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A SHORT REVIEW ON THE THEORY OF GENERALIZED CONICS

ÁBRIS NAGY

Dedicated to Professor Lajos Tamássy on the occasion of his 90th birthday

Abstract. Generalized conics are the level sets of functions measuring the average distance from a given set of points. This involves an extension of the concept of conics to the case of infinitely many focuses. The measuring of the average distance is realized via integration. Generalized conics recently have many interesting applications from Finsler geometry to geometric tomography. The aim of this survey paper is to collect the most important results concerning generalized conics.

1. Introduction

Generalized conics are the level sets of functions measuring the average distance from a given set of points. Polyellipses as the level sets of the function measuring the arithmetic mean of distances from the elements of a finite point set are one of the most important examples for generalized conics [7], [10]. They appear in optimization problems in a natural way [3]. The original formulation is due to P. Fermat: find the point P in the plane of the triangle $\triangle ABC$ such that the sum $PA + PB + PC$ is minimal. Polyellipses with three focuses are also called trifocal curves. They have applications in architecture, urban and spatial planning [8]. The characterization of the minimizer of the function measuring the sum of distances from finitely many given points is due to E. Vázsonyi [15]. He also posed the problem of the approximation of convex plane curves with polyellipses. P. Erdős and I. Vincze [1] proved that it is impossible for regular triangles, see also [11].

It is natural to take any other type of mean instead of the standard arithmetic one. To include hyperbolas we can admit simple weighted sum of distances. Classical conics can be considered as equidistant sets to suitable plane circles. As plane circles play the role of the foci in this plot it is also natural to

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replace these circles by more complicated planar sets, hence equidistant sets are generalizations of conics [9]. Lemniscates are sets all of whose points have the same geometric mean of the distances (i.e. their product is constant). Lemniscates play a central role in the theory of approximation. The polynomial approximation of a holomorphic function can be interpreted as the approximation of the level curves with lemniscates. The product of distances corresponds to the absolute value of the root-decomposition of polynomials in the complex plane.

In the case of an infinite set of points we can use integration over the set of foci to calculate the average distance. This concept was introduced by C. Gross and T.-K. Strempel [2] and they posed the problem whether *which results* (of the classical case) *can be extended to the case of infinitely many focal points or to continuous set of foci.*

The aim of this paper is to give a short review of the theory of generalized conics and their applications based on the works [4, 5, 6, 7, 10, 13, 14] and [12].

2. Preliminaries

Let $\mathbb{R}^N$ be the N-dimensional real coordinate space with the standard basis $\underline{e}_1, \dots, \underline{e}_N$ ($N \in \mathbb{N}$, $N > 0$). Vectors of the form $\underline{x} = (x_1, x_2, \dots, x_N)$ denote elements of $\mathbb{R}^N$. Throughout this paper λ_N will denote the N-dimensional Lebesgue measure. $\mathbb{R}^N$ is equipped with the canonical inner product

$$\langle\ ,\ \rangle : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}, \quad \big(\underline{x}, \underline{y}\big) \mapsto \big\langle \underline{x}, \underline{y} \big\rangle = \sum_{i=1}^{N} x_i y_i,$$

and $\mathbb{R}^3$ is equipped with the cross product

$$\times \colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3, \quad \big(\underline{x}, \underline{y}\big) \mapsto \underline{x} \times \underline{y} = \left(x_2 y_3 - x_3 y_2, -x_1 y_3 + x_3 y_1, x_1 y_2 - x_2 y_1\right).$$

Let $\|\underline{x}\|_p$ be the p-norm of $\underline{x}$ and consider the distance function d_p induced by the p-norm:

$$\|\underline{x}\|_p = \sqrt[p]{\sum_{i=1}^{N} |x_i|^p}, \quad d_p(\underline{x}, \underline{y}) = \left\|\underline{x} - \underline{y}\right\|_p.$$

Definition 1. Let $d \colon \mathbb{R}^N \to \mathbb{R}$ be a metric and μ be a measure on a compact set $K \subset \mathbb{R}^N$ with $\mu(K) > 0$. The unweighted *generalized conic function* f_K associated to K is

$$f_K \colon \mathbb{R}^N \to \mathbb{R}, \quad \underline{x} \mapsto f_K(\underline{x}) := \int_K g(\underline{x}, \underline{y}) d(\underline{x}, \underline{y})\, d_\mu \underline{y}, \tag{1}$$

where $g\colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is the kernel function for f_K. The set K is called the set of foci. The *weighted generalized conic function* F_K associated to K is

$$F_K\colon \mathbb{R}^N \to \mathbb{R}, \quad \underline{x} \mapsto F_K(\underline{x}) := \frac{1}{\mu(K)} \int\limits_K g(\underline{x}, \underline{y}) d(\underline{x}, \underline{y})\, d_\mu \underline{y}.$$

The level sets $C_K = \left\{\underline{x} \in \mathbb{R}^N | f_K(\underline{x}) \le c\right\}$ are called *generalized conics.*

3. Polyellipses

Polyellipses are one of the most important examples of generalized conics with many applications. Basic properties of polyellipses and important results are collected in this section along the works due to Sekino [10] and Nie, Parrilo, Sturmfels [7].

