by:

$$D_n := \sum_{k=0}^{n-1} \chi_k, \quad D_n^{\kappa} := \sum_{k=0}^{n-1} \kappa_k,$$

$$K_n := \frac{1}{n} \sum_{k=1}^{n} D_k, \quad K_n^{\kappa} := \frac{1}{n} \sum_{k=1}^{n} D_k^{\kappa},$$

where $n \geq 1$.

2. On Walsh-Paley system

Lemma 2.1. *The Fejér kernel satisfies the following formula*

$$nK_n == \sum_{j=1}^{N-1} D_{2^{A_j}} \left(\sum_{i=j+1}^{N} 2^{A_i} \right) \chi_{2^{A_1}} \chi_{2^{A_2}} \dots \chi_{2^{A_{j-1}}} + \sum_{j=1}^{N} (\chi_{2^{A_1}} \chi_{2^{A_2}} \dots \chi_{2^{A_{j-1}}}) 2^{A_j} K_{2^{A_j}}.$$

Proof. Suppose that the number n has the form $n = 2^{A_1} + 2^{A_2} + \dots + 2^{A_N}$, $A_1 > A_2 > \ldots > A_N$.

$$nK_n = \sum_{k=1}^{n} D_k = \sum_{k=1}^{2^{A_1}} D_k + \sum_{k=2^{A_1}+1}^{2^{A_1}+2^{A_2}} D_k + \dots + \sum_{k=n-2^{A_N}+1}^{n} D_k.$$

Notice that for $l = 1, 2, \ldots, N-1$,

$$\sum_{k=2^{A_1}+\ldots+2^{A_l}+1}^{2^{A_1}+\ldots+2^{A_{l+1}}} D_k = \sum_{k=1}^{2^{A_{l+1}}} D_{2^{A_1}+\ldots+2^{A_l}+k} =$$

$$= \sum_{k=1}^{2^{A_{l+1}}} \left(\sum_{i=0}^{2^{A_1}-1} \chi_i + \sum_{i=2^{A_1}}^{2^{A_1}+2^{A_2}-1} \chi_i + \ldots + \sum_{i=2^{A_1}+\ldots+2^{A_{l-1}}}^{2^{A_1}+\ldots+2^{A_l}-1} \chi_i + \sum_{i=2^{A_1}+\ldots+2^{A_l}}^{2^{A_1}+\ldots+2^{A_l}+k-1} \chi_i \right)$$

$$= \sum_{k=1}^{2^{A_{l+1}}} \left(\sum_{i=0}^{2^{A_1}-1} \chi_i + \chi_{2^{A_1}} \sum_{i=0}^{2^{A_2}-1} \chi_i + \cdots + \chi_{2^{A_1}+\cdots+2^{A_{l-1}}} \sum_{i=0}^{2^{A_l}-1} \chi_i + \chi_{2^{A_1}+\cdots+2^{A_l}} \sum_{i=0}^{k-1} \chi_i \right)$$

$$= 2^{A_{l+1}} \left[D_{2^{A_1}} + \chi_{2^{A_1}} D_{2_{A_2}} + \cdots + \chi_{2^{A_1}} \chi_{2^{A_2}} \ldots \chi_{2^{A_{l-1}}} D_{2_{A_l}} \right]$$

$$+ \sum_{k=1}^{2^{A_{l+1}}} \chi_{2^{A_1}} \chi_{2^{A_2}} \ldots \chi_{2^{A_l}} D_k.$$

Where we have considered that the product $\chi_{2^{A_1}} \chi_{2^{A_2}} \ldots \chi_{2^{A_{j-1}}} = 1$, for $j = 1$.

