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ITERATIVE ALGORITHMS FOR DETERMINING MULTIPLICATIVE INVERSES IN BANACH ALGEBRAS

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Abstract. By using Banach's fixed point theorem and properties of strong derivatives of nonlinear mappings, we prove some convergence theorems and error estimations of a family of iterative methods of arbitrary order for determining the inverse elements in a unital Bananch algebra.

1. Introduction

Finding the inverse of an element occurs quite often when solving mathematical problems. In most cases it is not possible to find an exact value of the inverse so numerical methods have to be used. The iterative method introduced by Schulz in 1963 to find the inverse of a non-singular matrix is defined as follows.

Let A be a non-singular square matrix and B_0 be the initial approximation of A^{-1}. (B_0 can be obtained by Gauss- Jordan method or any other direct method for inverting matrices). One forms $C_0 = I - AB_0$ with $\|C_0\| \leq 1$ (for some matrix norm) and then

$$\begin{aligned}
B_1 &= B_0(I + C_0),\ C_1 = I - AB_1\\
B_2 &= B_1(I + C_1),\ C_2 = I - AB_2\\
&\ \vdots \qquad\qquad\quad \vdots\\
B_n &= B_{n-1}(I + C_{n-1}),\ C_n = I - AB_n\\
&\ \vdots \qquad\qquad\quad \vdots
\end{aligned}$$

One can prove that $\|C_n\| \leq \|C_0\|^n$, and

$$\lim_{n\to\infty} B_n = A^{-1}.$$

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Researchers have since extended this method to ones which converge faster (see [4]). Szabó [5] in 1985 extended this method to one of finding the multiplicative inverse of a non singular element of a Banach algebra and the methods were of order 2 and 3. In this study we generalize the iterative methods in [5] to iterative methods of any integer order $p \geq 2$. The concept of order of convergence $p(\geq 1)$ of a convergent (one-point and stationary) iterative method

$$x_{n+1} = F(x_n), \quad n = 0, 1, 2, \ldots \tag{1}$$

with $\lim_{n\to\infty} x_n = x$ is well known: if the non zero and finite limit

$$\lim_{n\to\infty} \frac{\|x_{n+1} - x\|}{\|x_n - x\|^p} = C$$

exists, then the method (1) is said to be of order p [2]. Here C is called asymptotic error constant and, in a sufficiently small neighbourhood of x, the higher the value of p is the faster our method (1) converges to x [3].

If $\|x_{n+1} - x\| \leq K \cdot \|x_n - x\|^p$, for sufficiently high values of n (say for $n \geq n_0$) and for some constant $K \geq 0$, then the order of convergence is at least p.

2. Methods of order 2 and order 3

Szabó in [5] investigated the generalizations and extensions of Hotelling-Schulz method (mentioned above):

starting with a "good"(in some sense, to be described later) approximation $x_0 \in X$ of the inverse a^{-1} of a non-singular element $a \in X$, the following iteration was formed

$$\begin{aligned} c_0 &= e - a \cdot x_0, & x_1 &= x_0 \cdot (e + c_0) \\ c_1 &= e - a \cdot x_1, & x_2 &= x_1 \cdot (e + c_1) \\ &\vdots & &\vdots \\ c_n &= e - a \cdot x_n, & x_{n+1} &= x_n \cdot (e + c_n), \ (n = 0, 1, 2, \ldots) \end{aligned} \tag{2}$$

This method proved to be quadratic ($p = 2$) and , with the help of the iteration function.

$$F(x) = x \cdot (2e - a \cdot x), \tag{3}$$

it could be re-written in the form (1), since $c_n = e - a \cdot x_n$ and $x_{n+1} = x_n \cdot (e + c_n) = x_n \cdot (2e - a \cdot x_n)$ [5].

