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# ON A SUFFICIENT AND NECESSARY CONDITION FOR A MULTIVARIATE POLYNOMIAL TO HAVE ALGEBRAICALLY DEPENDENT ROOTS - AN ELEMENTARY PROOF

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Abstract. In the paper we prove that a multivariate polynomial has algebraically dependent roots iff the coefficients are algebraic numbers up to a common proportional term. A complex analytic proof can be found in [2] with applications in the theory of linear functional equations, see also [3, an open problem, section 4.4] and [1]. Here we present an elementary proof involving cardinality properties and basic linear algebra.

## 1. Introduction

Let $\mathbb{C}$ be the field of complex numbers. The element $(\beta_1, \ldots, \beta_m) \in \mathbb{C}^m$ is called an algebraically dependent system over the field $\mathbb{Q}$ of the rational numbers if there exists a not identically zero polynomial $Q \in \mathbb{Q}[x_1, \ldots, x_m]$ such that $Q(\beta_1, \ldots, \beta_m) = 0$. Otherwise we say that it is an algebraically independent system. If $m = 1$ then we speak about algebraic and transcendental numbers instead of algebraically dependent and independent systems. Since Viéta's formulas provide direct relationships between the roots and the coefficients (up to a common proportional term) it can be easily seen that if a univariate polynomial $p(x) \in \mathbb{C}[x]$ has algebraic roots then the coefficients are algebraic numbers up to a constant proportional term. The proof of the converse statement is based on a symmetrization process by taking the product as the coefficients of $p(x)$ runs through their algebraic conjugates. The fundamental theorem of symmetric polynomials shows that the product polynomial belongs to the polynomial ring $\mathbb{Q}[x]$ and the product vanishes at each root of the polynomial $p(x)$. Therefore its roots are algebraic. The symmetrization

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process can be generalized for the case of multivariate polynomials in a more or less direct way, see [2]. Therefore we are going to prove that if a multivariate polynomial has algebraically dependent roots then the coefficients of the polynomial are algebraic numbers up to a common proportional term. A complex analytic proof can be found in [2] with applications in the theory of linear functional equations, see also [3, an open problem, section 4.4] and [1]. Here we present an elementary proof involving cardinality properties and basic linear algebra.

## 2. The main theorem

*Theorem* 1. Let $P \in \mathbb{C}[x_1, \ldots, x_m]$ be a not identically zero polynomial; the solutions of equation

$$P(x_1, \ldots, x_m) = 0 \tag{1}$$

are algebraically dependent over the rationals if and only if the coefficients of $P$ are algebraic numbers over the rationals up to a common proportional term.

The criteria says that the coefficients of the polynomial have the following special form:

$$p_{i_1 \ldots i_m} = \lambda \omega_{i_1 \ldots i_m}$$

for some algebraic numbers $\omega_{i_1 \ldots i_m}$'s, $\lambda \in \mathbb{C}$ and $0 \leq i_1 \leq d_1$, …, $0 \leq i_m \leq d_m$, where

$$d_1 := \deg_1 P, \ \ldots, \ d_m := \deg_m P$$

denotes the degree of the polynomial $P \in \mathbb{C}[x_1, \ldots, x_m]$ with respect to the variable $x_j$, $j = 1, \ldots, m$. In what follows we prove that if equation (1) has algebraically dependent roots then the coefficients of the polynomial are algebraic numbers up to a common proportional term (for the converse statement and the special case of univariate polynomials see section 1). At first the special case of polynomials in two variables will be discussed to avoid the technical difficulties of the multivariable setting. Since the proof contains an inductive argument we note again that the statement is obvious in case of $m = 1$ (univariate polynomials) because Viéta's formulas provide direct relationships between the roots and the coefficients up to a common proportional term. To prove the general statement we adopt the basic ideas of section 3 to the case of multivariate polynomials in general.

## 3. Polynomials in two variables

Suppose that all solutions of equation

$$P(x_1, x_2) = 0 \tag{2}$$

are algebraically dependent over the rationals. For any solution $(w_1, w_2)$ of equation (2) let $Q_{w_1,w_2} \in \mathbb{Q}[x_1, x_2]$ be a nonzero polynomial such that $Q_{w_1,w_2}(w_1, w_2) = 0$.