Since

$$\sum_{k=1}^{2^{A_{l+1}}} \chi_{2^{A_1}} \chi_{2^{A_2}} \ldots \chi_{2^{A_l}} D_k = \chi_{2^{A_1}} \chi_{2^{A_2}} \ldots \chi_{2^{A_l}} 2^{A_{l+1}} K_{2^{A_{l+1}}},$$

for $l = 0$ we get $\sum\limits_{k=1}^{2^{A_1}} D_k = 2^{A_1} K_{2^{A_1}}$.

Now, summing over $l = 0, 1, \ldots, N-1$, completes the proof of Lemma 2.1. $\square$

Theorem 2.2. *The integral of the maximal function of Fejér kernel can be estimated by*

$$\int_{G \setminus I_k} \sup_{|n| \geq A} |K_n(x)| dx \leq C \frac{A-k}{2^{A-k}},$$

where $k < A$.

Proof. Using Lemma 2.1, we estimate $\sup\limits_{|n| \geq A} |K_n(x)|$ if $x \in I_{s_2+1}(e_{s_1} + e_{s_2})$, where $s_1 < k \leq A$ and $s_1 < s_2$. Let $n = 2^{A_1} + 2^{A_2} + \ldots + 2^{A_N}$, $A_1 > A_2 > \ldots > A_N$, and $A_1 \geq A$. We have

$$\begin{aligned}
K_n(x) =& \frac{1}{n}\sum_{j=1}^{N-1} D_{2^{A_j}}(\sum_{i=j+1}^{N} 2^{A_i})\chi_{2^{A_1}}(x)\chi_{2^{A_2}}(x)\ldots\chi_{2^{A_{j-1}}}(x)\\
&+\frac{1}{n}\sum_{j=1}^{N}(\chi_{2^{A_1}}(x)\chi_{2^{A_2}}(x)\ldots\chi_{2^{A_{j-1}}}(x))2^{A_j}K_{2^{A_j}}(x)\\
\leq& \frac{1}{n}\sum_{(A_j\leq s_1, j\leq N-1)} 4^{A_j} + \frac{1}{n}\sum_{A_j\leq s_2, j\leq N} 2^{A_j}2^{s_1}\\
\leq& \frac{C}{2^{A_1}}(4^{s_1}+2^{\min(s_2,A_1)}2^{s_1}+4^{s_1})\\
\leq& \frac{C}{2^{A_1}}2^{\min(s_2,A_1)+s_1} \leq \begin{cases} C\frac{2^{s_2+s_1}}{2^{A_1}}, & \text{if } s_2\leq A_1;\\ C2^{s_1}, & \text{if } s_2 > A_1.\end{cases}
\end{aligned}$$

Hence, $\sup\limits_{|n|\geq A}|K_n(x)| \leq 2^{s_1}$, for every $x\in I_{s_2+1}(e_{s_1}+e_{s_2})$, and if $s_2\leq A$, then $s_2\leq |n|$ for all $|n|\geq A$, which means that $\frac{2^{s_1+s_2}}{2^{|n|}} \leq \frac{2^{s_1+s_2}}{2^{A}}$.

Now,

$$\begin{aligned}
\int_{I_{s_1+1}(e_{s_1})}\sup_{|n|\geq A}|K_n(x)|dx =& \sum_{s_2\geq s_1+1}\int_{I_{s_2+1}(e_{s_1}+e_{s_2})}\sup_{|n|\geq A}|K_n(x)|dx\\
&\leq C\sum_{s_1+1\leq s_2\leq A}\frac{2^{s_1+s_2}}{2^A}|I_{s_2+1}| + C\sum_{s_2\geq A+1}2^{s_1}|I_{s_2+1}|\\
&\leq C\left(\sum_{s_1+1\leq s_2\leq A}\frac{2^{s_1}}{2^A}+\sum_{s_2\geq A+1}\frac{2^{s_1}}{2^{s_2}}\right)\leq C\frac{A-s_1}{2^{A-s_1}}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\int_{G\setminus I_k}\sup_{|n|\geq A}|K_n(x)|dx =& \sum_{s=0}^{k-1}\int_{I_{s+1}(e_s)}\sup_{|n|\geq A}|K_n(x)|dx \leq C\sum_{s=0}^{k-1}\frac{A-s}{2^{A-s}}\\
&\leq C\frac{A-k}{2^{A-k}}.
\end{aligned}$$