The following method was described and analysed in [5] and found to be of order 3:

$$(4)\qquad \begin{aligned} c_0 &= e - a\cdot x_0, & x_1 &= x_0\cdot(e+c_0+c_0^2)\\ c_1 &= e - a\cdot x_1, & x_2 &= x_1\cdot(e+c_1+c_1^2)\\ &\vdots & &\vdots\\ c_n &= e - a\cdot x_n, & x_{n+1} &= x_n\cdot(e+c_n+c_n^2)\ (n=0,1,2,\dots) \end{aligned}$$

The corresponding iteration function has the for

$$F(x) = x\cdot(3e - 3a\cdot x + (a\cdot x)^2), \tag{5}$$

since

$$\begin{aligned} x_{n+1} &= x_n\cdot(e+c_n+c_n^2) = x_n\cdot[e+(e-a\cdot x_n)+(e-a\cdot x_n)^2]\\ &= x_n\cdot[3e-3a\cdot x_n + a\cdot x_n\cdot a\cdot x_n]. \end{aligned}$$

Similarly, the method of order 4 is

$$x_0 \approx a^{-1},\ c_0 = e - a\cdot x_0$$
$$c_n = e - a\cdot x_n, x_{n+1} = x_n\cdot[e+c_n+c_n^2+c_n^3],\ \text{for } n = 0,1,2,\dots$$

and the corresponding iteration function has the form

$$F(x) = x\cdot[4e - 6a\cdot x + 4(a\cdot x)^2 - (a\cdot x)^3]. \tag{6}$$

Here the terms in braces have the (alternating) coefficients of Pascal's triangle (except its first column).

Now, we consider the general method of order $p \in [2,\infty)\bigcap\mathbb{N}$:

$$(7)\qquad \begin{aligned} & & x_0 &\approx a^{-1}\\ c_0 &= e - a\cdot x_0, & x_1 &= x_0\cdot[e+c_0+c_0^2+\cdots+c_0^{p-1}]\\ &\vdots & &\vdots\\ c_n &= e - a\cdot x_n, & x_{n+1} &= x_n\cdot[e+c_n+c_n^2+\cdots+c_n^{p-1}] \end{aligned}$$

for $n = 0,1,2,\dots$. The corresponding iteration function is

$$F(x) = x\cdot[e+(e-a\cdot x)+(e-a\cdot x)^2+\cdots+(e-a\cdot x)^{p-1}]. \tag{8}$$

Notation 1: The p-sum $e+(e-v)+(e-v)^2+\cdots+(e-v)^{p-1}$ will be denoted by $S_v^{(p)}$. Thus (8) has the form

$$F(x) = x\cdot S_{a\cdot x}^{(p)}. \tag{9}$$

3. Main results

In order to re-write the form of F and use the contraction principle we need the following lemmas.

Lemma 3.1. *In any real or complex algebra X with identity e, we have*

$$a\cdot(e-x\cdot a)^j = (e-a\cdot x)^j\cdot a,\ \textit{for } j\in\mathbb{N}.$$

Proof. We prove by induction on j. For $j = 1$ we have

$$a \cdot (e - x \cdot a) = a - a \cdot x \cdot a = (e - a \cdot x) \cdot a$$

We assume the claim holds for $j = k$ that is $a \cdot (e - x \cdot a)^k = (e - a \cdot x)^k \cdot a$. For $j = k + 1$ we have

$$\begin{aligned} a \cdot (e - x \cdot a)^{k+1} &= a \cdot (e - x \cdot a)^k \cdot (e - x \cdot a) \\ &= (e - a \cdot x)^k \cdot a \cdot (e - x \cdot a) \text{ by the hypothesis} \\ &= (e - a \cdot x)^k \cdot (a - a \cdot x \cdot a) = (e - a \cdot x)^k \cdot (e - a \cdot x) \cdot a \\ &= (e - a \cdot x)^{k+1} \cdot a. \end{aligned}$$

□

Lemma 3.2. *In any real or complex algebra X with identity e, we have that*

$$(e - v)^p = e - v \cdot S_v^{(p)} \text{ for } v \in X, \quad p \in \mathbb{N}.$$

Proof. We prove by induction on p. The claim evidently holds for $p = 1$ since $S_v^{(1)} = e$. Assume the claim holds for $p = k$, that is $(e - v)^k = e - v \cdot S_v^{(k)}$. For $p = k + 1$, we have

$$\begin{aligned} (e - v)^{k+1} &= (e - v) \cdot (e - v)^k \\ &= (e - v) \cdot (e - v \cdot S_v^{(k)}) \text{ by the hypothesis} \\ &= (e - v) - (v - v^2) \cdot S_v^{(k)} = (e - v) - v \cdot (e - v) \cdot S_v^{(k)} \\ &= (e - v) - v \cdot [(e - v) + (e - v)^2 + \cdots + (e - v)^k] \\ &= e - v \cdot [e + (e - v) + (e - v)^2 + \cdots + (e - v)^k] = e - v \cdot S_v^{(k+1)}. \end{aligned}$$