Definition 2. Let $\underline{p}_1, \underline{p}_1, \dots, \underline{p}_n \in \mathbb{R}^2$ be points on the plane. The function

$$f\colon \mathbb{R}^2 \to \mathbb{R}, \quad \underline{x} \mapsto f(\underline{x}) := \sum_{i=1}^n \left\| \underline{x} - \underline{p}_i \right\|$$

is called the *distance sum function.* Level sets of the form $f(\underline{x})$ =const. are called *polyellipses* with foci $\underline{p}_1, \underline{p}_1, \dots, \underline{p}_n$. By n-ellipse we mean a polyellipse with n focal points.

It is easy to see that every polyellipse is a generalized conic with $K = \left\{\underline{p}_1, \underline{p}_1, \dots, \underline{p}_n\right\}$, $d = d_2$, μ to be the counting measure and kernel function $g(\underline{x}, \underline{p}_i) = 1$.

Theorem 1. *The distance sum function f has a global minimizer.*

Proof. Let D be a closed disk containing all the foci in its interior and let $\underline{c}$ denote the center of D. Since f is a continuous function, f attains its minimum value M on the compact set D at some point $\underline{s}$. We show that M is the global minimum value of f. Let $\underline{r}$ be an arbitrary point in $\mathbb{R}^2 \setminus D$ and let $\underline{q}$ denote the intersection of the boundary of D with the line segment connecting $\underline{r}$ and $\underline{c}$. Then the distance of $\underline{q}$ from any of the foci is less then the distance of $\underline{r}$ from the same focal point. Thus $f(\underline{r}) > f(\underline{q}) \ge f(\underline{s}) = M$. □

Theorem 2 ([10])**.** *Let $M \in \mathbb{R}$ be the the global minimum value of the distance function f. Then every polyellipse with the distance sum greater than M is a piecewise smooth Jordan curve and its interior is a nonempty compact convex set.*

The following theorem gives the degree of an n-ellipse as an algebraic curve.

Theorem 3 ([7])**.** *Every polyellipse is an algebraic curve on the plane. The polynomial equation defining an n-ellipse has degree 2^n if n is odd and $2^n - \binom{n}{n/2}$ if n is even.*

Finally we would like to mention a result due to Sekino on the uniqueness of the global minimizer of f. We say that the point $\underline{s} \in \mathbb{R}^2$ is the center of the distance sum function f if $\underline{s}$ is the unique point at which f attains its global minimum. By a critical point, we mean a point $\underline{r} \in \mathbb{R}^2$ where $\nabla f(\underline{r}) = 0$. This includes the assumption that $\underline{r}$ is not one of the foci.

Theorem 4 ([10])**.** *Let an n-ellipse be given. (A) Suppose the foci are non-collinear. If a critical point exists then it is the center; otherwise one of the foci coincides with the center. (B) Suppose the foci are collinear. If n is even, then f has no center, and instead f attains its global minimum at every point in the closed line segment joining the middle two foci; if n is odd, then the middle focus is is the center.*

4. Awnings

We would like to give an extended overview of [4] in this section. The central problem is the characterization of the minimizer of the function (1) under special choices of K and the kernel function.

Definition 3. Let $\gamma\colon [a, b] \to \mathbb{R}^3$ be a continuous, piecewise smooth curve under the partition $a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$. The *generalized cone* $\mathcal{C}_\gamma$ with directrix γ and vertex $\underline{x}$ is the set

$$\mathcal{C}_\gamma(\underline{x}) = \left\{ s\underline{x} + (1-s)\gamma(t) \middle| t \in [a, b], s \in [0, 1] \right\}$$

Consider the function

$$\mathcal{A}\colon \mathbb{R}^3 \to \mathbb{R}, \quad \underline{x} \mapsto \mathcal{A}(\underline{x}) := \lambda_2\left(\mathcal{C}_\gamma(\underline{x})\right)$$

measuring the area of $\mathcal{C}_\gamma(\underline{x})$. Then the level sets of the form $\mathcal{A}(\underline{x})$ =const. are called *awnings* spanned by γ.

The area function can be calculated by the formula

$$\mathcal{A}(\underline{x}) = \frac{1}{2} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} |(\underline{x} - \gamma(t)) \times \gamma'(t)| \; dt$$

where $\underline{u} \times \underline{v}$ denotes the cross product of the vectors $\underline{u}$ and $\underline{v}$ in $\mathbb{R}^3$.

Theorem 5. *Every awning is the boundary of a generalized conic with the set of foci γ, $d = d_2$, $\mu = \lambda_1$ and kernel function*

$$g(\underline{x}, \gamma(t)) = \sin\left(\cos^{-1}\left(\frac{\langle \underline{x} - \gamma(t), \gamma'(t)\rangle}{\|\underline{x} - \gamma(t)\|_2 \cdot \|\gamma'(t)\|_2}\right)\right)$$

i.e. the sin of the angle of $\underline{x} - \gamma(t)$ and the tangent line of γ at $\gamma(t)$.

Theorem 6. *The area function $\mathcal{A}$ is convex. Consequently the sets of the form*

$$\left\{ \underline{x} \in \mathbb{R}^3 \middle| \mathcal{A}(\underline{x}) \leq c, c \in \mathbb{R}^+ \right\}$$

are convex closed subsets of $\mathbb{R}^3$. *Except the case of a line segment as the focal curve, any awning spanned by* γ *is a convex, compact subset of* $\mathbb{R}^3$ *and the area function* $\mathcal{A}$ *has a global minimizer.*

In the following subsections we discuss the problem of the minimizer. We formulate the analogues of Weissfeld's theorem [15] for the regular minimizer of a function measuring the sum of distances (the arithmetic mean) from finitely many given points.