□

3. On Walsh-Kaczmarz system

Lemma 3.1. *For every $x\in I_{s_2+1}(e_{s_1}+e_{s_2})$, $s_1<s_2$, the inequality*

$$\sup_n\frac{1}{A_n}\left|\sum_{i=0}^{|A_n|-1}2^i r_i(x)K_{2^i}(\tau_i(x))\right| \leq C\max(2^{s_1},2^{s_2-s_1})$$

holds

Proof. First, we take $|A_n| \leq s_1$. We have

$$\frac{1}{A_n}|\sum_{i=0}^{|A_n|-1} 2^i r_i(x) K_{2^i}(\tau_i(x))| \leq \frac{1}{A_n}\sum_{i=0}^{|A_n|-1} 4^i \leq \frac{4^{|A_n|}}{2^{|A_n|}} = 2^{|A_n|} \leq 2^{s_1}.$$

Now if $s_1 < |A_n|$, we have

$$\begin{aligned}
&\frac{1}{A_n}\left|\sum_{i=0}^{|A_n|-1} 2^i r_i(x) K_{2^i}(\tau_i(x))\right| = \\
&= \frac{1}{A_n}\left|\sum_{i=0}^{s_1-1} 2^i\left(2^{i-1}+\frac{1}{2}\right) - 2^{s_1}\left(2^{s_1-1}+\frac{1}{2}\right) + \sum_{i=s_1+1}^{\min(|A_n|-1,s_2)} 2^i 2^{i-s_1-2}\right| \\
&= \frac{1}{A_n}\left|\frac{4^{s_1}-1}{6} - \frac{1}{2} - \frac{4^{s_1}}{2} + \frac{4^{\min(|A_n|,s_2+1)|} - 4^{s_1+1}}{3\cdot 2^{s_1+2}}\right| \\
&\leq C\frac{1}{2^{|A_n|}}\max\left(4^{s_1}, \frac{4^{\min(|A_n|,s_2)}}{2^{s_1}}\right) \leq C\max(2^{s_1}, 2^{s_2-s_1}).
\end{aligned}$$

□

Theorem 3.2. *For every natural number k and every increasing sequence $(A_n)_n$ of natural numbers, we have*

$$\int_{I_k(e_j)} \sup_n |K^{\kappa}_{A_n}(x)| dx = +\infty,$$

for all $j \leq k-1$.

Proof. Let j be arbitrary, but fixed. We may assume that $k \geq 4j$.

Clearly,

$$\int_{I_k(e_j)} \sup_n |K^{\kappa}_{A_n}(x)| dx = \sum_{s=k}^{\infty} \int_{I_s(e_j)\setminus I_{s+1}(e_j)} \sup_n |K^{\kappa}_{A_n}(x)| dx.$$

However, it suffices to consider the sum over $s \geq k$ being such that $s-1 = |A_n| \geq 16$, for some n. Then, we have $r_{|A_n|}(x) = 1$ if $x \in I_s(e_j)\setminus I_{s+1}(e_j)$.

Moreover,

$$\begin{aligned}
\sum_{i=0}^{|A_n|-1} 2^i r_i(x) K_{2^i}(\tau_i(x)) &= \sum_{i=0}^{j-1} 2^i K_{2^i}(\tau_i(x)) - 2^j K_{2^j}(\tau_j(x)) + \sum_{i=j+1}^{|A_n|-1} 2^i K_{2^i}(\tau_i(x)) \\
&= \sum_{i=0}^{j-1} 2^i\left(2^{i-1}+\frac{1}{2}\right) - 2^j\left(2^{j-1}+\frac{1}{2}\right) + \sum_{i=j+1}^{|A_n|-1} 2^i 2^{i-j-2} \\
&= \frac{4^j-1}{6} - \frac{4^j}{2} + \frac{4^{|A_n|}-4^{j+1}}{2^{j+2}} - \frac{1}{2} \\
&\geq -4^{j+1} + \frac{4^{|A_n|-1}}{2^{j+2}} \geq \frac{4^{|A_n|-4}}{2^{j+2}}.
\end{aligned}$$