□

Lemma 3.3. *Our iteration function F in* (8) *has the forms*

$$\begin{aligned} F(x) &= x \cdot S_{a\cdot x}^{(p)} = S_{x\cdot a}^{(p)} \cdot x = x \cdot \sum_{j=1}^{p} (-1)^{j-1} \cdot \binom{p}{j} \cdot (a \cdot x)^{j-1} \\ &= \left[\sum_{j=1}^{p} (-1)^{j-1} \cdot \binom{p}{j} \cdot (x \cdot a)^{j-1} \right] \cdot x \\ &= a^{-1} \cdot [e - (e - a \cdot x)^p] = [e - (e - x \cdot a)^p] \cdot a^{-1}. \end{aligned}$$

Proof. Lemma 3.2 gives $v \cdot S_v^{(p)} = e - (e - v)^p$, putting $v = a \cdot x$ and using (8) and (9) we obtain

$$\begin{aligned} a \cdot F(x) &= a \cdot x \cdot S_{a\cdot x}^{(p)} = e - (e - a \cdot x)^p \\ &= \binom{p}{1} \cdot (a \cdot x) - \binom{p}{2} \cdot (a \cdot x)^2 + \binom{p}{3} \cdot (a \cdot x)^3 - \ldots \\ &\quad + (-1)^{p-1} \cdot (a \cdot x)^p \\ &= a \cdot \sum_{j=1}^{p} (-1)^{j-1} \cdot \binom{p}{j} \cdot x \cdot (a \cdot x)^{j-1} \end{aligned} \tag{10}$$

and

$$F(x) = x \cdot \sum_{j=1}^{p} (-1)^{j-1} \cdot \binom{p}{j} \cdot (a \cdot x)^{j-1} = a^{-1} \cdot [e - (e - a \cdot x)^p],$$

$$e - a \cdot F(x) = (e - a \cdot x)^p.$$

Lemma 3.1 applies to give

$$a \cdot S^{(p)}_{x \cdot a} = a \cdot \sum_{j=1}^{p} (e - x \cdot a)^{j-1} = \sum_{j=1}^{p} a \cdot (e - x \cdot a)^{j-1} \tag{11}$$

$$= \sum_{j=1}^{p} (e - a \cdot x)^{j-1} \cdot a = S^{(p)}_{x \cdot a} \cdot a$$

By exchanging a and x one gets

$$F(x) = x \cdot S^{(p)}_{a \cdot x} = S^{(p)}_{x \cdot a} \cdot x. \tag{12}$$

By putting $v = a \cdot x$ in Lemma 3.2, (9) and (11) yield

$$(e - a \cdot x)^p = e - x \cdot a \cdot S^{(p)}_{x \cdot a} = e - x \cdot S^{(p)}_{a \cdot x} \cdot a = e - F(x) \cdot a \tag{13}$$

and

$$F(x) \cdot a = e - (e - x \cdot a)^p = \sum_{j=1}^{p} (-1)^{j-1} \cdot \binom{p}{j} \cdot (x \cdot a)^j$$

$$= \sum_{j=1}^{p} (-i)^{j-1} \cdot \binom{p}{j} \cdot (x \cdot a)^{j-1} \cdot x \cdot a.$$

Thus,

$$F(x) = \left[\sum_{j=1}^{p} (-1)^{j-1} \cdot \binom{p}{j} \cdot (x \cdot a)^{j-1} \right] \cdot x = [e - (e - x \cdot a)^p] \cdot a^{-1}. \qquad \square$$

Theorem 3.4. *Let X be an arbitrary real or complex Banach algebra with identity e. Let $a \in X$ be any non singular element, $q \in [0, 1)$ and*

$$G := \{x \in X : D_x + E_x \le q\},$$

where $D_x := \|e - a \cdot x\|$ and $E_x := \|e - x \cdot a\|$.

