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A SHARP GENERAL L_2 INEQUALITY OF OSTROWSKI TYPE

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Abstract. A sharp general L_2 inequality of Ostrowski type is established, which provides a generalization of some previous results and gives some other interesting results as special cases.

1. Introduction

In [1] and [2], we may find the following interesting sharp trapezoid type inequality and midpoint type inequality:

Theorem 1. *Let $f\colon [a,b] \to \mathbf{R}$ be such that f' is absolutely continuous on $[a,b]$ and $f'' \in L_2[a,b]$. Then we have sharp inequality*

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)]\right| \leq \frac{(b-a)^{\frac{5}{2}}}{12\sqrt{5}}\sqrt{\sigma(f'')}, \tag{1}$$

where $\sigma(\cdot)$ is defined by

$$\sigma(f) = \|f\|_2^2 - \frac{1}{b-a}\left(\int_a^b f(t)\,dt\right)^2 \tag{2}$$

and $\|f\|_2 := [\int_a^b f^2(t)\,dt]^{\frac{1}{2}}$.

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Theorem 2. *Under the assumptions of Theorem 1, we have sharp inequality*

$$\left|\int_a^b f(t)\,dt-(b-a)f(\frac{a+b}{2})-\frac{(b-a)^2}{24}[f'(b)-f'(a)]\right| \leq \frac{(b-a)^{\frac{5}{2}}}{12\sqrt{5}}\sqrt{\sigma(f'')}. \tag{3}$$

In [5], the author provided a Sharp L_2 inequality of Ostrowski type as follows:

Theorem 3. *Let the assumptions of Theorem 1 hold. Then for any $\theta \in [0,1]$ and $x \in [a,b]$ we have sharp inequality*

$$\begin{aligned} &\left|\int_a^b f(t)dt-(b-a)\left[(1-\theta)f(x)+\theta\frac{f(a)+f(b)}{2}\right]\right. \\ &+(1-\theta)(b-a)\left(x-\frac{a+b}{2}\right)f'(x) \\ &\left.-\left[\frac{1-\theta}{2}\left(x-\frac{a+b}{2}\right)^2+\frac{1-3\theta}{24}(b-a)^2\right][f'(b)-f'(a)]\right| \\ &\leq \left[\frac{\theta(1-\theta)}{4}(b-a)\left(x-\frac{a+b}{2}\right)^4\right. \\ &+\frac{3\theta^2-5\theta+2}{24}(b-a)^3\left(x-\frac{a+b}{2}\right)^2 \\ &\left.+\frac{15\theta^2-15\theta+4}{2880}(b-a)^5\right]^{\frac{1}{2}}\sqrt{\sigma(f'')}. \end{aligned} \tag{4}$$

The inequality (4) not only provide a generalization of inequalities (1) and (3), but also give some other interesting sharp inequalities as special cases. Moreover, it has been shown that the corrected Simpson rule (see [8],[11] and [10]) gives better result than Simpson rule and, in particular, the corrected averaged midpoint-trapezoid quadrature rule is optimal.

In this work, we will derive a new sharp general inequality of Ostrowski type for functions whose $(n-1)$th derivatives are absolutely continuous and whose nth derivatives belong to $L_2(a,b)$. This will not only provide a generalization of the inequality (4), but also give some other interesting sharp inequalities as special cases.

2. The results

In [6], we may find the identity

$$(5)\qquad (-1)^n \int_a^b K_n(t,x,\theta) f^{(n)}(t)\,dt$$
$$= \int_a^b f(t)\,dt - \frac{b-a}{2}[\theta f(a) + 2(1-\theta) f(x) + \theta f(b)]$$
$$- \sum_{k=1}^{n-1} \left\{ \frac{(-1)^k (x-a)^{k+1} + (b-x)^{k+1}}{(k+1)!} - \frac{\theta(b-a)[(-1)^k (x-a)^k + (b-x)^k]}{2k!} \right\} f^{(k)}(x)$$

where $K_n(t,x,\theta)$ is the kernel given by

$$(6)\qquad K_n(t,x,\theta) = \begin{cases} \frac{(t-a)^n}{n!} - \frac{\theta(b-a)(t-a)^{n-1}}{2(n-1)!}, & \text{if } t \in [a,x], \\ \frac{(t-b)^n}{n!} + \frac{(b-a)(t-b)^{n-1}}{2(n-1)!}, & \text{if } t \in (x,b]. \end{cases}$$