It follows immediately that

$$|K^{\kappa}_{A_n}(x)| \geq \frac{2^{|A_n|}}{2^{j+11}},$$

for every $x \in I_s(e_j) \setminus I_{s+1}(e_j)$, and that

$$\int_{I_s(e_j)\setminus I_{s+1}(e_j)} \sup_n |K^{\kappa}_{A_n}(x)| dx \geq \frac{2^{|A_n|}}{2^{s+j+12}} = \frac{1}{8 \cdot 2^{j+11}}.$$

Since the set $\{|A_n| \geq 16\}$ is infinite, the result follows. □

Lemma 3.3. *Let $(N_n)_n$ be any sequence of nonnegative integers with $N_n \leq n$. Define the sets*

$$E_{n,N_n} = \{(x_i)_i \in G, x_{n-k} = 0, \forall k \in \{1, 2, \dots, N_n\}\}.$$

Let S denote the set of integers n such that $N_n \neq 0$, $n - N_n > n - k - N_{n-k}$, for every $k \leq n-1$ such that $n - k \in S$, where the least integer from S is the least nonnegative integer n for which $N_n \neq 0$. Then

$$\prod_{n\in S}\left(1 - \frac{1}{2^{N_n}}\right) \leq \left|\bigcap_{n\in\mathbb{N}} E^c_{n,N_n}\right| \leq \prod_{n\in S}\left(1 - \frac{1}{2^{N_n+1}}\right)$$

Proof. For every natural number i let u_i denote the number of I_{i+1}-cosets included in $\bigcap_n E^c_{n,N_n}$, and z_i denote the number of I_{i+1}-cosets included in $\bigcap_n E^c_{n,N_n}$ having 0 as i-th component.

Clearly, $(u_i)_i$ and $(z_i)_i$ are increasing sequences, $u_{i+1} \leq 2u_i$ and $z_i \leq \frac{u_i}{2}$, for every i.

Notice that if $n + k - N_{n+k} \leq n - N_n$, for some $k \geq 1$, $N_n, N_{n+k} \neq 0$, then $E^c_{n,N_n} \subset E^c_{n+k,N_{n+k}}$. Therefore, $\bigcap_{n\in S} E^c_{n,N_n} = \bigcap_{n\in S} E^c_{n,N_n}$.

Also, notice that $u_i = z_i + u_{i-1}$, for every $i \geq 1$, because u_{n-1} represents the number of I_{n+1}-cosets not contained in $\bigcup_n E_n$ having 1 as n-th component.

Clearly, if $i + 1$ is not an element from S, then $z_i = u_{i-1}$, which gives $u_i = 2u_{i-1}$.

We prove that for every i being such that $(i + 1) \in S$, we have

$$2\left(1 - \frac{1}{2^{N_{i+1}}}\right) u_{i-1} \leq u_i \leq 2\left(1 - \frac{1}{2^{N_{i+1}+1}}\right) u_{i-1}.$$

It can be seen that for $N_{i+1} \geq 3$, $i + 1 \in S$, we have

$$z_i = u_{i-1} - z_{i+1-N_{i+1}} + \sum_{k=i+2-N_{i+1}}^{i-1} u_{k-1} - z_k,$$

because the number $(u_{k-1} - z_k)$ represents the number of chains of zeros beginning at some $k - N_{k+1}$, where $(k + 1) \in \{i + 3 - N_{i+1}, \dots, i\} \cap S$.