Then given $p \in [2, \infty) \bigcap \mathbb{N}$ and $x_0 \in G$, the sequence $\{x_n\}$ generated by the method (7) and the corresponding iteration function F in (8) have the following properties:

1. *F has exactly one fixed point in G: $F(a^{-1}) = a^{-1}$;*
2. *$\lim_{n \to \infty} x_n = a^{-1}$, for any $x_0 \in G$;*
3. *the real sequence $\{\|x_n - a^{-1}\|\}$ is decreasing;*
4. *$\|x_n - a^{-1}\| \le \dfrac{\|x_0\|}{1 - q} \cdot q^{p^n}$, for $n = 0, 1, 2, \ldots$ ("a priori" error estimate), and the order of convergence is not less than p;*

(5) *if* $q \leq \frac{1}{2}$*, then* $\|x_n - a^{-1}\| \leq \|x_n - x_{n-1}\|$*, for* $n = 0, 1, 2, \ldots$ *("a posteriori" error estimate).*

Proof. We are going to use the fixed point theorem by Banach [1].

G is non empty, since $a^{-1} \in G$. X is a normed algebra over $\mathbb{R}$ or $\mathbb{C}$, so it is also a topological algebra, and the scalar multiplication, addition and multiplication are continuous operations. The set $G \subset X$ is closed because of continuity of the norm.

In order to show convexity of G, we choose $x, y \in G$ arbitrarily. Then

$$D_x + E_x \leq q \text{ and } D_y + E_y \leq q.$$

If $t \in (0, 1) \subset \mathbb{R}$, then $z := ty + (1-t)x \in G$, since

$$\begin{aligned} D_z + E_z &= \|e - ta \cdot y + (1-t)a \cdot x\| \\ &\quad + \|e - ty \cdot a + (1-t)x \cdot a\| \\ &= \|te - ta \cdot y + (1-t)e - (1-t)a \cdot x\| \\ &\quad + \|te - ty \cdot a + (1-t)e + (1-t)x \cdot a\| \\ &\leq t\|e - a \cdot y\| + (1-t)\|e - a \cdot x\| \\ &\quad + t\|e - y \cdot a\| + (1-t)\|e - x \cdot a\| \\ &= t(D_y + E_y) + (1-t)(D_x + E_x) \leq tq + (1-t)q = q \end{aligned} \tag{14}$$

and G is convex.

In order to show $F(G) \subset G$, let $x \in G$ i.e. $D_x + E_x \leq q$.

The formulae (10) and (13) apply to give

$$\begin{aligned} D_{F(x)} + E_{F(x)} &= \|e - a \cdot F(x)\| + \|e - F(x) \cdot a\| \\ &= \|(e - a \cdot x)^p\| + \|(e - x \cdot a)^p\| \leq D_x^p + E_x^p \\ &\leq (D_x + E_x)^p < q^p < q \end{aligned}$$

and $F(x) \in G$. In order to prove contraction property of F in G, the value of the differential $F'(x)h$ as the linear term in

$$\begin{aligned} F(x+h) - F(x) = (x+h) \cdot \sum_{j=1}^{p} (-1)^{j-1} \cdot \binom{p}{j} \cdot (a \cdot x + a \cdot h)^{j-1} \\ -x \cdot \sum_{j=1}^{p} (-1)^{j-1} \cdot \binom{p}{j} \cdot (a \cdot x)^{j-1} \end{aligned}$$

shall first be found. This linear (in h) term is

$$
\begin{aligned}
F'(x)h = {} & h \cdot \sum_{j=1}^{p} (-1)^{j-1} \binom{p}{j} (a \cdot x)^{j-1} \\
& + x \cdot \sum_{j=1}^{p} (-1)^{j-1} \binom{p}{j} \sum_{\substack{r,s \geq 0 \\ r+s=j-2}} (a \cdot x)^r \cdot (a \cdot h) \cdot (a \cdot x)^s \\
= {} & \sum_{j=1}^{p} (-1)^{j-1} \binom{p}{j} h \cdot (a \cdot x)^{j-1} \\
& + \sum_{j=1}^{p} (-1)^{j-1} \binom{p}{j} \sum_{\substack{r,s \geq 0 \\ r+s=j-2}} x \cdot (a \cdot x)^r \cdot a \cdot h \cdot (a \cdot x)^s
\end{aligned}
$$

and here $x \cdot (a \cdot x)^r \cdot a \cdot h \cdot (a \cdot x)^s = (x \cdot a)^{r+1} \cdot h \cdot (a \cdot x)^s$.