4.1. **Awnings spanned by simple polygonal chains.** Let $\mathcal{P}$ be a closed polygonal chain in $\mathbb{R}^3$ with vertices $\underline{y}_0, \underline{y}_1 \dots, \underline{y}_n = \underline{y}_0$ ($n \geq 3$) such that no three of them are collinear. Then the area function reduces to the finite sum

$$\mathcal{A}(\underline{x}) = \frac{1}{2}\sum_{i=1}^{n} \left|(\underline{x} - \underline{y}_{i-1}) \times (\underline{x} - \underline{y}_i)\right| \tag{2}$$

A point $\underline{x}$ is called regular if $\underline{x}, \underline{y}_{i-1}, \underline{y}_i$ are not collinear for any $i \in \{1, \dots, n\}$. The directional derivative of $\mathcal{A}$ in the regular point $\underline{x}$ along a vector $\underline{v} \in \mathbb{R}^3$ is

$$D_{\underline{v}}\mathcal{A}(\underline{x}) = \sum_{i=1}^{n} \left\langle (\underline{y}_{i-1} - \underline{y}_i) \times \underline{n}_i(\underline{x}), \underline{v} \right\rangle, \tag{3}$$

where

$$\underline{n}_i(\underline{x}) := \frac{(\underline{x} - \underline{y}_{i-1}) \times (\underline{x} - \underline{y}_i)}{\left|(\underline{x} - \underline{y}_{i-1}) \times (\underline{x} - \underline{y}_i)\right|} \tag{4}$$

is a unit vector orthogonal to the plane spanned by $\underline{x}, \underline{y}_{i-1}$ and $\underline{y}_i$. To characterize the regular minimizers we need only to state the first order condition because of the convexity of the function. The analogue of the Weissfeld's theorem [15] for the regular minimizer can be formulated as follows.

Theorem 7. *A regular point* $\underline{x}$ *is a global minimizer of* $\mathcal{A}$ *if and only if*

$$\sum_{i=1}^{n} (\underline{y}_{i-1} - \underline{y}_i) \times \underline{n}_i(\underline{x}) = 0$$

Example 1. *The point* $\underline{x}_0 = (1/2, 1/2, 1/2)$ *is the global minimizer of* $\mathcal{A}$ *for the closed polynomial chain* $\mathcal{P}_6$ *with vertices* $(0,0,0)$, $(0,0,1)$, $(0,1,1)$, $(1,1,1)$, $(1,1,0)$, $(1,0,0)$. *The chain represents a closed path along the edges of a cube.*

The following example shows that the minimizer is not unique in general. At the same time we present a non-regular case because the formula for the directional derivative is far from being linear in the variable $\underline{v}$.

Example 2. *Let the vertices of the polygonal chain* $\mathcal{P}_{12}$ *be*

$$\underline{y}_0 = (0,0,0)\,,\ \underline{y}_1 = (0,1,0)\,,\ \underline{y}_2 = (0,1,2)\,,\ \underline{y}_3 = (0,0,2)\,,\ \underline{y}_4 = (0,0,1)\,,$$
$$\underline{y}_5 = (0,-1,1)\,,\ \underline{y}_6 = (0,-1,-1)\,,\ \underline{y}_7 = (0,0,-1)\,,\ \underline{y}_8 = (1,0,-1)\,,$$

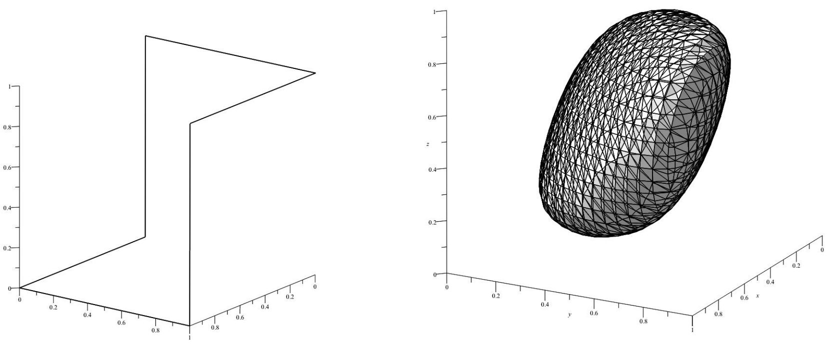

FIGURE 1. Example 1 - $\mathcal{P}_6$ and an awning spanned by $\mathcal{P}_6$

$$\underline{y}_9 = (1,0,3)\,,\ \underline{y}_{10} = (-1,0,3)\,,\ \underline{y}_{11} = (-1,0,0) \quad \textit{and} \ \ \underline{y}_{12} = \underline{y}_0.$$

If $\underline{x}$ is a point between $\underline{y}_0$ and $\underline{y}_4$ then for all $\underline{v} \in \mathbb{R}^3$

$$D_{\underline{v}}\mathcal{A}(\underline{x}) = |\underline{v} \times \underline{r}| + \langle \underline{v}, \underline{n}(\underline{x})\rangle\,,$$

where $\underline{r} = \underline{y}_4 - \underline{y}_0 = (0,0,1)$,

$$\underline{n}(\underline{x}) = \sum_{\substack{1 \le i \le 12\\ i \ne 4}} (\underline{y}_{i-1} - \underline{y}_i) \times \underline{n}_i(\underline{x});$$

see (4). It is easy to check that $\langle \underline{n}(\underline{x}), \underline{r}\rangle = 0$ and $|\underline{n}(\underline{x})| = |\underline{r}| = 1$. Then we get

$$D_{\underline{v}}\mathcal{A}(\underline{x}) \ge 0$$

for all $\underline{v} \in \mathbb{R}^3$. This means that the origin belongs to the set of subgradients of $\mathcal{A}$ at $\underline{x}$ and the general theory of convex functions says that $\underline{x}$ is a global minimizer of $\mathcal{A}$.