By elementary calculus, it is not difficult to get

$$(7)\qquad \int_a^b K_n(t,x,\theta)\,dt = \frac{(x-a)^{n+1} + (-1)^n (b-x)^{n+1}}{(n+1)!} - \frac{\theta(b-a)[(x-a)^n + (-1)^n (b-x)^n]}{2n!}$$

and

$$(8)\qquad \int_a^b K_n^2(t,x,\theta) dt = \frac{(x-a)^{2n+1} + (b-x)^{2n+1}}{(2n+1)(n!)^2} - \frac{\theta(b-a)[(x-a)^{2n} + (b-x)^{2n}]}{2(n!)^2} + \frac{\theta^2 (b-a)^2 [(x-a)^{2n-1} + (b-x)^{2n-1}]}{4(2n-1)[(n-1)!]^2}.$$

For brevity, in what follows, we will use the notations

$$(9)\qquad G_n(x,\theta) := \sum_{k=1}^{n-1} \left\{ \frac{(-1)^k (x-a)^{k+1} + (b-x)^{k+1}}{(k+1)!} - \frac{\theta(b-a)[(-1)^k (x-a)^k + (b-x)^k]}{2k!} \right\} f^{(k)}(x),$$

$$(10)\qquad H_n(x,\theta) := \frac{(x-a)^{n+1} + (-1)^n (b-x)^{n+1}}{(n+1)!} - \frac{\theta(b-a)[(x-a)^n + (-1)^n (b-x)^n]}{2n!},$$

$$(11)\quad I_n(x,\theta) := \frac{(x-a)^{2n+1}+(b-x)^{2n+1}}{(2n+1)(n!)^2} - \frac{\theta(b-a)[(x-a)^{2n}+(b-x)^{2n}]}{2(n!)^2} + \frac{\theta^2(b-a)^2[(x-a)^{2n-1}+(b-x)^{2n-1}]}{4(2n-1)[(n-1)!]^2}$$

and

$$(12)\qquad D_n := \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}.$$

Theorem 4. *Let $f\colon [a,b]\to \mathbf{R}$ be such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ and $f^{(n)}\in L_2[a,b]$. Then for any $\theta\in[0,1]$ and $x\in[a,b]$ we have*

$$(13)\quad \left|(-1)^n\left\{\int_a^b f(t)dt-(b-a)\left[(1-\theta)f(x)+\theta\frac{f(a)+f(b)}{2}\right]-G_n(x,\theta)\right\} - D_nH_n(x,\theta)\right| \le \left[I_n(x,\theta)-\frac{H_n^2(x,\theta)}{b-a}\right]^{\frac{1}{2}}\sqrt{\sigma(f^{(n)})}.$$

The inequality (13) is sharp.

Proof. From (5),(7)-(10),(12) we get

$$(14)\quad \int_a^b\left[K_n(t,x,\theta)-\frac{1}{b-a}\int_a^b K_n(s,x,\theta)ds\right]\left[f^{(n)}(t)-\frac{1}{b-a}\int_a^b f^{(n)}(s)ds\right]dt$$
$$=(-1)^n\left\{\int_a^b f(t)\,dt-(b-a)\left[(1-\theta)f(x)+\theta\frac{f(a)+f(b)}{2}\right]-G_n(\theta,x)\right\}$$
$$-D_nH_n(\theta,x).$$

By using the Cauchy-Schwarz-Bunyakowski inequality, we have

$$(15)\quad \left|\int_a^b\left[K_n(t,x,\theta)-\frac{1}{b-a}\int_a^b K_n(s,x,\theta)ds\right]\left[f^{(n)}(t)-\frac{1}{b-a}\int_a^b f^{(n)}(s)ds\right]dt\right|$$
$$\le \left\|K(\cdot,x,\theta)-\frac{1}{b-a}\int_a^b K_n(s,x,\theta)ds\right\|_2\left\|f^{(n)}-\frac{1}{b-a}\int_a^b f^{(n)}(s)ds\right\|_2.$$