Thus

$$
\begin{aligned}
F'(x)h &= \sum_{j=1}^{p} (-1)^{j-1} \binom{p}{j} \sum_{\substack{r,s \geq 0 \\ r+s=j-1}} (x \cdot a)^r \cdot h \cdot (a \cdot x)^s \\
&= \sum_{k=0}^{p-1} (-1)^k \binom{p}{k+1} \sum_{\substack{r,s \geq 0 \\ r+s=k}} (x \cdot a)^r \cdot h \cdot (a \cdot x)^s .
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
\|F'(x)\| &= \sup_{\|h\|=1} \|F'(x)h\| \\
&= \sup_{\|h\|=1} \| \sum_{k=0}^{p} (-1)^k \binom{p}{k+1} \sum_{\substack{r,s \geq 0 \\ r+s=k}} (x \cdot a)^r \cdot h \cdot (a \cdot x)^s \| \\
&= \sup_{\|h\|=1} \| \sum_{k=0}^{p-1} \sum_{\substack{r,s \geq 0 \\ r+s=k}} \binom{p}{k+1} (-x \cdot a)^r \cdot h \cdot (-a \cdot x)^s \| \\
&= \sup_{\|h\|=1} \| \sum_{\substack{i,j \geq 0 \\ i+j=p-1}} (e - x \cdot a)^i \cdot h \cdot (e - a \cdot x)^j \| \\
&\leq \sup_{\|h\|=1} \sum_{\substack{i,j \geq 0 \\ i+j=p-1}} \|(e - x \cdot a)^i \cdot h \cdot (e - a \cdot x)^j \| \\
&\leq \sup_{\|h\|=1} \sum_{\substack{i,j \geq 0 \\ i+j=p-1}} \|(e - x \cdot a\|^i \, \|h\| \, \|e - a \cdot x\|^j
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\substack{i,j\geq 0\\ i+j=p-1}} D_x^i\, E_x^j\\
&= D_x^{p-1} + D_x^{p-2}E_x + D_x^{p-3}E_x^2 + \cdots + D_xE_x^{p-1} + E_x^{p-1}\\
&\leq (D_x+E_x)^{p-1} \leq q^{p-1} \leq q < 1, \text{ for } x\in G, h\in X.
\end{aligned}
$$

Let x and y be two arbitrary elements in G. By using the generalization of Lagrange's mean value theorem in normed spaces [1], we obtain

$$\|F(x)-F(y)\| \leq \|F'(v)\|\|x-y\|,$$

where $v = x + t(y-x)$, $t \in (0,1)$.

We have that $v = ty + (1-t)x \in G$, since G is convex, and $\|F'(v)\| \leq q$. Thus our map is a contraction and the above mentioned fixed point theorem applies to give that

(1) there exist a unique $u \in G$ with $F(u) = u$, and $u = a^{-1}$, since $a^{-1} = F(a^{-1})$;
(2) the sequence $\{x_n\}$ generated by the iteration $x_{n+1} = F(x_n)$, $\quad n = 0,1,2,\ldots$ converges to the fixed point a^{-1} of F in (8), for any $x_0 \in G$;
(3) the sequence of absolute errors $\{\|x_n - a^{-1}\|\}_{n=0}^{\infty}$ decreases.

First we claim the

$$c_n = c_0^{p^n}, \qquad \text{for } n = 0,1,2,\ldots. \tag{15}$$

This can be proved using induction on n.