To provide the unicity of the minimizer, the basic idea is to require the strict convexity of the function $\mathcal{A}$. The conditions of the following illustrative theorems guarantee that for any different points $\underline{x}_1, \underline{x}_2 \in \mathbb{R}^3$ we can find an index i such that the term

$$h\colon \mathbb{R}^3 \to \mathbb{R}, \quad \underline{x} \mapsto h(\underline{x}) := \left(\underline{x} - \underline{y}_{i-1}\right) \times \left(\underline{x} - \underline{y}_i\right)$$

in the sum (2) is strictly convex along the line l of $\underline{x}_1$ and $\underline{x}_2$. This obviously happens if l and the line through $\underline{y}_{i-1}$ and $\underline{y}_i$ are skew lines.

Theorem 8. *If $\mathcal{P}_n$ is a simple closed polygonal chain with vertices $\underline{y}_0, \underline{y}_1, \ldots, \underline{y}_n = \underline{y}_0$ and*

(1) n is odd, $n \ge 5$

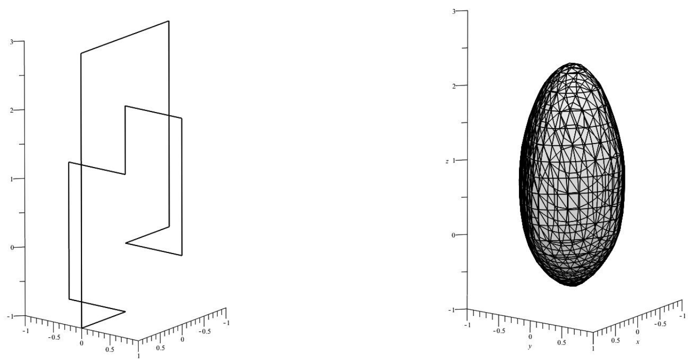

FIGURE 2. Example 2 - $\mathcal{P}_{12}$ and an awning spanned by $\mathcal{P}_{12}$

(2) $\underline{y}_i$, $\underline{y}_{i+1}$, $\underline{y}_{i+2}$, $\underline{y}_{i+3}$ *are in general position for all* $i \in \{0, 1, \ldots, n-3\}$
then $\mathcal{A}$ *has a unique global minimizer.*

Theorem 9. *If* $\mathcal{P}_n$ *is a simple closed polygonal chain with vertices* $\underline{y}_0, \underline{y}_1, \ldots, \underline{y}_n = \underline{y}_0$ *and*

(1) $n \geq 5$
(2) $\underline{y}_i$, $\underline{y}_{i+2}$, $\underline{y}_{i+4}$ *are not collinear for all* $i \in \{0, 1, \ldots, n\}$
(3) $\underline{y}_i$, $\underline{y}_{i+1}$, $\underline{y}_{i+2}$, $\underline{y}_{i+3}$ *are in general position for all* $i \in \{0, 1, \ldots, n\}$

then $\mathcal{A}$ *has a unique global minimizer.*

The conditions of Theorem 9 can be directly checked for the following example: the chain represents a closed path along the edges of an octahedron.

Example 3. *The point* $\underline{x} = (1, 1, 0)$ *is the unique global minimizer of* $\mathcal{A}$ *for the closed polynomial chain* $\mathcal{P}_6$ *with vertices* $(0, 0, 0)$, $(2, 0, 0)$, $(1, 1, -2)$, $(0, 2, 0)$, $(2, 2, 0)$, $(1, 1, 2)$.

4.2. **Awnings spanned by smooth curves.** In the case of awnings spanned by smooth curves the unicity of the minimizer can be also guaranteed in such a way that we require the strict convexity. Since we have an 'infinite sum' instead of (2) we should change the principle of existence of the 'strict convex term' in a suitable way:

(P) in the case of polygonal chains for any different points $\underline{x}_1, \underline{x}_2 \in \mathbb{R}^3$ we should find an index i such that $\underline{x}_1$, $\underline{x}_2$ and $\underline{y}_{i-1}$, $\underline{y}_i$ determine skew lines.

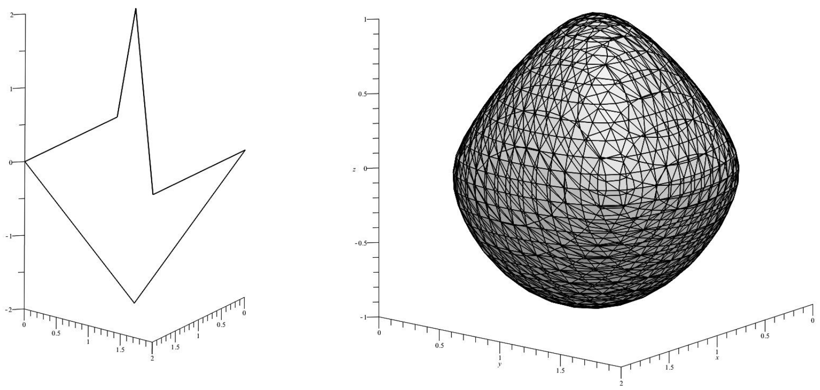

FIGURE 3. Example 3 - $\mathcal{P}_6$ and an awning spanned by $\mathcal{P}_6$

(C) in the case of smooth curves for any different points $\underline{x}_1, \underline{x}_2 \in \mathbb{R}^3$ we should find a parameter t such that $\underline{x}_1$, $\underline{x}_2$ and the tangent at $\gamma(t)$ determine skew lines.