3.1. **The cardinality argument and the Vandermonde process.** Consider the set

$$\mathcal{M} := \{(z_1, z_2) \in \mathbb{C}^2 \mid P(z_1, z_2) \neq 0\};$$

it is an open subset in $\mathbb{C}^2$. Since the projections are open mappings we can choose open subsets $U_1$, $U_2 \subset \mathbb{C}$ such that $U_1 \times U_2 \subset \mathcal{M}$. In what follows we restrict our investigations to the product $U_1 \times U_2$. Since $P(z_1, z_2) \neq 0$ ($z_1 \in U_1$, $z_2 \in U_2$) we have that equation $P(x_1, z_2) = 0$ has only finitely many roots $w_{1i_1}$ ($i_1 < \infty$) for any given $z_2 \in U_2$. Let us define the polynomial

$$Q_1^{\hat{z}_1, z_2} := \prod_{i_1 < \infty} Q_{w_{1i_1}, z_2}. \tag{3}$$

The ˆ - operator deletes the argument which means that the polynomial $Q_1^{\hat{z}_1, z_2}$ does not depend on $z_1 \in U_1$. Since $Q_1^{\hat{z}_1, z_2} \in Q[x_1, x_2]$ and $z_2 \in U_2 \subset \mathbb{C}$, a cardinality argument shows that we can choose a non-finite subset $\mathcal{N}_2 \subset U_2$ such that

$$Q_1^{\hat{z}_1, z_2} = Q_1^{\hat{z}_1, \hat{z}_2} \quad (z_2 \in \mathcal{N}_2), \tag{4}$$

i.e. the same polynomial $Q_1^{\hat{z}_1, \hat{z}_2} \in \mathbb{Q}[x_1, x_2]$ occurs for any $z_2 \in \mathcal{N}_2$. By (3) the polynomial $P(x_1, z_2)$ divides $Q_1^{\hat{z}_1, \hat{z}_2}(x_1, z_2)$ in the polynomial ring $\mathbb{C}[x_1]$ for any $z_2 \in \mathcal{N}_2$. In a similar way we can introduce a polynomial $Q_2^{\hat{z}_1, \hat{z}_2} \in \mathbb{Q}[x_1, x_2]$ such that $P(z_1, x_2)$ divides $Q(z_1, x_2)$ in the polynomial ring $\mathbb{C}[x_2]$ for any $z_1 \in \mathcal{N}_1$, where $\mathcal{N}_1 \subset U_1$ is not a finite subset. Taking the product

$$Q_{12}^{\hat{z}_1, \hat{z}_2} := Q_1^{\hat{z}_1, \hat{z}_2} \cdot Q_2^{\hat{z}_1, \hat{z}_2} \tag{5}$$

we can write that

$$Q_{12}^{\hat{z}_1, \hat{z}_2}(x_1, z_2) = P(x_1, z_2) \sum_{j_1=0}^{N_1} r_{1j_1}(\hat{z}_1, z_2) x_1^{j_1} \quad (z_2 \in \mathcal{N}_2), \tag{6}$$

$$Q_{12}^{\hat{z}_1, \hat{z}_2}(z_1, x_2) = P(z_1, x_2) \sum_{j_2=0}^{N_2} r_{2j_2}(z_1, \hat{z}_2) x_2^{j_2} \quad (z_1 \in \mathcal{N}_1). \tag{7}$$

Therefore

$$\sum_{j_1=0}^{N_1} r_{1j_1}(\hat{z}_1, z_2) z_1^{j_1} = \sum_{j_2=0}^{N_2} r_{2j_2}(z_1, \hat{z}_2) z_2^{j_2} \quad (z_1 \in \mathcal{N}_1, z_2 \in \mathcal{N}_2) \tag{8}$$

because of $P(z_1, z_2) \neq 0$ ($z_1 \in \mathcal{N}_1 \subset U_1, z_2 \in \mathcal{N}_2 \subset U_2$). Since $\mathcal{N}_1$ contains more than finitely many elements we can choose different values $z_{10}, \ldots, z_{1N_1}$ to satisfy (8). In terms of a linear system of equations:

$$V(z_{10}, \ldots, z_{1N_1}) \begin{pmatrix} r_{10}(\hat{z}_1, z_2) \\ r_{11}(\hat{z}_1, z_2) \\ \cdot \\ r_{1N_1}(\hat{z}_1, z_2) \end{pmatrix} = \sum_{j_2=0}^{N_2} \begin{pmatrix} r_{2j_2}(z_{10}, \hat{z}_2) \\ r_{2j_2}(z_{11}, \hat{z}_2) \\ \cdot \\ r_{2j_2}(z_{1N_1}, \hat{z}_2) \end{pmatrix} z_2^{j_2},$$

where

$$V(z_{10},\dots,z_{1N_1}) := \begin{pmatrix} 1 & z_{10} & \dots & z_{10}^{N_1} \\ 1 & z_{11} & \dots & z_{11}^{N_1} \\ . & . & . & . \\ 1 & z_{1N_1} & \dots & z_{1N_1}^{N_1} \end{pmatrix}$$

is the usual Vandermonde matrix. By Cramer's rule

$$r_{1j_1}(\hat{z}_1, z_2) = \sum_{j_2=0}^{N_2} r_{12j_1j_2}(\hat{z}_1, \hat{z}_2) z_2^{j_2} \quad (z_2 \in \mathcal{N}_2, j_1 = 0, \dots, N_1), \tag{9}$$

where the coefficient $r_{12j_1j_2}(\hat{z}_1, \hat{z}_2)$ is independent of the choice of $z_1$ and $z_2$. Using (9), equation (6) can be written as

$$Q_{12}^{\hat{z}_1,\hat{z}_2}(x_1, z_2) = P(x_1, z_2) \sum_{j_1=0}^{N_1} \sum_{j_2=0}^{N_2} r_{12j_1j_2}(\hat{z}_1, \hat{z}_2) x_1^{j_1} z_2^{j_2} \quad (z_2 \in \mathcal{N}_2). \tag{10}$$

Since both sides are polynomials in the second variable for fixed $x_1$ and $\mathcal{N}_2$ is not finite it follows that

$$Q_{12}^{\hat{z}_1,\hat{z}_2}(x_1, x_2) = P(x_1, x_2) \sum_{j_1=1}^{N_1} \sum_{j_2=1}^{N_2} r_{12j_1j_2}(\hat{z}_1, \hat{z}_2) x_1^{j_1} x_2^{j_2}, \tag{11}$$

i.e. $P(x_1, x_2)$ divides the polynomial $Q := Q_{12}^{\hat{z}_1,\hat{z}_2} \in \mathbb{Q}[x_1, x_2]$:

$$Q(x_1, x_2) = P(x_1, x_2) R(x_1, x_2), \quad \text{where} \quad R(x_1, x_2) \in \mathbb{C}[x_1, x_2]. \tag{12}$$

3.2. **The comparison of the coefficients.** The next step is to compare the coefficients of the polynomials on different sides of (12). Let $d_1 := \deg_1 P$ be the degree of the polynomial with respect to the first variable. The coefficient of $x_1^{d_1}$ can be written as the polynomial

$$P_{d_1}(x_2) := \sum_{i_2=0}^{d_2} p_{d_1 i_2} x_2^{i_2}$$

of the second variable and the substitution of each root $w_2$ of $P_{d_1}$ causes a decreasing in the degree of the right hand side of (12) with respect to $x_1$. Therefore the polynomial $Q$ also has to decrease its maximal degree $D_1$ with respect to $x_1$. This means that the polynomial $Q_{D_1}(x_2)$ (the coefficient of $x_1^{D_1}$ in the polynomial $Q$) vanishes at each root of $P_{d_1}(x_2)$, i.e. each root is an algebraic number. By the inductive hypothesis (the case of univariate polynomials is well-known),

$$p_{d_1 i_2} = \lambda \omega_{i_2} \quad (i_2 = 0, \dots, d_2), \tag{13}$$

where $\lambda \in \mathbb{C}$ is a common proportional term and $\omega_{i_2}$'s are algebraic numbers over the rationals. Let us choose an algebraic number $a_2$ such that $Q(x_1, a_2) \in \mathbb{C}[x_1]$ is not identically zero. Then the solutions of equation

