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REST BOUNDED SECOND VARIATION SEQUENCES AND p-TH POWER INTEGRABILITY OF SOME FUNCTIONS RELATED TO SUMS OF FORMAL TRIGONOMETRIC SERIES

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Abstract. In this paper we have studied p-th power integrability of functions $\sin x g(x)$ and $\sin x f(x)$ with a weight, where $g(x)$ and $f(x)$ denote the formal sum functions of sine and cosine trigonometric series respectively. This study may be taken as a continuation for some recent foregoings results proven by L. Leindler [3] and S. Tikhonov [7] employing the so-called rest bounded second variation sequences.

1. Introduction

Many authors have studied the integrability of the formal series

$$g(x) := \sum_{n=1}^{\infty} \lambda_n \sin nx \tag{1.1}$$

and

$$f(x) := \sum_{n=1}^{\infty} \lambda_n \cos nx \tag{1.2}$$

imposing certain conditions on the coefficients λ_n.

Some classical results of this type are obtained by Young-Boas-Haywood (see [1], [2], [8]) which deal with above mentioned trigonometric series whose coefficients are monotone decreasing.

Theorem 1.1. *Let $\lambda_n \downarrow 0$. If $0 \leq \alpha \leq 2$, then*

$$x^{-\alpha} g(x) \in L(0,\pi) \Longleftrightarrow \sum_{n=1}^{\infty} n^{\alpha-1}\lambda_n < \infty.$$

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If $0 < \alpha < 1$, *then*

$$x^{-\alpha} f(x) \in L(0,\pi) \Longleftrightarrow \sum_{n=1}^{\infty} n^{\alpha-1}\lambda_n < \infty.$$

The monotonicity condition on the sequence $\{\lambda_n\}$ was replaced by L. Leindler [3] to a more general ones $\{\lambda_n\} \in R_0^+ BVS$.

A sequence $c := \{c_n\}$ of positive numbers tending to zero is of rest bounded variation, or briefly $R_0^+ BVS$, if it possesses the property

$$\sum_{n=m}^{\infty} |c_n - c_{n+1}| \le K(c)c_m \tag{1.3}$$

for all natural numbers m, where $K(c)$ is a constant depending only on c.

His theorems on integrability of the sum functions of the sine and the cosine trigonometric series state as follows:

Theorem 1.2. *Suppose that* $\{\lambda_n\} \in R_0^+ BVS$, $1 < p < \infty$, *and* $1/p - 1 < \theta < 1/p$. *Then*

$$x^{-p\theta}|\psi(x)|^p \in L(0,\pi) \Longleftrightarrow \sum_{n=1}^{\infty} n^{p\theta+p-2}\lambda_n^p < \infty,$$

where $\psi(x)$ *represents either* $f(x)$ *or* $g(x)$.

Later on, J. Németh [4] considered weight functions more general than power one and obtained some sufficient conditions for the integrability of the sine series with such weights. Namely, he proved:

Theorem 1.3. *Suppose that* $\{\lambda_n\} \in R_0^+ BVS$ *and the sequence* $\gamma := \{\gamma_n\}$ *satisfies the condition: there exists an* $\epsilon > 0$ *such that the sequence* $\{\gamma_n n^{-2+\epsilon}\}$ *is almost decreasing. Then*

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n}\lambda_n < \infty \Longrightarrow \gamma(x)g(x) \in L(0,\pi).$$

A sequence $\gamma := \{\gamma_n\}$ of positive terms will be called almost increasing (decreasing) if there exists a constant $C := C(\gamma) \ge 1$ such that

$$C\gamma_n \ge \gamma_m \quad (\gamma_n \le C\gamma_m)$$

holds for any $n \ge m$.

Here and in the sequel, a function $\gamma(x)$ is defined by the sequence γ in the following way: $\gamma\left(\frac{\pi}{n}\right) := \gamma_n$, $n \in \mathbb{N}$ and there exist positive constants C_1 and C_2 such that $C_1\gamma_n \le \gamma(x) \le C_2\gamma_{n+1}$ for $x \in \left(\frac{\pi}{n+1}, \frac{\pi}{n}\right)$.