The following theorem gives a system of conditions to imply condition (C). Roughly speaking condition 2 in Theorem 9 corresponds to the nonzero curvature and condition 3 in Theorem 9 corresponds to the nonzero torsion (i.e. the curve does not belong to any plane of the space).

Theorem 10. *Let $\gamma\colon [a,b] \to \mathbb{R}^3$ be a smooth curve with never vanishing curvature and suppose that γ is not a plane curve. Then $\mathcal{A}$ has a unique global minimizer.*

A point $\underline{x}$ is said to be regular for γ if it is not an element of the set

$$\left\{\gamma(t) + s\gamma'(t) \middle| t \in [a,b], s \in \mathbb{R}\right\}$$

Theorem 11. *A regular point $\underline{x}$ is a global minimizer of $\mathcal{A}$ if and only if*

$$\int_a^b \frac{(\underline{x} - \gamma(t)) \times \gamma'(t)}{|(\underline{x} - \gamma(t)) \times \gamma'(t)|} \times \gamma'(t)\, dt = 0$$

As an application of the cited results let us investigate the following problem.

Example 4. *Consider now the curve*

$$\gamma(t) = (\cos t, \sin t, \sin(3t)) \quad t \in [0, 2\pi]$$

Then

$$\gamma'(t) \times \gamma''(t) = (-24\cos^3 t \sin t, 24\cos^4 t - 36\cos^2 t + 9, 1)$$

which means that the curvature is nonzero for all $t \in [a,b]$. On the other hand

$$\gamma'''(t) = (\sin t, -\cos t, -27\cos(3t))$$

thus

$$\langle \gamma'(t) \times \gamma''(t), \gamma'''(t)\rangle = -24(4\cos^2 t - 3)\cos t.$$

This shows that the torsion is nonzero in at least one point, and we can apply Theorem 10, which says $\mathcal{A}$ has a unique global minimizer. We show that this minimizer is the origin.

The origin is a global minimizer if and only if

$$\int_0^{2\pi} \frac{\gamma(t) \times \gamma'(t)}{|\gamma(t) \times \gamma'(t)|} \times \gamma'(t)\, dt = 0.$$

Notice that $\gamma(t+\pi) = -\gamma(t)$ holds for all $t \in [0, \pi]$. Then

$$\int_0^{2\pi} \frac{\gamma(t) \times \gamma'(t)}{|\gamma(t) \times \gamma'(t)|} \times \gamma'(t)\, dt =$$

$$= \int_0^{\pi} \frac{\gamma(t) \times \gamma'(t)}{|\gamma(t) \times \gamma'(t)|} \times \gamma'(t)\, dt + \int_{\pi}^{2\pi} \frac{\gamma(t) \times \gamma'(t)}{|\gamma(t) \times \gamma'(t)|} \times \gamma'(t)\, dt =$$

$$= \int_0^{\pi} \frac{\gamma(t) \times \gamma'(t)}{|\gamma(t) \times \gamma'(t)|} \times \gamma'(t)\, dt - \int_0^{\pi} \frac{\gamma(t) \times \gamma'(t)}{|\gamma(t) \times \gamma'(t)|} \times \gamma'(t)\, dt = 0.$$

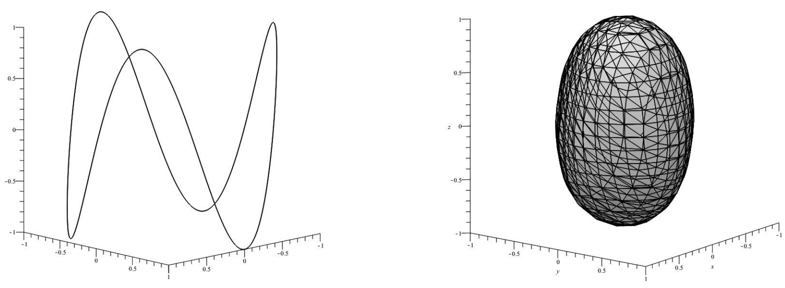

FIGURE 4. Example 4 - γ and an awning spanned by γ

5. MINKOWSKI FUNCTIONALS AND GENERALIZED CONICS

L. Bieberbach proved that the holonomy group of any flat compact Riemannian manifold is finite. If v is a non-zero element in the tangent space T_pM then its orbit is a finite set which is invariant under the holonomy group. As a focal set the finite invariant system determines invariant polyellipses/pollyellipsoids. Using parallel transports we can transfer the invariant polyellipse/polyellipsoid to any tangent space. They form a smoothly varying family of compact convex bodies in the tangent spaces. This is the general structure of the alternative of the Riemannian geometry for the Lévi-Civita connection. *Finsler geometry is* *a non-Riemannian geometry in a finite number of dimensions. The differentiable structure is the same as the Riemannian one but distance is not uniform in all directions. Instead of the Euclidean spheres in the tangent spaces, the unit vectors form the boundary of general convex sets containing the origin in their interiors.* (M. Berger). Since the holonomy group is typically not finite we should extend the notion of polyellipses to construct holonomy group - invariant conics in the tangent spaces of a Riemannian manifold if possible.