From (7),(8),(10),(11) we also have

$$(16)\qquad \left\|K(\cdot,x,\theta)-\frac{1}{b-a}\int_a^b K_n(t,s,\theta)\,ds\right\|_2^2 = I_n(\theta,x)-\frac{H_n^2(\theta,x)}{b-a}.$$

and by (2),

$$(17)\quad \left\|f^{(n)}-\frac{1}{b-a}\int_a^b f^{(n)}(s)\,ds\right\|_2^2 = \|f^{(n)}\|_2^2-\frac{(f^{(n-1)}(b)-f^{(n-1)}(a))^2}{b-a}=\sigma(f^{(n)}).$$

Consequently, the inequality (13) follows from (14)-(17).

It is clear that the inequality (13) is sharp, since in its proof only Cauchy-Schwarz-Bunyakowski inequality is used, and so the equality condition follows from the well-known inequality. □

Remark 1. If we take $n = 1$ in Theorem 4, then we have

$$G_1(x,\theta) = 0,$$
$$H_1(x,\theta) = (1-\theta)(b-a)\left(x - \frac{a+b}{2}\right),$$
$$I_1(x,\theta) = \frac{1-3\theta+3\theta^2}{12}(b-a)^3 + (1-\theta)(b-a)\left(x - \frac{a+b}{2}\right)^2,$$

and

$$I_1(x,\theta) - \frac{H_1^2(x,\theta)}{b-a} = \theta(1-\theta)(b-a)\left(x - \frac{a+b}{2}\right)^2 + \frac{1-3\theta+3\theta^2}{12}(b-a)^3.$$

Thus we can derived a sharp L_2 inequality of Ostrowski type as

$$\left|(b-a)\left[(1-\theta)f(x) + \theta\frac{f(a)+f(b)}{2}\right] - (1-\theta)\left(x - \frac{a+b}{2}\right)[f(b)-f(a)] - \int_a^b f(t)dt\right|$$
$$\leq \left[\theta(1-\theta)(b-a)\left(x - \frac{a+b}{2}\right)^2 + \frac{1-3\theta+3\theta^2}{12}(b-a)^3\right]^{\frac{1}{2}}\sqrt{\sigma(f')},$$

which has been first proved in Theorem 1 of [4].

Remark 2. If we take $n = 2$ in Theorem 4, then we have

$$G_2(x,\theta) = (1-\theta)(b-a)\left(\frac{a+b}{2} - x\right)f'(x),$$
$$H_2(x,\theta) = \frac{(1-3\theta)(b-a)^3}{24} + \frac{(1-\theta)(b-a)}{2}\left(x - \frac{a+b}{2}\right)^2,$$

$$I_2(x,\theta) = \frac{3-15\theta+20\theta^2}{960}(b-a)^5 + \frac{1-3\theta+2\theta^2}{8}(b-a)^3\left(x - \frac{a+b}{2}\right)^2 - \frac{1-\theta}{4}(b-a)\left(x - \frac{a+b}{2}\right)^4,$$

and

$$I_2(x,\theta) - \frac{H_2^2(x,\theta)}{b-a} = \frac{\theta(1-\theta)}{4}(b-a)\left(x - \frac{a+b}{2}\right)^4 + \frac{3\theta^2 - 5\theta + 2}{24}(b-a)^3\left(x - \frac{a+b}{2}\right)^2 + \frac{15\theta^2 - 15\theta + 4}{2880}(b-a)^5$$

Thus we can derived a sharp L_2 inequality of Ostrowski type as

$$\begin{aligned}&\left|\int_a^b f(t)dt - (b-a)\left[(1-\theta)f(x) + \theta\frac{f(a)+f(b)}{2}\right]\right. \\ &+(1-\theta)(b-a)\left(x - \frac{a+b}{2}\right)f'(x) \\ &\left.-\left[\frac{1-\theta}{2}\left(x - \frac{a+b}{2}\right)^2 + \frac{1-3\theta}{24}(b-a)^2\right][f'(b) - f'(a)]\right| \\ &\leq \left[\frac{\theta(1-\theta)}{4}(b-a)\left(x - \frac{a+b}{2}\right)^4 + \frac{3\theta^2 - 5\theta + 2}{24}(b-a)^3\left(x - \frac{a+b}{2}\right)^2\right. \\ &\left.+\frac{15\theta^2 - 15\theta + 4}{2880}(b-a)^5\right]^{\frac{1}{2}}\sqrt{\sigma(f'')}.\end{aligned}$$

which is just the inequality (4).