In 2005 S. Tikhonov [7] has proved two theorems providing necessary and sufficient conditions for the $p-$th power integrability of the sums of sine and cosine series with weight γ. His results refine the assertions of Theorems 1.2-1.3 which show that such conditions depend on the behavior of the sequence γ.

We present Tikhonov's results below.

Theorem 1.4. *Suppose that* $\{\lambda_n\} \in R_0^+BVS$ *and* $1 \leq p < \infty$.

(A) *If the sequence* $\{\gamma_n\}$ *satisfies the condition: there exists an* $\varepsilon_1 > 0$ *such that the sequence* $\{\gamma_n n^{-p-1+\varepsilon_1}\}$ *is almost decreasing, then the condition*

$$\sum_{n=1}^{\infty} \gamma_n n^{p-2} \lambda_n^p < \infty \tag{1.4}$$

is sufficient for the validity of the condition

$$\gamma(x)|g(x)|^p \in L(0,\pi). \tag{1.5}$$

(B) *If the sequence* $\{\gamma_n\}$ *satisfies the condition: there exists an* $\varepsilon_2 > 0$ *such that the sequence* $\{\gamma_n n^{p-1-\varepsilon_2}\}$ *is almost increasing, then the condition (1.4) is necessary for the validity of condition (1.5).*

Theorem 1.5. *Suppose that* $\{\lambda_n\} \in R_0^+BVS$ *and* $1 \leq p < \infty$.

(A) *If the sequence* $\{\gamma_n\}$ *satisfies the condition: there exists an* $\varepsilon_3 > 0$ *such that the sequence* $\{\gamma_n n^{-1+\varepsilon_3}\}$ *is almost decreasing, then the condition*

$$\sum_{n=1}^{\infty} \gamma_n n^{p-2} \lambda_n^p < \infty \tag{1.6}$$

is sufficient for the validity of the inclusion

$$\gamma(x)|f(x)|^p \in L(0,\pi). \tag{1.7}$$

(B) *If the sequence* $\{\gamma_n\}$ *satisfies the condition: there exists an* $\varepsilon_4 > 0$ *such that the sequence* $\{\gamma_n n^{p-1-\varepsilon_4}\}$ *is almost increasing, then the condition (1.6) is necessary for the validity of condition (1.7).*

In 2009 B. Szal [6] introduced a new class of sequences as follows.

Definition 1.1. A sequence $\alpha := \{c_k\}$ of nonnegative numbers tending to zero is called of Rest Bounded Second Variation sequence, or briefly $\{c_k\} \in RBSVS$, if it has the property

$$\sum_{k=m}^{\infty} |c_k - c_{k+2}| \leq K(\alpha) c_m$$

for all natural numbers m, where $K(\alpha)$ is positive, depending only on sequence $\{c_k\}$, and we assume it to be bounded.

Before we state the purpose of this paper we give the following definition:

Definition 1.2. A sequence $\alpha := \{c_k\}$ of nonnegative numbers tending to zero is called of Mean Rest Bounded Second Variation sequence, or briefly $\{c_k\} \in MRBSVS$, if it has the property

$$\sum_{k=2m}^{\infty} k|c_k - c_{k+2}| \leq \frac{K(\alpha)}{m} \sum_{k=m}^{2m-1} k|c_k - c_{k+2}|$$

for all natural numbers m, where $K(\alpha)$ is positive, depending only on sequence $\{c_k\}$, and we assume it to be bounded.

The aim of this paper is to extend Tikhonov's results (as well as Leindler's result) so that the sequence $\{\lambda_n\}$ belongs the class $MRBSVS$ or $RBSVS$ which is a wider one than $RBVS$ class. To achieve this goal we need some helpful statements given in next section.

Closing this section we shall assume, throughout this paper, that $\lambda_1 = \lambda_2 = 0$.

2. Helpful Lemmas

We shall use the following lemmas for the proof of the main results.