Let Γ be a bounded orientable submanifold of $\mathbb{R}^N$ ($N \geq 2$). Consider the function

$$F_\Gamma\colon \mathbb{R}^N \to \mathbb{R}, \quad F_\Gamma(\underline{x}) = \frac{1}{\mathrm{Vol}(\Gamma)} \int\limits_\Gamma h(d_2(\underline{x},\gamma))\, d\gamma \tag{5}$$

where the integral is taken with respect to the induced Riemannian volume form, and $h\colon \mathbb{R} \to \mathbb{R}$ is a strictly monotone increasing convex function with $h(0) = 0$. Then h satisfies

$$\lim_{t\to 0} \frac{h(t)}{t} =: t_0 < +\infty.$$

F_Γ is clearly a weighted generalized conic function associated to Γ with

$$g(\underline{x},\underline{y}) = \begin{cases} \frac{h(d_2(\underline{x},\underline{y}))}{d_2(\underline{x},\underline{y})} \text{ if } \underline{x} \neq \underline{y} \\ t_0 \text{ if } \underline{x} = \underline{y} \end{cases}$$

Now we give the basic properties and an application of the above generalized function originally presented in [13].

Theorem 12. *The function F_Γ is convex and satisfies the growth condition*

$$\liminf_{\|\underline{x}\|_2\to\infty} \frac{F_\Gamma(\underline{x})}{\|\underline{x}\|_2} > 0$$

Consequently the sets of the form

$$\left\{\underline{x} \in \mathbb{R}^N \middle| F_\Gamma(\underline{x}) \leq c,\ c \in \mathbb{R}\right\}$$

are convex, compact subsets of $\mathbb{R}^N$.

Let G be a closed and, consequently, compact subgroup of $\mathcal{O}(N)$ the orthogonal group of $\mathbb{R}^N$. We would like to find alternatives of the Euclidean geometry for the subgroup G. By an alternative for the subgroup G we mean a convex body K containing the origin in its interior, having smooth boundary and invariant under the elements of G. Such a convex body induces the Minkowski functional

$$L\colon \mathbb{R}^N \to \mathbb{R}, \quad L(\underline{x}) = \begin{cases} \inf\left\{\lambda \middle| \underline{x} \in \lambda K\right\} \text{ if } \underline{x} \neq 0 \\ 0 \text{ if } \underline{x} = 0 \end{cases}$$

An alternative K is called non-trivial if the induced Minkowski functional doesn't arrive from an inner product, i.e. the boundary of K is not a quadratic hypersurface in $\mathbb{R}^N$.

Definition 4. A subgroup $G \subset \mathcal{O}(N)$ is *dense* if for all units x and y there is a sequence $g_m \in G$ such that $\lim_{n\to\infty} g_n(x) = y$. Dense and closed subgroup of $\mathcal{O}(N)$ are called *transitive.*

Definition 5. A linear mapping $\varphi\colon \mathbb{R}^N \to \mathbb{R}^N$ is called linear isometry with respect to the Minkowski functional L, if $L \circ \varphi = L$.

It is clear that if Γ is invariant under some element $g \in G$ then g is a linear isometry with respect to the Minkowski functional induced by generalized conics associated to Γ. In the case of dense subgroups of $\mathcal{O}(N)$ the only possible invariant compact convex body is the unit ball with respect to the canonical inner product. The following theorem states the converse of this statement.

Theorem 13 ([13])**.** *Let $G \subset \mathcal{O}(N)$ be a closed subgroup; if G is not transitive then there exists a non-trivial alternative for the group G.*

The proof of the above theorem consists of two main parts depending on the reducibility of the group G. The key step of the construction is to find an invariant set under G as the foci of a generalized conic.

5.1. **The case of reducible subgroups.** If $N = 2$ then it is easy to construct a polyellipse which induces a non-Euclidean Minkowski functional L such that G is a subgroup of the linear isometries with respect to L. If the dimension is not less than 3, then, by the reducibility of G, we can take one of the Euclidean spheres

$$S_1 \subset S_2 \subset \ldots \subset S_{N-2}$$

as the invariant set under the elements of G (in the case of one dimensional invariant subspace consider its orthogonal complement). Using the same notation as in (5), we have

$$F_{S_1}(\underline{x}) = \frac{1}{2\pi}\int_0^{2\pi} \sqrt{(x_1-\cos t)^2 + (x_2-\sin t)^2 + x_3^2 + \cdots + x_N^2}\,dt.$$

Theorem 14 ([5])**.** *The generalized conic*

$$C_{S_1} = \left\{ \underline{x} \in \mathbb{R}^N \middle| F_{S_1}(\underline{x}) \le \frac{8}{2\pi} \right\}$$

is not an ellipsoid (as a body).

Furthermore we have the following theorem.

Theorem 15 ([13])**.** *Let $N \ge 4$ and $2 \le k \le N-2$ be fixed integers. The generalized conic*

$$\left\{ \underline{x} \in \mathbb{R}^N \middle| F_{S_k}(\underline{x}) \le \frac{c(k-1)}{\mathrm{Vol}(S_k)} \right\} \text{ where } c(l) := \frac{2^{l+2}\cdot l!}{1\cdot 3\cdot \ldots \cdot (2l+1)}$$

is not an ellipsoid.