$$P(x_1, a_2) = 0$$

are algebraic because of (12) and, using the inductive hypothesis, the coefficients of the polynomial

$$P(x_1, a_2) = \sum_{i_1=0}^{d_1} \left( \sum_{i_2=0}^{d_2} p_{i_1 i_2} a_2^{i_2} \right) x_1^{i_1} \tag{14}$$

can be written as

$$\sum_{i_2=0}^{d_2} p_{i_1 i_2} a_2^{i_2} = c_{i_1}(a_2) \underbrace{\sum_{i_2=0}^{d_2} p_{d_1 i_2} a_2^{i_2}}_{\text{the main coeff. of (14).}} ,$$

where $i_1 = 0, \ldots, d_1$ and $c_{i_1}(a_2)$ is an algebraic number depending on $a_2$; especially $c_{d_1}(a_2) = 1$. According to (13)

$$\sum_{i_2=0}^{d_2} p_{i_1 i_2} a_2^{i_2} = \lambda \omega_{i_1}(a_2), \quad \omega_{i_1}(a_2) = c_{i_1}(a_2) \sum_{i_2=0}^{d_2} \omega_{i_2} a_2^{i_2}, \tag{15}$$

where $i_1 = 0, \ldots, d_1$ and $\omega_{i_1}(a_2)$'s are algebraic numbers depending on $a_2$. Since the cardinality of the algebraic numbers is not finite we can choose different values $a_{20}, \ldots, a_{2N}$ to satisfy (15). In terms of a linear system of equations

$$V(a_{20}, \ldots, a_{2N}) \begin{pmatrix} p_{00} & \ldots & p_{N0} \\ p_{01} & \ldots & p_{N1} \\ . & . & . \\ p_{0N} & \ldots & p_{NN} \end{pmatrix} = \lambda \begin{pmatrix} \omega_0(a_{20}) & \ldots & \omega_N(a_{20}) \\ \omega_0(a_{21}) & \ldots & \omega_N(a_{21}) \\ . & . & . \\ \omega_0(a_{2N}) & \ldots & \omega_N(a_{2N}) \end{pmatrix}, \tag{16}$$

where

$$V(a_{20}, \ldots, a_{2N}) := \begin{pmatrix} 1 & a_{20} & \ldots & a_{20}^N \\ 1 & a_{21} & \ldots & a_{21}^N \\ . & . & . & . \\ 1 & a_{2N} & \ldots & a_{2N}^N \end{pmatrix}$$

is the usual Vandermonde matrix, $N := \max\{d_1, d_2\}$; note that we allow zero elements too, i.e. if the monomial term $x_1^k x_2^l$ is missing for some values of $k$ and $l$ in the polynomial $P$ then $p_{kl} := 0$. Since the algebraic numbers form a field the matrix $V^{-1}(a_{20}, \ldots, a_{2N})$ contains algebraic numbers and we have that

$$p_{i_1 i_2} = \lambda \omega_{i_1 i_2}, \tag{17}$$

where $\omega_{i_1 i_2}$'s are algebraic numbers, $\lambda \in \mathbb{C}$ and $0 \le i_1 \le d_1, 0 \le i_2 \le d_2$.

## 4. Polynomials in more than two variables

In what follows we illustrate how to generalize the process in section 3 for more than two variables. Suppose that all solutions of equation

$$P(x_1, \ldots, x_m) = 0 \tag{18}$$

are algebraically dependent over the rationals. For any solution $(w_1, \ldots, w_m)$ of equation (18) let $Q_{w_1,\ldots,w_m} \in \mathbb{Q}[x_1, \ldots, x_m]$ be a nonzero polynomial such that $Q_{w_1,\ldots,w_m}(w_1, \ldots, w_m) = 0$.

4.1. **The cardinality argument and the Vandermonde process.** Consider the set

$$\mathcal{M} := \{(z_1, \ldots, z_m) \in \mathbb{C}^m \mid P(z_1, \ldots, z_m) \neq 0\};$$

it is an open subset in $\mathbb{C}^m$. Since the projections are open mappings we can choose open subsets $U_1$, ..., $U_m \in \mathbb{C}$ such that $U_1 \times \cdots \times U_m \subset \mathcal{M}$. In what follows we restrict our investigations to the product $U_1 \times \cdots \times U_m$.