Lemma 2.1 ([5])**.** *Let $\lambda_n > 0$ and $a_n \geq 0$. Then*

$$\sum_{n=1}^{\infty} \lambda_n \left(\sum_{\nu=1}^{n} a_\nu \right)^p \leq p^p \sum_{n=1}^{\infty} \lambda_n^{1-p} a_n^p \left(\sum_{\nu=n}^{\infty} \lambda_\nu \right)^p, \quad p \geq 1$$

and

$$\sum_{n=1}^{\infty} \lambda_n \left(\sum_{\nu=n}^{\infty} a_\nu \right)^p \leq p^p \sum_{n=1}^{\infty} \lambda_n^{1-p} a_n^p \left(\sum_{\nu=1}^{n} \lambda_\nu \right)^p, \quad p \geq 1.$$

Lemma 2.2. *The following representations of $g(x)$ and $f(x)$ hold true:*

$$2 \sin x g(x) = -\sum_{k=1}^{\infty} (\lambda_k - \lambda_{k+2}) \cos(k+1)x$$

and

$$2 \sin x f(x) = \sum_{k=1}^{\infty} (\lambda_k - \lambda_{k+2}) \sin(k+1)x,$$

where we have assumed that $\lambda_1 = \lambda_2 = 0$.

Proof. We start from obvious equality

$$\sum_{k=1}^{\infty} \lambda_k \cos kx = \frac{1}{2} \sum_{k=1}^{\infty} (\lambda_k + \lambda_{k+1}) \cos kx + \frac{1}{2} \sum_{k=1}^{\infty} (\lambda_k - \lambda_{k+1}) \cos kx,$$

or

$$\frac{1}{2} \sum_{k=1}^{\infty} \lambda_k \cos kx$$

$$= \frac{1}{2} \sum_{k=1}^{\infty} (\lambda_k + \lambda_{k+1}) \cos kx - \frac{1}{2} \cos x \sum_{k=2}^{\infty} \lambda_k \cos kx - \frac{1}{2} \sin x \sum_{k=2}^{\infty} \lambda_k \sin kx.$$

Thus we have

$$\frac{1 + \cos x}{2} \sum_{k=2}^{\infty} \lambda_k \cos kx$$

$$= \frac{1}{2} \sum_{k=1}^{\infty} (\lambda_k + \lambda_{k+1}) \cos kx - \frac{1}{2} \sin x \sum_{k=2}^{\infty} \lambda_k \sin kx - \frac{1}{2} \lambda_1 \cos x$$

or since $\lambda_1 = 0$ we obtain

$$\sum_{k=2}^{\infty} \lambda_k \cos kx = \frac{1}{2\cos^2 \frac{x}{2}} \left\{ \sum_{k=1}^{\infty} (\lambda_k + \lambda_{k+1}) \cos kx - \sin x \sum_{k=2}^{\infty} \lambda_k \sin kx \right\}. \tag{2.1}$$

Similarly as above we obtain

$$\sum_{k=1}^{\infty} \lambda_k \sin kx = \frac{1}{2} \sum_{k=1}^{\infty} (\lambda_k + \lambda_{k+1}) \sin kx + \frac{1}{2} \sum_{k=1}^{\infty} (\lambda_k - \lambda_{k+1}) \sin kx,$$

or

$$\frac{1}{2} \sum_{k=1}^{\infty} \lambda_k \sin kx = \frac{1}{2} \sum_{k=1}^{\infty} (\lambda_k + \lambda_{k+1}) \sin kx$$

$$-\frac{1}{2} \cos x \sum_{k=2}^{\infty} \lambda_k \sin kx + \frac{1}{2} \sin x \sum_{k=2}^{\infty} \lambda_k \cos kx. \tag{2.2}$$