In this case

$$\sum_{k=i+1-N_{i+1}}^{i} z_k = \sum_{k=i+1-N_{i+1}}^{i-1} u_k,$$

from which we get

$$\sum_{k=i+1-N_{i+1}}^{i-1} z_k + z_i + u_{i-1} = u_{i-1} + \sum_{k=i+1-N_{i+1}}^{i-1} u_k,$$

and

$$u_i = u_{i-1} + \sum_{k=i+1-N_{i+1}}^{i-1} (u_k - z_k) = \sum_{k=i-N_{i+1}}^{i-1} u_k.$$

Now, if $N_{i+1} = 2$, $i+1 \in S$, then $z_i = u_{i-1} - z_{i-1}$.

The expression

$$u_i = \sum_{k=i-N_{i+1}}^{i-1} u_k,$$

is easily verified in this case as

$$u_i = 2u_{i-1} - z_{i-1} = u_{i-1} + u_{i-2}.$$

Therefore, we obtain in the general case that

$$u_i - u_{i-1} \geq \sum_{k=i-N_{i+1}}^{i-2} \frac{1}{2^{i-1-k}} u_{i-1} = \left(1 - \frac{1}{2^{N_{i+1}-1}}\right) u_{i-1}.$$

It follows that

$$2\left(1 - \frac{1}{2^{N_{i+1}}}\right) u_{i-1} \leq u_i.$$

In order to prove the second inequality for $(i+1) \in S$, let j be the largest integer such that $j \leq i-1$ and $(j+1) \in S$.

If $j \leq i - N_{i+1}$, then for every $k \in \{i - N_{i+1}, \dots, i-2\} \cap S$, we have $u_k = \frac{1}{2^{i-1-k}} u_{i-1}$.

Hence,

$$\begin{aligned} u_i - u_{i-1} &= \sum_{k=i-N_{i+1}}^{i-2} u_k = \frac{1}{2^{i-1}} \sum_{k=i-N_{i+1}}^{i-2} 2^k u_{i-1} \\ &= \frac{2^{i-1} - 2^{i-N_{i+1}}}{2^{i-1}} u_{i-1} = \left(1 - \frac{1}{2^{N_{i+1}-1}}\right) u_{i-1}. \end{aligned}$$

We get

$$u_i = 2\left(1 - \frac{1}{2^{N_{i+1}}}\right) u_{i-1}.$$

Suppose that $j > i - N_{i+1}$. We have for $j \leq i-2$,

$$\begin{aligned}
u_i - u_{i-1} &= \sum_{k=i-N_{i+1}}^{j-1} u_k + \sum_{k=j}^{i-2} u_k = u_j - \sum_{k=j-N_{j+1}}^{i-N_{i+1}-1} u_k + \frac{1}{2^{i-1}} \sum_{k=j}^{i-2} 2^k u_{i-1} \\
&= \frac{1}{2^{i-1-j}} u_{i-1} - \sum_{k=j-N_{j+1}}^{i-N_{i+1}-1} u_k + \frac{2^{i-1} - 2^j}{2^{i-1}} u_{i-1} = u_{i-1} - \sum_{k=j-N_{j+1}}^{i-N_{i+1}-1} u_k \\
&\leq u_{i-1} - \sum_{k=j-N_{j+1}}^{i-N_{i+1}-1} \frac{2^k}{2^{i-1}} u_{i-1} = u_{i-1} - \frac{2^{i-N_{i+1}} - 2^{j-N_{j+1}}}{2^{i-1}} u_{i-1} \\
&= \left(1 - \frac{1}{2^{N_{i+1}-1}} + \frac{2^j}{2^{i-1}} \frac{1}{2^{N_{j+1}}}\right) u_{i-1} \leq \left(1 - \frac{1}{2^{N_{i+1}+1}}\right) u_{i-1}.
\end{aligned}$$