For $n=0$, the statement is true since $c_0 = c_0^{p^0} = c_0^1$. Assume the statement is true for $n=k$, ie $c_k = c_0^{p^k}$. Then we show that the statement is true for $n = k+1$ ie $c_{k+1} = c_0^{p^{k+1}}$.

$$
\begin{aligned}
c_{k+1} &= e - a\cdot x_{k+1} = e - a\cdot x_k \cdot [e + c_k + c_k^2 + \ldots + c_k^{p-1}]\\
&= e - (e - c_k)\cdot[e + c_k + c_k^2 + \cdots + c_k^{p-1}] = c_k^p = (c_0^{p^k})^p = c_0^{p^{k+1}}.
\end{aligned}
$$

One can use (7) to obtain

$$a\cdot x_n = e - c_n, \quad x_n = a^{-1}\cdot(e-c_n) = a^{-1} - a^{-1}\cdot c_n$$

and

$$
\begin{aligned}
\|x_n - a^{-1}\| &= \|(a^{-1} - a^{-1}\cdot c_n) - a^{-1}\| = \|a^{-1}\cdot c_n\|\\
&\leq \|a^{-1}\|\ \ \|c_n\| \leq \|a^{-1}\|\ \ \|c_0^{p^n}\| \leq \|a^{-1}\|\ \ \|c_0\|^{p^n}\\
&= \|a^{-1}\|\ \ \|e - a\cdot x\|^{p^n} \leq \|a^{-1}\|\ \ (D_{x_0} + E_{x_0})^{p^n}\\
&\leq \|a^{-1}\| q^{p^n}, \text{ for } n \in \mathbb{N}.
\end{aligned}
$$

The Sandwich theorem implies $\lim_{n\to\infty} x_n = a^{-1}$, since $\lim_{n\to\infty} q^{p^n} = 0$. (7) applies to give $a\cdot x_0 = e - c_0$ and $a^{-1} = x_0\cdot(e-c_0)^{-1}$. Due to Banach's theorem on bounded inverse (see [1] pp. 61), $(e-c_0)^{-1}$ exist, and

$$\|(e - c_0)^{-1}\| \leq \frac{1}{1 - \|c_0\|} \leq \frac{1}{1 - q} \tag{16}$$

since

$$\|c_0\| = \|e - a \cdot x_0\| = D_{x_0} \leq D_{x_0} + E_{x_0} \leq q < 1.$$

Thus, $\|a^{-1}\| \leq \|x_0\| \|(e - c_0)^{-1}\| \leq \frac{\|x_0\|}{1-q}$, and $\|x_n - a^{-1}\| \leq \frac{\|x_0\|}{1-q} \cdot q^{p^n}$, for $n \in \mathbb{N}$.

The order of convergence can be read from this error estimation. Beyond this fact, a lower bound for the order of convergence can directly be obtained by using Lemma 3.3:

$$\begin{aligned}
\|x_{n+1} - a^{-1}\| &= \|F(x_n) - a^{-1}\| = \|[a^{-1} - a^{-1} \cdot (e - a \cdot x_n)^p] - a^{-1}\| \\
&= \|a^{-1} \cdot (e - a \cdot x_n)^p\| \leq \|a^{-1}\| \cdot \|(e - a \cdot x_n)^p\| \\
&\leq \|a^{-1}\| \cdot \|(e - a \cdot x_n)\|^p \leq \|a^{-1}\| \cdot \|a \cdot (a^{-1} - x_n)\|^p \\
&\leq \|a^{-1}\| \cdot \|a\|^p \cdot \|x_n - a^{-1}\|^p \leq \frac{\|x_0\| \cdot \|a\|^p}{1-q} \cdot \|x_n - a^{-1}\|^p \\
&= K \cdot \|x_n - a^{-1}\|^p, \text{ for } n \in \mathbb{N},
\end{aligned}$$

where $K = \frac{\|x_0\| \cdot \|a\|^p}{1-q}$ depends neither on n nor on x_n. So, our method has an order of convergence not less than p.

(4) Let $q \leq \frac{1}{2}$, we have

$$\begin{aligned}
\|x_n - a^{-1}\| &= \|(x_n - x_{n+1}) + (x_{n+1} - a^{-1})\| \\
&\leq \|x_n - x_{n+1}\| + \|x_{n+1} - a^{-1}\| \\
&= \|F(x_{n-1}) - F(x_n)\| + \|F(x_n) - F(a^{-1})\| \\
&\leq \|F'(v)\| \cdot \|x_{n-1} - x_n\| + \|F'(w)\| \cdot \|x_n - a^{-1}\|,
\end{aligned}$$