Corollary 1. *The generalized conics* C_{S_k} $(k = 1, 2, \ldots, n-1)$ *induces non-Euclidean Minkowski functionals* L *such that* G *is a subgroup of the linear isometries with respect to* L.

5.2. **The case of irreducible subgroups.** S_{N-1} as the set of foci gives generalized conics which are invariant under the whole orthogonal group because of the invariance of the set of their foci. Therefore they are balls of dimension $N-1$ and induce trivial Minkowski functionals.

Let us consider the orbits of points with respect to the closed, irreducible group G instead of S_{N-1}. If one of the convex hulls of a non-trivial orbit is an ellipsoid (as a body) centered at the origin, then it must be ball in Euclidean sense according to the irreducibility of G. Then G is transitive on the unit sphere and all of the possible Minkowski functional must be Euclidean. On the other hand if G is not transitive on the unit sphere, then the convex hull of any nontrivial orbit induces a non-Euclidean Minkowski functional L such that G is a subgroup of the linear isometries with respect to L.

Unfortunately the boundary of the convex hull of a nontrivial orbit is not necessarily smooth. Now we show how to avoid singularities.

Definition 6. Let $z \in S_{N-1}$ be a fixed point and consider its orbit Γ_z. The *minimax point* of Γ_z is the point $z^* \in S_{N-1}$ where the minimum

$$a := \min_{\|\underline{x}\|_2 = 1} \max_{\gamma \in \operatorname{conv}(\Gamma_z)} d_2(\underline{x}, \gamma)$$

is attained.

Consider the function

$$h\colon \mathbb{R} \to \mathbb{R}, \quad t \mapsto h(t) := \begin{cases} t + (t-a)e^{-\frac{1}{t-a}} \text{ if } t > a \\ t \text{ if } t \le a \end{cases}$$

By the help of standard calculus it can be seen that h is a smooth, strictly monotone increasing convex function with $h(0) = 0$. Then take the functions

$$F(\underline{x}) = \int\limits_{\operatorname{conv}(\Gamma_z)} d_2(\underline{x}, \gamma)\, d\gamma \quad \text{and} \quad \widehat{F}(\underline{x}) = \int\limits_{\operatorname{conv}(\Gamma_z)} h\left(d_2(\underline{x}, \gamma)\right) d\gamma$$

It is clear that F and $\widehat{F}$ are generalized conic function associated to $\operatorname{conv}\Gamma_z$. Furthermore

$$F(z^*) = \widehat{F}(z^*) =: c^*$$

and one of the sets defined by $F(\underline{x}) = c^*$ or $\widehat{F}(\underline{x}) = c^*$ must be different from the sphere unless the mapping

$$\underline{x} \in S_{N-1} \mapsto \max_{\gamma \in \operatorname{conv}(\Gamma_z)} d_2(\underline{x}, \gamma)$$

is constant. Since $\Gamma_z \subset S_{n-1}$, this is possible only if Γ_z itself is the unit sphere and G is transitive. Therefore we have the following theorem of alternatives.

Theorem 16 ([13])**.** *If $G \subset \mathcal{O}(N)$ is non-transitive on the unit sphere, closed and irreducible, then one of the generalized conics*

$$\left\{\underline{x} \in \mathbb{R} \middle| F(\underline{x}) \leq c^*\right\} \ \textit{and} \ \left\{\underline{x} \in \mathbb{R} \middle| \widehat{F}(\underline{x}) \leq c^*\right\}$$

is different from a ball. Consequently one of them induces a non-Euclidean Minkowski functional L such that G is a subgroup of the linear isometries with respect to L.

6. Generalized conics with the taxicab metric

In the previous sections we have seen some special types of generalized conic function where the standard Euclidean distance was used. Now we choose the taxicab metric d_1 instead. This is the starting point of a nice application of generalized conics in the theory of geometric tomography.

Definition 7. Let $K \subset \mathbb{R}^N$ a compact subset. The *generalized 1-conic function* associated to K is the mapping

$$f_1K \colon \mathbb{R}^N \to \mathbb{R}, \quad \underline{x} \mapsto f_1K(\underline{x}) := \int\limits_K d_1(\underline{x}, \underline{y})\, d\underline{y}.$$

Level sets of generalized 1-conic functions are called generalized 1-conics.

Theorem 17 ([14])**.** *f_1K is a convex function satisfying the growth condition*

$$\liminf_{\|\underline{x}\|_2} \frac{f_1K(\underline{x})}{\|\underline{x}\|_2} > 0.$$

Consequently generalized 1-conics are compact convex subsets of $\mathbb{R}^N$.

Parallel X-rays are fundamental objects in geometric tomography. They measure the sections of a given measurable set with hyperplanes parallel to a fixed 1-codimensional subspace. The formal definition is the following.

Definition 8. Let $\mathcal{H}$ be an $N-1$ dimensional subspace of $\mathbb{R}^N$ and let $E \subset \mathbb{R}^N$ be a bounded, measurable set. Consider an orthonormal basis $\left(\underline{v}_1, \ldots, \underline{v}_{N-1}\right)$ of $\mathcal{H}$. The *X-ray* of E parallel to $\mathcal{H}$ is the mapping

$$X_{\mathcal{H}}E \colon \mathbb{R} \to \mathbb{R}, \quad t \mapsto X_{\mathcal{H}}E(t) := \lambda_1\left((t\underline{w} + \mathcal{H}) \cap E\right)$$

where $\left(\underline{v}_1, \ldots, \underline{v}_{N-1}, \underline{w}\right)$ is an orthonormal basis of $\mathbb{R}^N$ having the same orientation as the the standard basis $(\underline{e}_1, \ldots, \underline{e}_N)$.