If $j = i-1$, then in a similar way we obtain

$$\begin{aligned}
u_i - u_{i-1} &= u_{i-1} - \sum_{k=i-1-N_i}^{i-N_{i+1}-1} u_k \leq u_{i-1} - \frac{1}{2^{i-1}} (2^{i-N_{i+1}} - 2^{i-1-N_i}) u_{i-1} \\
&= \left(1 - \frac{1}{2^{N_{i+1}-1}} + \frac{1}{2^{N_i}}\right) u_{i-1} \leq \left(1 - \frac{1}{2^{N_{i+1}}}\right) u_{i-1}.
\end{aligned}$$

Therefore, we obtain

$$u_i \leq \left(1 - \frac{1}{2^{N_{i+1}+1}}\right) u_{i-1}.$$

Since

$$\left| \bigcap_{n \in \mathbb{N}} E^c_{n,N_n} \right| = \lim_{N \to \infty} \prod_{n=1}^{N} \frac{u_n}{2u_{n-1}} = \prod_{(n+1) \in S} \frac{u_n}{2u_{n-1}},$$

we have

$$\prod_{(n+1) \in S} \frac{2\left(1 - \frac{1}{2^{N_{n+1}}}\right) u_{n-1}}{2u_{n-1}} \leq \left| \bigcap_{n \in \mathbb{N}} E^c_{n,N_n} \right| \leq \prod_{(n+1) \in S} \frac{2\left(1 - \frac{1}{2^{N_{n+1}+1}}\right) u_{n-1}}{2u_{n-1}}.$$

□

Theorem 3.4. *Let k be a natural number and $(A_n)_n$ an increasing sequence of natural numbers. For every $M > 0$ define the sets (E_{l,N_l}) such that $N_l = 0$ if $l \neq |A_n|$ for every n, or if $M > \frac{(A_n - 2^{|A_n|})^2}{A_n}$ for every n such that $l = |A_n|$.*

If $l = |A_n|$ for some n such that $M \leq \frac{(A_n - 2^{|A_n|})^2}{A_n}$, then

$$N_l = \min_{n: l = |A_n|, M \leq \frac{(A_n - 2^{|A_n|})^2}{A_n}} \left[\log_2 \frac{M A_n}{A_n - 2^{|A_n|}}\right] - 1.$$

Then define the sets S_M as in Lemma 3.3. If

$$R_M := \sum_{|A_n| \in S_M} \frac{A_n - 2^{|A_n|}}{A_n} + \sum_{k=1}^{\infty} \frac{1}{M+k} \sum_{|A_n| \in S_{M+k}} \frac{A_n - 2^{|A_n|}}{A_n} < +\infty,$$

where in the sum every $|A_n| \in S$ is taken at most only once with the largest value of $\frac{A_n - 2^{|A_n|}}{A_n}$ satisfying $M \leq \frac{(A_n - 2^{|A_n|})^2}{A_n}$, then we have

$$\int_{G \setminus \bigcup_{j=0}^{k} I_k(e_j)} \sup_n |K^{\kappa}_{A_n}(x)| dx < +\infty.$$

Proof. We use a formula proved in [6]

$$\begin{aligned} K^{\kappa}_n(x) =& \frac{1}{n} + \frac{1}{n} \sum_{i=0}^{|n|-1} 2^i D_{2^i}(x) + \frac{1}{n} \sum_{i=0}^{|n|-1} 2^i r_i(x) K_{2^i}(\tau_i(x)) \\ &+ \frac{1}{n}(n - 2^{|n|}) D_{2^{|n|}}(x) + \frac{1}{n}(n - 2^{|n|}) r_{|n|}(x) K_{n-2^{|n|}}(\tau_{|n|}(x)), \end{aligned}$$

where $\tau_n(x_0, \dots, x_n, \dots) = (x_{n-1}, x_{n-2}, \dots, x_0, x_n, x_{n+1}, \dots)$.