Inserting (2.1) into (2.2) we have ($\lambda_1 = 0$)

$$\begin{aligned} \frac{1}{2} \sum_{k=1}^{\infty} \lambda_k \sin kx &= \frac{1}{2} \sum_{k=1}^{\infty} (\lambda_k + \lambda_{k+1}) \sin kx - \frac{1}{2} \cos x \sum_{k=2}^{\infty} \lambda_k \sin kx \\ &\quad + \frac{\sin \frac{x}{2}}{2\cos \frac{x}{2}} \sum_{k=1}^{\infty} (\lambda_k + \lambda_{k+1}) \cos kx - \frac{\sin \frac{x}{2} \sin x}{2\cos \frac{x}{2}} \sum_{k=2}^{\infty} \lambda_k \sin kx \\ &= \frac{1}{2} \sum_{k=1}^{\infty} (\lambda_k + \lambda_{k+1}) \sin kx + \frac{\sin \frac{x}{2}}{2\cos \frac{x}{2}} \sum_{k=1}^{\infty} (\lambda_k + \lambda_{k+1}) \cos kx \\ &\quad - \left(\frac{\cos x}{2} + \frac{\sin \frac{x}{2} \sin x}{2\cos \frac{x}{2}} \right) \sum_{k=2}^{\infty} \lambda_k \sin kx \end{aligned}$$

or

$$\sum_{k=1}^{\infty} \lambda_k \sin kx = \frac{1}{2\cos \frac{x}{2}} \sum_{k=1}^{\infty} (\lambda_k + \lambda_{k+1}) \sin \left(k + \frac{1}{2} \right) x$$

Applying the summation by parts to above equality and taking into account that $\lambda_1 = \lambda_2 = 0$ we obtain

$$\sum_{k=1}^{\infty} \lambda_k \sin kx = \frac{1}{2\cos \frac{x}{2}} \sum_{k=1}^{\infty} (\lambda_k - \lambda_{k+2}) \sum_{i=0}^{k} \sin \left(i + \frac{1}{2} \right) x,$$

or finally, noting that

$$\sum_{i=0}^{k} 2 \sin \left(i + \frac{1}{2} \right) x \sin \frac{x}{2} = 1 - \cos(k+1)x$$

we get

$$\sum_{k=1}^{\infty} \lambda_k \sin kx = -\frac{1}{2\sin x} \sum_{k=1}^{\infty} (\lambda_k - \lambda_{k+2}) \cos(k+1)x,$$

which clearly proves the first part of this lemma.

For the proof of the second part of this lemma, it is enough to put $n = 1$ to the equality (3.10), see page 167 of [6]. □

3. Main Results

Our first result deals with $p-$th power integrability of the function $\sin x f(x)$ with weight γ.

Theorem 3.1. *Suppose that* $1 \le p < \infty$. *Let* $\{\lambda_n\} \in MRBSVS$. *If the sequence* $\{\gamma_n\}$ *satisfies the condition: there exists an* $\varepsilon_1 > 0$ *such that the sequence* $\{\gamma_n n^{-p-1+\varepsilon_1}\}$ *is almost decreasing, then the condition*

$$\sum_{n=1}^{\infty} \gamma_n n^{p-2} |\lambda_n - \lambda_{n+2}|^p < \infty \tag{3.1}$$

is sufficient for the validity of the condition

$$\gamma(x)|\sin x f(x)|^p \in L(0,\pi). \tag{3.2}$$

Proof. For the proof we shall use the idea of Tikhonov which he used for his results. For this, let $x \in \left(\frac{\pi}{n+1}, \frac{\pi}{n}\right]$. Based on Lemma 2.2 and applying the summation by parts we obtain

$$\begin{aligned}
2|\sin x f(x)| &\le x \sum_{k=1}^{n} (k+1)|\lambda_k - \lambda_{k+2}| + \left| \sum_{k=n+1}^{\infty} (\lambda_k - \lambda_{k+2}) \sin(k+1)x \right| \\
&\ll \frac{1}{n} \sum_{k=1}^{n} k|\lambda_k - \lambda_{k+2}| + \sum_{k=n}^{\infty} |\triangle^2 \lambda_k + \triangle^2 \lambda_{k+1}| \left|\widetilde{D}^*_k(x)\right| + |\lambda_{n+1} - \lambda_{n+3}| \left|\widetilde{D}^*_n(x)\right|
\end{aligned}$$