For every $i \in \{1, \ldots, N\}$ let X_iK denote the X-ray of the compact set $K \subset \mathbb{R}^N$ parallel to $\underline{e}_i^{\perp}$ (the orthogonal complement of $\underline{e}_i$). The X-rays X_iK are called coordinate X-rays of K.

Theorem 18 ([6],[14])**.** *For compact subsets K and K^* of $\mathbb{R}^N$ $f_1K = f_1K^*$ pointwise if and only if $X_iK =_{a.e.} X_iK^*$ $(i = 1, \ldots, N)$, i.e. the corresponding coordinate X-rays are equal to each other almost everywhere.*

This theorem shows that generalized 1-conic functions carry all the informations of the coordinate X-rays. Moreover the coordinate X-rays can be expressed explicitly by the generalized 1-conic function and vice versa.

$$\frac{\partial^2}{\partial x_i} f_1 K(\underline{x}) = 2 X_i K(x_i) \quad (i = 1, \ldots, N),$$

$$f_1 K(\underline{x}) = \sum_{i=1}^{N} \int_{-\infty}^{\infty} |x_i - t| \, X_i K(t) \, dt.$$

The above formulas allow us to work with generalized 1-conic functions instead of coordinate X-rays. Generalized 1-conic functions of compact sets are convex functions on $\mathbb{R}^N$, while X-rays of compact sets may have infinitely many discontinuities. The following example and theorem illustrates that generalized 1-conic functions also have a better behavior under limits than X-rays.

Example 5. *Consider the set* $K = \operatorname{conv}\{(-1,-1),(2,-1),(2,1),(-1,1)\}$. *For all* $n \in \mathbb{N} \setminus \{0\}$ *let* k_n *be the smallest integer such that*

$$\frac{k_n(k_n+1)}{2} \geq n \quad \textit{and} \quad d_n := \sum_{i=0}^{k_n-1} i = \frac{(k_n-1)k_n}{2}$$

Let

$$L_n = \operatorname{conv}\left\{ \left(\frac{n-d_n-1}{k_n}, 0\right), \left(\frac{n-d_n}{k_n}, 0\right), \left(\frac{n-d_n-1}{k_n}, 1\right), \left(\frac{n-d_n}{k_n}, 1\right) \right\}$$

and consider the sequence

$$K_n := \operatorname{cl}(K \setminus L_n)$$

containing the complements of L_n *with respect to the set* K. *It can be easily seen that* $K_n \to K$ *with respect to the Hausdorff metric and*

$$\lim_{n\to\infty} \lambda_2(K_n) = \lambda_2(K) - \lim_{n\to\infty} \lambda_2(L_n) = \lambda_2(K) - \lim_{n\to\infty} \frac{1}{k_n} = \lambda_2(K).$$

On the other hand the sequence of coordinate X-rays $X_2 K_n$ *is divergent in every irrational* $t \in [0,1]$.

Theorem 19 ([14]). *Let* $K \subset \mathbb{R}^N$ *and suppose that* $K_n \to K$ *with respect to the Hausdorff metric and* $\lim_{n\to\infty} \lambda_2(K_n) = \lambda_2(K)$ *also holds. Then* $f_1 K_n \to f_1 K$ *pointwise.*

If we have some additional information on the compact set K then the condition $\lim_{n\to\infty} \lambda_2(K_n) = \lambda_2(K)$ can be omitted. Let $B \subset \mathbb{R}^2$ be a rectangle having sides parallel to the coordinate axes. The set $\mathcal{M}_B^{hv}$ consists of all nonempty compact connected hv-convex sets contained in B.

Theorem 20 ([12]). *The mapping*

$$\Phi\colon \mathcal{M}_B^{hv} \to L^\infty(B), \quad K \mapsto \Phi(K) := f_1K$$

is continuous between $\mathcal{M}_B^{hv}$ equipped with the Hausdorff metric, and the functions space $L^\infty(B)$ equipped with the norm

$$\|f_1K\| = \sup_{(x_1,x_2)\in B} |f_1K(x_1,x_2)|\,.$$

Theorem 21 ([12]). *Let $K_n \subset \mathbb{R}^2$ $(n \in \mathbb{N})$ be a sequence of non-empty compact connected hv-convex sets contained in the rectangle B. If $f_1K_n \to f_1K$ with respect to the supremum norm on B then any convergent subsequence of K_n tends to a set K^* with respect to the Hausdorff metric, such that K^* has the same coordinate X-rays as K almost everywhere. If K is uniquely determined by the coordinate X-rays then the symmetric difference of K and K^* is a set of measure zero.*

The above theorems also hold if $L^\infty(B)$ is replaced by $L^1(B)$. A reconstruction algorithm is presented in [6] with a full proof of convergence based on the these theorems.

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Institute of Mathematics,
MTA-DE Research Group "Equations Functions and Curves",
Hungarian Academy of Sciences and University of Debrecen,
P. O. Box 12, 4010 Debrecen, Hungary
E-mail address: abris.nagy@science.unideb.hu