The first sum and the fourth term clearly represent integrable functions. We only need to study the integrability of the second sum and the last term.

Let $x \in G \setminus \bigcup_{j=0}^{k} I_k(e_j)$. Then, $x \in I_{s_2+1}(e_{s_1} + e_{s_2})$, where $s_1 < s_2 \leq k-1$. Using Lemma 3.1, we get

$$\begin{aligned} &\int_{G \setminus \bigcup_{j=0}^{k} I_k(e_j)} \sup_n \frac{1}{A_n} \left| \sum_{i=0}^{|A_n|-1} 2^i r_i(x) K_{2^i}(\tau_i(x)) \right| dx = \\ &= \sum_{s_1=0}^{k-2} \sum_{s_2=s_1+1}^{k-1} \int_{I_{s_2+1}(e_{s_1}+e_{s_2})} \frac{1}{A_n} \left| \sum_{i=0}^{|A_n|-1} 2^i r_i(x) K_{2^i}(\tau_i(x)) \right| dx \\ &\leq \sum_{s_1=0}^{k-2} \left(\sum_{s_2=s_1+1}^{2s_1} \frac{2^{s_1}}{2^{s_2+1}} + \sum_{s_2=2s_1+1}^{k-1} \frac{2^{s_2-s_1}}{2^{s_2+1}} \right) \\ &\leq \sum_{s_1=0}^{k-2} \left(\frac{2^{s_1}}{2^{s_1+1}} + \frac{k}{2^{s_1}} \right) \leq \frac{k}{2} + k \sum_{s_1=0}^{k-2} \frac{1}{2^{s_1}} \leq 3k. \end{aligned}$$

Let

$$F_M = \left\{ x \in G : \sup_n \frac{A_n - 2^{|A_n|}}{A_n} |K_{A_n - 2^{|A_n|}}(\tau_{|A_n|}(x))| \geq M \right\}.$$

If $x \in F_M$, then $|K_{A_n-2^{|A_n|}}(\tau_{|A_n|}(x))| \geq \frac{MA_n}{A_n - 2^{|A_n|}}$ for some n, which implies that $M \leq \frac{(A_n - 2^{|A_n|})^2}{A_n}$ and $x \in E_{|A_n|, s_n}$, for $s_n \geq [\log_2 \frac{MA_n}{A_n - 2^{|A_n|}}] - 1$.

This gives that

$$F_M \subset \bigcup_{l \in S_M} E_{l, N_l}.$$

By Lemma 3.3, we get

$$\left|\bigcap_{l\in S_M} E^c_{l,N_l}\right| \geq \prod_{l\in S_M}\left(1-\frac{1}{2^{N_l}}\right) = \prod_{|A_n|\in S_M}\left(1-\frac{A_n-2^{|A_n|}}{MA_n}\right),$$

where in the last product every $|A_n|$ is taken at most only once with the largest value of $\frac{A_n-2^{|A_n|}}{A_n}$ satisfying $M \leq \frac{(A_n-2^{|A_n|})^2}{A_n}$.

Since the convergence of R_M implies that $R_{M'}$ converges for every $M' > M$. Then we can choose M sufficiently large so that

$$\left|\bigcap_{l} E^c_{l,N_l}\right| \geq \exp\left(-\frac{2}{M}\sum_{|A_n|\in S_M}\frac{A_n-2^{|A_n|}}{A_n}\right).$$

It follows

$$|F_M| \leq 1-\exp\left(-\frac{2}{M}\sum_{|A_n|\in S_M}\frac{A_n-2^{|A_n|}}{A_n}\right) \leq \frac{C}{M}\sum_{|A_n|\in S_M}\frac{A_n-2^{|A_n|}}{A_n}.$$