where $\widetilde{D}^*_k(x)$ are defined by

$$\widetilde{D}^*_k(x) := \sum_{i=0}^{k} \sin(i+1)x = \frac{\cos\frac{x}{2} - \cos\left(k+\frac{3}{2}\right)x}{2\sin\frac{x}{2}}, \quad k \in \mathbb{N},$$

and $\triangle^2 \lambda_k = \lambda_k - 2\lambda_{k+1} + \lambda_{k+2}$.

Taking into account that $|\widetilde{D}^*_k(x)| = O\left(\frac{1}{x}\right)$ and $\{\lambda_n\} \in MRBSVS$ we have that

$$\begin{aligned}
2|\sin x f(x)| &\ll \frac{1}{n} \sum_{k=1}^{n} k|\lambda_k - \lambda_{k+2}| + \sum_{k=n}^{\infty} k|\lambda_k - \lambda_{k+2}| + n|\lambda_{n+1} - \lambda_{n+3}| \\
&\ll \frac{1}{n} \sum_{k=1}^{n} k|\lambda_k - \lambda_{k+2}| + \frac{1}{n} \sum_{k=\frac{n}{2}}^{n-1} k|\lambda_k - \lambda_{k+2}| + n|\lambda_{n+1} - \lambda_{n+3}| \\
&\ll \frac{1}{n} \sum_{k=1}^{n} k|\lambda_k - \lambda_{k+2}|,
\end{aligned}$$

where we have used the fact that from $\{\lambda_n\} \in MRBSVS$ it follows

$$n|\lambda_{n+1} - \lambda_{n+3}| \ll \sum_{k=n+1}^{\infty} k|\lambda_k - \lambda_{k+2}| \ll \frac{1}{n} \sum_{k=\frac{n}{2}}^{n-1} k|\lambda_k - \lambda_{k+2}|.$$

Hence, we get

$$\int_0^{\pi} \gamma(x)|\sin x f(x)|^p dx$$
$$\ll \sum_{n=1}^{\infty} \int_{\pi/(n+1)}^{\pi/n} \gamma(x)|\sin x f(x)|^p dx \ll \sum_{n=1}^{\infty} \frac{\gamma_n}{n^{p+2}} \left(\sum_{k=1}^{n} k|\lambda_k - \lambda_{k+2}| \right)^p .$$

Applying Lemma 2.1 with $\lambda_n = \frac{\gamma_n}{n^{p+2}} > 0$ and $a_n = n|\lambda_n - \lambda_{n+2}|$ we obtain

$$\int_0^{\pi} \gamma(x)|\sin x f(x)|^p dx \ll \sum_{n=1}^{\infty} (n|\lambda_n - \lambda_{n+2}|)^p \left(\frac{\gamma_n}{n^{p+2}} \right)^{1-p} \left(\sum_{\nu=n}^{\infty} \frac{\gamma_\nu}{\nu^{p+2}} \right)^p .$$

Moreover, by the assumption on $\{\gamma_n\}$, we get

$$\sum_{\nu=n}^{\infty} \frac{\gamma_\nu}{\nu^{p+2}} \ll \frac{\gamma_n}{n^{1+p-\varepsilon_1}} \sum_{\nu=n}^{\infty} \frac{1}{\nu^{1+\varepsilon_1}} \ll \frac{\gamma_n}{n^{1+p}},$$

which along with above inequality we have

$$\int_0^{\pi} \gamma(x)|\sin x f(x)|^p dx \ll \sum_{n=1}^{\infty} \gamma_n n^{p-2} |\lambda_n - \lambda_{n+2}|^p. \qquad \square$$