By keeping the variables $z_3 \in U_3$, ..., $z_m \in U_m$ constant we have polynomials in two variables to repeat the steps in subsection 3.1.

Namely, for any $z_2 \in U_2$ let us define the polynomial

$$Q_1^{\hat{z}_1,z_2,\ldots,z_m} := \prod_{i_1<\infty} Q_{w_{1i_1},z_2,\ldots,z_m}, \tag{19}$$

where $w_{1i_1}$ runs through the finitely many roots of equation $P(x_1, z_2, \ldots, z_m) = 0$. Since $Q_1^{\hat{z}_1,z_2,\ldots,z_m} \in \mathbb{Q}[x_1, \ldots, x_m]$ and $z_2 \in U_2 \subset \mathbb{C}$, a cardinality argument shows that we can choose a non-finite subset $\mathcal{N}_2 \subset U_2$ such that

$$Q_1^{\hat{z}_1,z_2,z_3,\ldots,z_m} = Q_1^{\hat{z}_1,\hat{z}_2,z_3,\ldots,z_m} \quad (z_2 \in \mathcal{N}_2), \tag{20}$$

i.e. the same polynomial $Q_1^{\hat{z}_1,\hat{z}_2,z_3,\ldots,z_m} \in \mathbb{Q}[x_1, \ldots, x_m]$ occurs for any $z_2 \in \mathcal{N}_2$. The polynomial $P(x_1, z_2, \ldots, z_m)$ divides $Q_1^{\hat{z}_1,\hat{z}_2,z_3,\ldots,z_m}(x_1, z_2, \ldots, z_m)$ in the polynomial ring $\mathbb{C}[x_1]$ for any $z_2 \in \mathcal{N}_2$. In a similar way we can introduce a polynomial $Q_2^{\hat{z}_1,\hat{z}_2,z_3\ldots,z_m} \in \mathbb{Q}[x_1, \ldots, x_m]$ such that $P(z_1, x_2, z_3, \ldots, z_m)$ divides $Q_2^{\hat{z}_1,\hat{z}_2,z_3\ldots,z_m}(z_1, x_2, z_3, \ldots, z_m)$ in the polynomial ring $\mathbb{C}[x_2]$ for any $z_1 \in \mathcal{N}_1$, where $\mathcal{N}_1 \subset U_1$ is not a finite subset. Taking the product

$$Q_{12}^{\hat{z}_1,\hat{z}_2,z_3,\ldots,z_m} := Q_1^{\hat{z}_1,\hat{z}_2,z_3,\ldots,z_m} \cdot Q_2^{\hat{z}_1,\hat{z}_2,z_3,\ldots,z_m} \tag{21}$$

we can write that

$$Q_{12}^{\hat{z}_1,\hat{z}_2,z_3,\ldots,z_m}(x_1, z_2, z_3, \ldots, z_m) = P(x_1, z_2, z_3, \ldots, z_m) \sum_{j_1=0}^{N_1} r_{1j_1}(\hat{z}_1, z_2, z_3, \ldots, z_m) x_1^{j_1} \quad (z_2 \in \mathcal{N}_2), \tag{22}$$

$$Q_{12}^{\hat{z}_1,\hat{z}_2,z_3,\ldots,z_m}(z_1, x_2, z_3, \ldots, z_m) = P(z_1, x_2, z_3, \ldots, z_m) \sum_{j_2=0}^{N_2} r_{2j_2}(z_1, \hat{z}_2, z_3, \ldots, z_m) x_2^{j_2} \quad (z_1 \in \mathcal{N}_1). \tag{23}$$