Corollary 1. *Let the assumptions of Theorem 4 hold. Then for any $\theta \in [0,1]$, we get a sharp inequality*

$$\begin{aligned}&\left|(-1)^n\left\{\int_a^b f(t)dt - (b-a)\left[(1-\theta)f(x) + \theta\frac{f(a)+f(b)}{2}\right]\right.\right. \\ &-\sum_{k=1}^{n-1}\frac{[(-1)^k+1][1-(k+1)\theta](b-a)^{k+1}}{2^{k+1}(k+1)!}f^{(k)}(x) \\ &\left.-\frac{[1+(-1)^n][1-(n+1)\theta](b-a)^{n+1}}{2^{n+1}(n+1)!}D_n\right| \\ &\leq \frac{c(n,\theta)}{(n+1)!}\left(\frac{b-a}{2}\right)^{n+\frac{1}{2}}\sqrt{\sigma(f^{(n)})},\end{aligned} \tag{18}$$

where $c(n,\theta) = [\frac{2(n+1)^2[(2n-1)-(4n^2-1)\theta+(2n+1)n^2\theta^2]-[1+(-1)^n](4n^2-1)[1-(n+1)\theta]^2}{4n^2-1}]^{\frac{1}{2}}$.

Proof. We set $x = \frac{a+b}{2}$ in (4) and observe that

$$\begin{aligned}G_n\left(\frac{a+b}{2},\theta\right) &= \sum_{k=1}^{n-1}\left\{\frac{[(-1)^k+1](b-a)^{k+1}}{2^{k+1}(k+1)!} - \frac{\theta[(-1)^k+1](b-a)^{k+1}}{2^{k+1}k!}\right\}f^{(k)}(x) \\ &= \sum_{k=1}^{n-1}\frac{[(-1)^k+1][1-(k+1)\theta](b-a)^{k+1}}{2^{k+1}(k+1)!}f^{(k)}(x),\end{aligned}$$

$$H_n\left(\frac{a+b}{2},\theta\right) = \frac{[1+(-1)^n](b-a)^{n+1}}{2^{n+1}(n+1)!} - \frac{[1+(-1)^n]\theta(b-a)^{n+1}}{2^{n+1}n!}$$
$$= \frac{[1+(-1)^n][1-(n+1)\theta](b-a)^{n+1}}{2^{n+1}(n+1)!},$$

$$I_n\left(\frac{a+b}{2},\theta\right) = \frac{(b-a)^{2n+1}}{2^{2n}(2n+1)(n!)^2} - \frac{\theta(b-a)^{2n+1}}{2^{2n}(n!)^2} + \frac{\theta^2(b-a)^{2n+1}}{2^{2n}(2n-1)[(n-1)!]^2}$$
$$= \frac{(2n-1)-(4n^2-1)\theta+(2n+1)n^2\theta^2}{2^{2n}(4n^2-1)(n!)^2}(b-a)^{2n+1},$$

and

$$I_n\left(\frac{a+b}{2},\theta\right) - \frac{H_n^2(\frac{a+b}{2},\theta)}{b-a} = \frac{c^2(n,\theta)}{2^{2n+1}[(n+1)!]^2}(b-a)^{2n+1},$$

then the inequality (18) follows. □

If n is an odd integer, then for any $\theta \in [0,1]$ we have

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{2}\left[\theta f(a) + 2(1-\theta)f\left(\frac{a+b}{2}\right) + \theta f(b)\right]\right.$$
$$\left. - \sum_{k=1}^{n-1} \frac{[(-1)^k+1][1-(k+1)\theta](b-a)^{k+1}}{2^{k+1}(k+1)!} f^{(k)}(x)\right|$$
$$\le \frac{(b-a)^{n+\frac{1}{2}}}{2^n n!}\sqrt{\frac{(2n-1)-(4n^2-1)\theta+(2n+1)n^2\theta^2}{4n^2-1}}\sqrt{\sigma(f^{(n)})}, \tag{19}$$

which also has been proved in Theorem 8 of [3] as well as Theorem 7 of [7], simultaneously.