Theorem 3.2. *Suppose that $\{\lambda_n\} \in MRBSVS$ and $1 \le p < \infty$. If the sequence $\{\gamma_n\}$ satisfies the condition: there exists an $\varepsilon_3 > 0$ such that the sequence $\{\gamma_n n^{-1+\varepsilon_3}\}$ is almost decreasing, then the condition*

$$\sum_{n=1}^{\infty} \gamma_n n^{p-2} |\lambda_n - \lambda_{n+2}|^p < \infty \tag{3.3}$$

is sufficient for the validity of the inclusion

$$\gamma(x)|\sin x g(x)|^p \in L(0, \pi). \tag{3.4}$$

Proof. Based on Lemma 2.2 and applying the summation by parts we obtain

$$2|\sin x g(x)| \le \sum_{k=1}^{n} |\lambda_k - \lambda_{k+2}| + \left| \sum_{k=n+1}^{\infty} (\lambda_k - \lambda_{k+2}) \cos(k+1)x \right|$$
$$\ll \sum_{k=1}^{n} |\lambda_k - \lambda_{k+2}| + \sum_{k=n}^{\infty} |\triangle^2 \lambda_k + \triangle^2 \lambda_{k+1}| \left| D_k^*(x) \right| + |\lambda_{n+1} - \lambda_{n+3}| \left| D_n^*(x) \right|$$

where $D_k^*(x)$ are defined by

$$D_k^*(x) := \sum_{i=0}^{k} \cos(i+1)x = \frac{\sin\left(k+\frac{3}{2}\right)x - \sin\frac{x}{2}}{2\sin\frac{x}{2}}, \quad k \in \mathbb{N}.$$

Since $|D_k^*(x)| = O\left(\frac{1}{x}\right)$ and $\{\lambda_n\} \in MRBSVS$ then

$$\begin{aligned} 2|\sin x g(x)| &\ll \sum_{k=1}^{n} |\lambda_k - \lambda_{k+2}| + n\sum_{k=n}^{\infty} |\lambda_k - \lambda_{k+2}| + n|\lambda_{n+1} - \lambda_{n+3}| \\ &\ll \sum_{k=1}^{n} |\lambda_k - \lambda_{k+2}| + \frac{1}{n}\sum_{k=\frac{n}{2}}^{n-1} k|\lambda_k - \lambda_{k+2}| + n|\lambda_{n+1} - \lambda_{n+3}| \\ &\ll \sum_{k=1}^{n} |\lambda_k - \lambda_{k+2}|, \end{aligned}$$

for $x \in \left(\frac{\pi}{n+1}, \frac{\pi}{n}\right]$, where we have used the fact that from $\{\lambda_n\} \in MRBSVS$ it follows

$$n|\lambda_{n+1} - \lambda_{n+3}| \le n\sum_{k=n+1}^{\infty} |\lambda_k - \lambda_{k+2}| \ll \frac{1}{n}\sum_{k=\frac{n}{2}}^{n-1} k|\lambda_k - \lambda_{k+2}| \ll \sum_{k=1}^{n} |\lambda_k - \lambda_{k+2}|.$$

Therefore, applying Lemma 2.1 and based on conditions imposed on γ_n we have

$$\begin{aligned} \int_0^{\pi} \gamma(x)|\sin x g(x)|^p dx &\ll \sum_{n=1}^{\infty} \int_{\pi/(n+1)}^{\pi/n} \gamma(x)|\sin x f(x)|^p dx \\ &\ll \sum_{n=1}^{\infty} \frac{\gamma_n}{n^2}\left(\sum_{k=1}^{n} |\lambda_k - \lambda_{k+2}|\right)^p \\ &\ll \sum_{k=1}^{\infty} |\lambda_k - \lambda_{k+2}|^p \left(\frac{\gamma_k}{k^2}\right)^{1-p} \left(\sum_{j=n}^{\infty} \frac{\gamma_j}{j^2}\right)^p \\ &\ll \sum_{k=1}^{\infty} \gamma_k k^{p-2} |\lambda_k - \lambda_{k+2}|^p < +\infty, \end{aligned}$$

which implies $\gamma(x)|\sin x g(x)|^p \in L(0,\pi)$. □

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