Therefore

$$\sum_{j_1=0}^{N_1} r_{1j_1}(\hat{z}_1, z_2, z_3, \ldots, z_m) z_1^{j_1} = \sum_{j_2=0}^{N_2} r_{2j_2}(z_1, \hat{z}_2, z_3, \ldots, z_m) z_2^{j_2} \quad (z_1 \in \mathcal{N}_1, z_2 \in \mathcal{N}_2) \tag{24}$$

because of $P(z_1, \ldots, z_m) \neq 0$ $(z_1 \in \mathcal{N}_1 \subset U_1, z_2 \in \mathcal{N}_2 \subset U_2)$; note that $z_3 \in U_3$, $\ldots$, $z_m \in U_m$ are fixed. Since $\mathcal{N}_1$ contains more than finitely many elements we can choose different values $z_{10}, \ldots, z_{1N_1}$ to satisfy (24):

$$V(z_{10}, \ldots, z_{1N_1}) \begin{pmatrix} r_{10}(\hat{z}_1, z_2, \ldots, z_m) \\ r_{11}(\hat{z}_1, z_2, \ldots, z_m) \\ \cdot \\ r_{1N_1}(\hat{z}_1, z_2, \ldots, z_m) \end{pmatrix} = \sum_{j_2=0}^{N_2} \begin{pmatrix} r_{2j_2}(z_{10}, \hat{z}_2, z_3, \ldots, z_m) \\ r_{2j_2}(z_{11}, \hat{z}_2, z_3, \ldots, z_m) \\ \cdot \\ r_{2j_2}(z_{1N_1}, \hat{z}_2, z_3, \ldots, z_m) \end{pmatrix} z_2^{j_2}.$$

By Cramer's rule

$$r_{1j_1}(\hat{z}_1, z_2, \ldots, z_m) = \sum_{j_2=0}^{N_2} r_{12j_1j_2}(\hat{z}_1, \hat{z}_2, z_3, \ldots, z_m) z_2^{j_2} \quad (z_2 \in \mathcal{N}_2, j_1 = 0, \ldots, N_1), \tag{25}$$

where the coefficient $r_{12j_1j_2}(\hat{z}_1, \hat{z}_2, z_3, \ldots, z_m)$ is independent of the choice of $z_1$ and $z_2$. Using (25), equation (22) can be written as

$$Q_{12}^{\hat{z}_1,\hat{z}_2,z_3,\ldots,z_m}(x_1, z_2, z_3, \ldots, z_m) = P(x_1, z_2, z_3, \ldots, z_m) \sum_{j_1=0}^{N_1} \sum_{j_2=0}^{N_2} r_{12j_1j_2}(\hat{z}_1, \hat{z}_2, z_3 \ldots, z_m) x_1^{j_1} z_2^{j_2} \tag{26}$$

for any $z_2 \in \mathcal{N}_2$. Since both sides are polynomials in the second variable for fixed $x_1$ and $\mathcal{N}_2$ is not finite we have the following generalization of (11).

$$Q_{12}^{\hat{z}_1,\hat{z}_2,z_3,\ldots,z_m}(x_1, x_2, z_3, \ldots, z_m) = P(x_1, x_2, z_3, \ldots, z_m) \sum_{j_1=1}^{N_1} \sum_{j_2=1}^{N_2} r_{12j_1j_2}(\hat{z}_1, \hat{z}_2, z_3, \ldots, z_m) x_1^{j_1} x_2^{j_2}, \tag{27}$$

i.e. for any $z_3 \in U_3, \ldots, z_m \in U_m$ there is a polynomial $Q_{12}^{\hat{z}_1,\hat{z}_2,z_3,\ldots,z_m} \in \mathbb{Q}[x_1, \ldots, x_m]$ which can be divided by $P(x_1, x_2, z_3, \ldots, z_m)$:

$$Q_{12}^{\hat{z}_1,\hat{z}_2,z_3,\ldots,z_m}(x_1, x_2, z_3, \ldots, z_m) = P(x_1, x_2, z_3, \ldots, z_m) R_{12}(x_1, x_2, z_3, \ldots, z_m), \tag{28}$$

where $R_{12}(x_1, x_2, z_3, \ldots, z_m) \in \mathbb{C}[x_1, x_2]$. Equation (28) corresponds to (12). It can be easily seen that a polynomial $Q_{ij}^{z_1,\ldots,\hat{z}_i,\ldots,\hat{z}_j,\ldots,z_m} \in \mathbb{Q}[x_1, \ldots, x_m]$ can be choosen for any pair of different indices in a similar way:

$$Q_{13}^{\hat{z}_1,z_2,\hat{z}_3,z_4,\ldots,z_m}(x_1,z_2,x_3,z_4,\ldots,z_m)= P(x_1,z_2,x_3,z_4,\ldots,z_m)R_{13}(x_1,z_2,x_3,z_4,\ldots,z_m), \tag{29}$$

$$Q_{23}^{z_1,\hat{z}_2,\hat{z}_3,z_4,\ldots,z_m}(z_1,x_2,x_3,z_4,\ldots,z_m)= P(z_1,x_2,x_3,z_4,\ldots,z_m)R_{23}(z_1,x_2,x_3,z_4,\ldots,z_m) \text{ and so on.} \tag{30}$$

By keeping the variables $z_4 \in U_4,\ldots,z_m \in U_m$ constant we can generalize formula (27) for the triplet $i=1$, $j=2$ and $k=3$ as follows. Since $Q_{12}^{\hat{z}_1,\hat{z}_2,z_3,z_4,\ldots z_m} \in Q[x_1,\ldots,x_m]$ and $z_3 \in U_3$, a cardinality argument shows that we can choose a non-finite subset $\mathcal{N}_3 \subset U_3$ such that

$$Q_{12}^{\hat{z}_1,\hat{z}_2,z_3,z_4,\ldots,z_m} = Q_{12}^{\hat{z}_1,\hat{z}_2,\hat{z}_3,z_4,\ldots,z_m} \quad (z_3 \in \mathcal{N}_3), \tag{31}$$

i.e. the same polynomial $Q_{12}^{\hat{z}_1,\hat{z}_2,\hat{z}_3,z_4,\ldots,z_m} \in \mathbb{Q}[x_1,\ldots,x_m]$ occurs for any $z_3 \in \mathcal{N}_3$. In a similar way we can introduce polynomials satisfying

$$Q_{13}^{\hat{z}_1,z_2,\hat{z}_3,z_4,\ldots,z_m} = Q_{13}^{\hat{z}_1,\hat{z}_2,\hat{z}_3,z_4,\ldots,z_m} \quad (z_2 \in \mathcal{N}_2), \tag{32}$$

$$Q_{23}^{z_1,\hat{z}_2,\hat{z}_3,z_4,\ldots,z_m} = Q_{23}^{\hat{z}_1,\hat{z}_2,\hat{z}_3,z_4,\ldots,z_m} \quad (z_1 \in \mathcal{N}_1), \tag{33}$$

where $\mathcal{N}_1 \subset U_1$ and $\mathcal{N}_2 \subset U_2$ are not finite subsets. Taking the product

$$Q_{123}^{\hat{z}_1,\hat{z}_2,\hat{z}_3,z_4,\ldots,z_m} := Q_{12}^{\hat{z}_1,\hat{z}_2,\hat{z}_3,z_4,\ldots,z_m} \cdot Q_{13}^{\hat{z}_1,\hat{z}_2,\hat{z}_3,z_4,\ldots,z_m} \cdot Q_{23}^{\hat{z}_1,\hat{z}_2,\hat{z}_3,z_4,\ldots,z_m} \tag{34}$$

it follows that

$$Q_{123}^{\hat{z}_1,\hat{z}_2,\hat{z}_3,z_4,\ldots,z_m}(x_1,x_2,z_3,z_4,\ldots,z_m)= P(x_1,x_2,z_3,z_4,\ldots,z_m)\sum_{j_1=0}^{N_1}\sum_{j_2=0}^{N_2} r_{12j_1j_2}(\hat{z}_1,\hat{z}_2,z_3,z_4,\ldots,z_m)x_1^{j_1}x_2^{j_2} \quad (z_3 \in \mathcal{N}_3), \tag{35}$$

$$Q_{123}^{\hat{z}_1,\hat{z}_2,\hat{z}_3,z_4,\ldots,z_m}(x_1,z_2,x_3,z_4,\ldots,z_m)= P(x_1,z_2,x_3,z_4,\ldots,z_m)\sum_{j_1=0}^{N_1}\sum_{j_3=0}^{N_3} r_{13j_1j_3}(\hat{z}_1,z_2,\hat{z}_3,z_4,\ldots,z_m)x_1^{j_1}x_3^{j_3} \quad (z_2 \in \mathcal{N}_2), \tag{36}$$