where $v = x_n + t(x_{n+1} - x_n)$ and $w = x_n + T(a^{-1} - x_n)$ for some $t, T \in (0,1)$. G is convex, so $v, w \in G$, and $\|x_n - a^{-1}\| \leq q\|x_n - x_{n-1}\| + q\|x_n - a^{-1}\|$. Thus,

$$(1-q)\|x_n - a^{-1}\| \leq q\|x_n - x_{n-1}\|$$

and since $q \leq \frac{1}{2}$, we obtain

$$\begin{aligned}
\|x_n - a^{-1}\| &\leq \frac{q}{1-q}\|x_n - x_{n-1}\| \\
&\leq \frac{\frac{1}{2}}{1 - \frac{1}{2}}\|x_n - x_{n-1}\| = \|x_n - x_{n-1}\|, \text{ for } n \in \mathbb{N}. \qquad \square
\end{aligned}$$

Remark 3.5. The functions in (3), (5) and (6) are the first three examples of F, for $p = 2, 3$ and 4, respectively.

Remark 3.6. Roughly speaking, the theorem claims that for any $p \in \mathbb{N} - \{1\}$, our iteration method (7), starting from any element $x_0 \in G$, monotonically converges to the uniquely determined inverse of the non singular element $a \in X$. The order of convergence is p, and this fact is reflected in the "a priori" error estimation.

Remark 3.7. Our result is not surprising since for every $x \in G$, we have $\|e - a \cdot x\| \leq q < 1$ and

$$\begin{aligned} x \cdot [e + (e - a \cdot x) + (e - a \cdot x)^2 + \ldots] &= x \cdot \sum_{n=0}^{\infty} (e - a \cdot x)^n \\ &= x[e - (e - a \cdot x)]^{-1} \\ &= x(a \cdot x)^{-1} = a^{-1}. \end{aligned}$$

It means that the members of the collection of our iteration functions of the form (8) and the corresponding iterative methods for $p = 2, 3, 4, \ldots$ are the p^{th} partial sums of the geometric series $\sum_{n=0}^{\infty} x \cdot (e - a \cdot x)^n$ of the inverse a^{-1} of a non singular element $a \in X$.

Remark 3.8. Any element $x_0 \in G$ has an inverse, and $x_0^{-1} = (a \cdot x_0)^{-1} \cdot a$.

Proof. Due to (16), $(e - c_0)^{-1}$ exists, and (7) yields $e - c_0 = a \cdot x_0$, $(a \cdot x_0)^{-1}$ exists. But a is non-singular, therefore $(a \cdot x_0)^{-1} \cdot a$ is also invertible and $[(a \cdot x_0)^{-1} \cdot a]^{-1} = a^{-1} \cdot (a \cdot x_0) = x_0$. Hence x_0 has an inverse

$$x_0^{-1} = (a \cdot x_0)^{-1} \cdot a.$$

□

Remark 3.9. For the iteration function F in Lemma 3 we have

$$\lim_{p \to \infty} F(x) = a^{-1},$$

since $\lim_{p \to \infty} q^p = 0$, and (15) applies to give $\lim_{p \to \infty} (e - a \cdot x)^p = 0$ and $\lim_{p \to \infty} F(x) = a^{-1}$.

Remark 3.10. In our main theorem, G can be considered as an "immediate" convergence neighbourhood of the attractive fixed point a^{-1} of the iteration function F of order $p \geq 2$.

4. Numerical results

We consider the vector space consisting of all $n \times n$ matrices equipped with the norm

$$\|A\| = \max \sum_{1 \leq j \leq n} a_{ij}.$$

The initial element is chosen as (see [4])

$$A_0 = \frac{A^T}{\|A\|_1 \|A\|_\infty}.$$

p	2	3	4	5	6	7	8	9	10	20	30
steps	12	8	6	6	5	5	5	4	4	3	3

Table 1. Number of steps till convergence.

The above technique was applied in evaluating the inverse of the matrix

$$A = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 4 & 3 & 2 \\ 2 & 3 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

for different values of p. Table 1 below shows the number of steps needed to estimate the inverse of A to within an error less than 0.0005, i.e

$$\|x_n - A^{-1}\| < 0.0005.$$

From the table above it is clear that as p increases the method converges faster.

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