We obtain

$$\begin{aligned}
&\int_{G\setminus\bigcup_{j=0}^k I_k(e_j)} \sup_n \frac{A_n-2^{|A_n|}}{A_n}|K_{A_n-2^{|A_n|}}(x)|dx \leq \\
&\qquad \leq M+(M+1)|F_M| + \sum_{k=1}^{\infty}(M+k+1-(M+k))|F_{M+k}| \\
&\qquad \leq M + CR_M < +\infty.
\end{aligned}$$

□

Example 3.5. Consider the sequence $(A_n = 2^n + 2^{\alpha n})_{\alpha n\in\mathbb{N}}$, for some fixed rational number α such that $1 > \alpha > \frac{1}{2}$. Applying Theorem 3.4, we get

$$\begin{aligned}
R_M &= \sum_{|A_n|\in S_M}\frac{A_n-2^{|A_n|}}{A_n} + \sum_{k=1}^{\infty}\frac{1}{M+k}\sum_{|A_n|\in S_{M+k}}\frac{A_n-2^{|A_n|}}{A_n} \\
&\leq \sum_{n\geq 1}\frac{2^{\alpha n}}{2^n+2^{\alpha n}} + \sum_{k=1}^{\infty}\frac{1}{M+k}\sum_{n\geq\frac{\log_2(M+k)}{2\alpha-1}}\frac{2^{\alpha n}}{2^n+2^{\alpha n}} \\
&\sim \sum_{k=1}^{\infty}\frac{1}{(M+k)^{1+\frac{1-\alpha}{2\alpha-1}}} < +\infty.
\end{aligned}$$

While, if we consider the sequence $A_n = 2^n + 2^{n-l-1}$, $n > 2l+2$, used by G. Gát in [2], where l is a fixed positive integer, we have for arbitrary $M > 0$,

$$R_M = \sum_{n \geq \log_2 M + 2l + 2} \frac{2^{n-l-1}}{2^n + 2^{n-l-1}} + \sum_{k=1}^{\infty} \frac{1}{M+k} \sum_{n \geq \log_2 (M+k) + 2l + 2} \frac{2^{n-l-1}}{2^n + 2^{n-l-1}}$$
$$> \sum_{n \geq \log_2 M + 2l + 2} \frac{2^{-l-1}}{1 + 2^{-l-1}} = +\infty.$$

References

[1] G. Gát. On the Fejér kernel functions with respect to the Walsh-Paley system. *Acta Acad. Paedagog. Agriensis, Sect. Mat. (N.S.)*, 24(v. 24):105–110, 1997.

[2] G. Gát. On the Fejér kernel functions with respect to the Walsh-Kaczmarz system. *Acta Math. Acad. Paedagog. Nyházi. (N.S.)*, 17(2):121–126 (electronic), 2001.

[3] U. Goginava. On the conjecture of Gát. *Acta Math. Acad. Paedagog. Nyházi. (N.S.)*, 20(1):39–42 (electronic), 2004.

[4] K. Nagy. On the L^1 norm of the weighted maximal function of Fejér kernels with respect to the Walsh-Kaczmarz system. *JIPAM. J. Inequal. Pure Appl. Math.*, 9(1):Article 16, 9, 2008.

[5] F. Schipp. Pointwise convergence of expansions with respect to certain product systems. *Anal. Math.*, 2(1):65–76, 1976.

[6] V. A. Skvorcov. On Fourier series with respect to the Walsh-Kaczmarz system. *Anal. Math.*, 7(2):141–150, 1981.

[7] A. A. Šneĭder. On series of Walsh functions with monotonic coefficients. *Izvestiya Akad. Nauk SSSR. Ser. Mat.*, 12:179–192, 1948.

[8] W. S. Young. On the a.e. convergence of Walsh-Kaczmarz-Fourier series. *Proc. Amer. Math. Soc.*, 44:353–358, 1974.

Received May 9, 2012.

Department of Mathematics,
University of Sarajevo,
Zmaja od Bosne 33-35,
Sarajevo, Bosnia and Herzegovina
E-mail address: nacima.o@gmail.com