If n is an even integer, then for any $\theta \in [0,1]$ we have

$$\left|\int_a^b f(x)dx - \frac{b-a}{2}\left[\theta f(a) + 2(1-\theta)f\left(\frac{a+b}{2}\right) + \theta f(b)\right]\right.$$
$$- \sum_{k=1}^{n-1} \frac{[(-1)^k+1][1-(k+1)\theta](b-a)^{k+1}}{2^{k+1}(k+1)!} f^{(k)}(x)$$
$$\left. - \frac{[1-(n+1)\theta](b-a)^{n+1}}{(n+1)!2^n} D_n\right|$$
$$\le \frac{(b-a)^{n+\frac{1}{2}}}{2^n(n+1)!}\sqrt{\sigma(f^{(n)})} \times$$
$$\times \sqrt{\frac{2n^3-n^2-(4n^4-5n^2+1)\theta+(2n^5+n^4-4n^3-2n^2+2n+1)\theta^2}{4n^2-1}}. \tag{20}$$

which also has been proved in Theorem 9 of [3] as well as Theorem 8 of [7], simultaneously.

Remark 3. If we take $\theta = 0$ in (19) and (20) then we get sharp midpoint type inequalities

$$\left|\int_a^b f(x)dx - (b-a)f\left(\frac{a+b}{2}\right) + \sum_{k=1}^{[\frac{n-1}{2}]} \frac{(b-a)^{2k+1}}{(2k+1)!2^{2k}} f^{(2k)}\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^{n+\frac{1}{2}}}{2^n n!\sqrt{2n+1}}\sqrt{\sigma(f^{(n)})} \tag{21}$$

for an odd n and

$$\left|\int_a^b f(x)dx - (b-a)f\left(\frac{a+b}{2}\right) + \sum_{k=1}^{[\frac{n-1}{2}]} \frac{(b-a)^{2k+1}}{(2k+1)!2^{2k}} f^{(2k)}\left(\frac{a+b}{2}\right) - \frac{(b-a)^n}{(n+1)!2^n}[f^{(n-1)}(b) - f^{(n-1)}(a)]\right| \leq \frac{n(b-a)^{n+\frac{1}{2}}}{2^n (n+1)!\sqrt{2n+1}}\sqrt{\sigma(f^{(n)})} \tag{22}$$

for an even n.

In particular, for $n = 1$ in (21), we have

$$\left|\int_a^b f(x)dx - (b-a)f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^{\frac{3}{2}}}{2\sqrt{3}}\sqrt{\sigma(f')},$$

and for $n = 2$ in (22), we have

$$\left|\int_a^b f(x)dx - (b-a)f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24}[f'(b) - f'(a)]\right| \leq \frac{(b-a)^{\frac{5}{2}}}{12\sqrt{5}}\sqrt{\sigma(f'')}.$$

Remark 4. If we take $\theta = 1$ in (19) and (20) then we get sharp trapezoid type inequalities

$$\left|\int_a^b f(x)dx - \frac{(b-a)}{2}[f(a) + f(b)] - \sum_{k=1}^{[\frac{n-1}{2}]} \frac{k(b-a)^{2k+1}}{(2k+1)!2^{2k-1}} f^{(2k)}\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^{n+\frac{1}{2}}}{2^n n!}\sqrt{\frac{2n^3 - 3n^2 + 2n}{4n^2 - 1}}\sqrt{\sigma(f^{(n)})} \tag{23}$$

for an odd n and

$$
\begin{aligned}
(24)\quad & \left| \int_a^b f(x)dx - \frac{(b-a)}{2}[f(a)+f(b)] - \sum_{k=1}^{[\frac{n-1}{2}]} \frac{k(b-a)^{2k+1}}{(2k+1)!2^{2k-1}} f^{(2k)}\left(\frac{a+b}{2}\right) \right. \\
& \left. + \frac{n(b-a)^n}{(n+1)!2^n}[f^{(n-1)}(b) - f^{(n-1)}(a)] \right| \\
& \leq \frac{(b-a)^{n+\frac{1}{2}}}{2^n(n+1)!} \sqrt{\frac{2n^5-3n^4-2n^3+2n^2+2n}{4n^2-1}} \sqrt{\sigma(f^{(n)})}
\end{aligned}
$$

for an even n.

In particular, for $n=1$ in (23), we have

$$
\left| \int_a^b f(x)dx - \frac{b-a}{2}[f(a)+f(b)] \right| \leq \frac{(b-a)^{\frac{3}{2}}}{2\sqrt{3}} \sqrt{\sigma(f')},
$$

and for $n=2$ in (24), we have

$$
\left| \int_a^b f(x)dx - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)] \right| \leq \frac{(b-a)^{\frac{5}{2}}}{12\sqrt{5}} \sqrt{\sigma(f'')}.
$$

Remark 5. If we take $\theta = \frac{1}{3}$ then we get sharp Simpson type inequalities

$$
\begin{aligned}
(25)\quad & \left| \int_a^b f(x)dx - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \right. \\
& \left. + \sum_{k=2}^{[\frac{n-1}{2}]} \frac{(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-1}} f^{(2k)}\left(\frac{a+b}{2}\right) \right| \\
& \leq \frac{1}{3}\frac{(b-a)^{n+\frac{1}{2}}}{2^n n!} \sqrt{\frac{2n^3-11n^2+18n-6}{4n^2-1}} \sqrt{\sigma(f^{(n)})}
\end{aligned}
$$

for an odd n and

$$
\begin{aligned}
(26)\quad & \left| \int_a^b f(x)dx - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \right. \\
& + \sum_{k=2}^{[\frac{n-1}{2}]} \frac{(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-1}} f^{(2k)}\left(\frac{a+b}{2}\right) \\
& \left. + \frac{(n-2)(b-a)^n}{3(n+1)!2^n}[f^{(n-1)}(b) - f^{(n-1)}(a)] \right| \\
& \leq \frac{1}{3}\frac{(b-a)^{n+\frac{1}{2}}}{2^n(n+1)!} \sqrt{\frac{2n^5-11n^4+14n^3+4n^2+2n-2}{4n^2-1}} \sqrt{\sigma(f^{(n)})}
\end{aligned}
$$

for an even n, which have been first proved in Theorem 5 and Theorem 6 of [9] in a more direct way.

In particular, for $n = 1, 3$ in (25) and $n = 2$ in (26), we have

$$\left|\int_a^b f(x)dx - \frac{b-a}{6}[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)]\right| \le C_n (b-a)^{n+\frac{1}{2}} \sqrt{\sigma(f^{(n)})},$$

where $C_1 = \frac{1}{6}$, $C_2 = \frac{1}{12\sqrt{30}}$, $C_3 = \frac{1}{48\sqrt{105}}$.

Remark 6. If we take $\theta = \frac{1}{2}$ then we have sharp averaged midpoint-trapezoid inequalities

$$\begin{aligned}(27)\qquad &\left|\int_a^b f(x)dx - \frac{b-a}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right]\right.\\ &\left.- \sum_{k=1}^{[\frac{n-1}{2}]} \frac{(2k-1)(b-a)^{2k+1}}{(2k+1)!2^{2k+1}} f^{(2k)}\left(\frac{a+b}{2}\right)\right|\\ &\le \frac{(b-a)^{n+\frac{1}{2}}}{2^{n+1}n!}\sqrt{\frac{2n^3 - 7n^2 + 8n - 2}{4n^2 - 1}}\sqrt{\sigma(f^{(n)})}\end{aligned}$$

for an odd n and

$$\begin{aligned}(28)\qquad &\left|\int_a^b f(x)dx - \frac{b-a}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right]\right.\\ &- \sum_{k=1}^{[\frac{n-1}{2}]} \frac{(2k-1)(b-a)^{2k+1}}{(2k+1)!2^{2k+1}} f^{(2k)}\left(\frac{a+b}{2}\right)\\ &\left.+ \frac{(n-1)(b-a)^n}{(n+1)!2^{n+1}}[f^{(n-1)}(b) - f^{(n-1)}(a)]\right|\\ &\le \frac{(b-a)^{n+\frac{1}{2}}}{2^{n+1}(n+1)!}\sqrt{\frac{2n^5 - 7n^4 + 4n^3 + 4n^2 + 2n - 1}{4n^2 - 1}}\sqrt{\sigma(f^{(n)})}\end{aligned}$$

for an even n.

In particular, for $n = 1$ in (27), we have

$$\left|\int_a^b f(x)dx - \frac{b-a}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \le \frac{(b-a)^{\frac{3}{2}}}{4\sqrt{3}}\sqrt{\sigma(f')},$$

and for $n = 2$, in (28), we have

$$\begin{aligned}&\left|\int_a^b f(x)dx - \frac{b-a}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right] + \frac{(b-a)^2}{48}[f'(b) - f'(a)]\right|\\ &\qquad \le \frac{(b-a)^{\frac{5}{2}}}{48\sqrt{5}}\sqrt{\sigma(f'')},\end{aligned}$$

Remark 7. If we take $\theta = \frac{7}{15}$ then we have sharp corrected Simpson type inequalities

$$\begin{aligned}
(29)\qquad & \left| \int_a^b f(x)dx - \frac{b-a}{30}\left[7f(a) + 16f\left(\frac{a+b}{2}\right) + 7f(b)\right] \right. \\
& \left. - \sum_{k=1}^{[\frac{n-1}{2}]} \frac{(1-14k)(b-a)^{2k+1}}{15(2k+1)!2^{2k}} f^{(2k)}\left(\frac{a+b}{2}\right) \right| \\
& \leq \frac{(b-a)^{n+\frac{1}{2}}}{n!2^n} \sqrt{\frac{98n^3 - 371n^2 + 450n - 120}{225(4n^2-1)}} \sqrt{\sigma(f^{(n)})}
\end{aligned}$$

for an odd n and

$$\begin{aligned}
(30)\qquad & \left| \int_a^b f(x)dx - \frac{b-a}{30}\left[7f(a) + 16f\left(\frac{a+b}{2}\right) + 7f(b)\right] \right. \\
& - \sum_{k=1}^{[\frac{n-1}{2}]} \frac{(1-14k)(b-a)^{2k+1}}{15(2k+1)!2^{2k}} f^{(2k)}\left(\frac{a+b}{2}\right) \\
& \left. + \frac{(7n-8)(b-a)^n}{15(n+1)!2^{n+1}}[f^{(n-1)}(b) - f^{(n-1)}(a)] \right| \\
& \leq \frac{(b-a)^{n+\frac{1}{2}}}{2^n(n+1)!} \sqrt{\frac{98n^5 - 371n^4 + 254n^3 + 202n^2 + 98n - 56}{225(4n^2-1)}} \sqrt{\sigma(f^{(n)})}
\end{aligned}$$

for an even n.

In particular, for $n = 1$ in (29), we have

$$\left| \int_a^b f(x)dx - \frac{b-a}{30}\left[7f(a) + 16f\left(\frac{a+b}{2}\right) + 7f(b)\right] \right| \leq \frac{\sqrt{19}(b-a)^{\frac{3}{2}}}{30}\sqrt{\sigma(f')},$$

and for $n = 2$, in (30), we have

$$\begin{aligned}
& \left| \int_a^b f(x)dx - \frac{b-a}{30}\left[7f(a) + 16f\left(\frac{a+b}{2}\right) + 7f(b)\right] \right. \\
& \qquad \left. + \frac{(b-a)^2}{60}[f'(b) - f'(a)] \right| \leq \frac{(b-a)^{\frac{5}{2}}}{60\sqrt{3}}\sqrt{\sigma(f'')},
\end{aligned}$$

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