$$Q_{123}^{\hat{z}_1,\hat{z}_2,\hat{z}_3,z_4,\ldots,z_m}(z_1,x_2,x_3,z_4,\ldots,z_m)= P(z_1,x_2,x_3,z_4,\ldots,z_m)\sum_{j_2=0}^{N_2}\sum_{j_3=0}^{N_3} r_{23j_2j_3}(z_1,\hat{z}_2,\hat{z}_3,z_4,\ldots,z_m)x_2^{j_2}x_3^{j_3} \quad (z_1 \in \mathcal{N}_1) \tag{37}$$

and we have the system of equations

$$
\begin{aligned}
(38)\qquad &\sum_{j_1=0}^{N_1}\sum_{j_2=0}^{N_2} r_{12j_1j_2}(\hat{z}_1,\hat{z}_2,z_3,z_4,\dots,z_m)z_1^{j_1}z_2^{j_2} = \\
&\qquad \sum_{j_1=0}^{N_1}\sum_{j_3=0}^{N_3} r_{13j_1j_3}(\hat{z}_1,z_2,\hat{z}_3,z_4,\dots,z_m)z_1^{j_1}z_3^{j_3} \\
&\qquad\qquad (z_1\in\mathcal{N}_1, z_2\in\mathcal{N}_2, z_3\in\mathcal{N}_3), \\
&\sum_{j_1=0}^{N_1}\sum_{j_2=0}^{N_2} r_{12j_1j_2}(\hat{z}_1,\hat{z}_2,z_3,z_4,\dots,z_m)z_1^{j_1}z_2^{j_2} = \\
&\qquad \sum_{j_2=0}^{N_2}\sum_{j_3=0}^{N_3} r_{23j_2j_3}(z_1,\hat{z}_2,\hat{z}_3,z_4,\dots,z_m)z_2^{j_2}z_3^{j_3} \\
&\qquad\qquad (z_1\in\mathcal{N}_1, z_2\in\mathcal{N}_2, z_3\in\mathcal{N}_3), \\
&\sum_{j_1=0}^{N_1}\sum_{j_3=0}^{N_3} r_{13j_1j_3}(\hat{z}_1,z_2,\hat{z}_3,z_4,\dots,z_m)z_1^{j_1}z_3^{j_3} = \\
&\qquad \sum_{j_2=0}^{N_2}\sum_{j_3=0}^{N_3} r_{23j_2j_3}(z_1,\hat{z}_2,\hat{z}_3,z_4,\dots,z_m)z_2^{j_2}z_3^{j_3} \\
&\qquad\qquad (z_1\in\mathcal{N}_1, z_2\in\mathcal{N}_2, z_3\in\mathcal{N}_3).
\end{aligned}
$$

According to the common polynomial terms, (38) can be simplified as

$$
\begin{aligned}
(39)\qquad &\sum_{j_2=0}^{N_2} r_{12j_1j_2}(\hat{z}_1,\hat{z}_2,z_3,z_4,\dots,z_m)z_2^{j_2} = \\
&\qquad \sum_{j_3=0}^{N_3} r_{13j_1j_3}(\hat{z}_1,z_2,\hat{z}_3,z_4,\dots,z_m)z_3^{j_3} \\
&\qquad\qquad (z_2\in\mathcal{N}_2, z_3\in\mathcal{N}_3, j_1=0,\dots,N_1), \\
&\sum_{j_1=0}^{N_1} r_{12j_1j_2}(\hat{z}_1,\hat{z}_2,z_3,z_4,\dots,z_m)z_1^{j_1} = \\
&\qquad \sum_{j_3=0}^{N_3} r_{23j_2j_3}(z_1,\hat{z}_2,\hat{z}_3,z_4,\dots,z_m)z_3^{j_3} \\
&\qquad\qquad (z_1\in\mathcal{N}_1, z_3\in\mathcal{N}_3, j_2=0,\dots,N_2), \\
&\sum_{j_1=0}^{N_1} r_{13j_1j_3}(\hat{z}_1,z_2,\hat{z}_3,z_4,\dots,z_m)z_1^{j_1} =
\end{aligned}
$$

$$\sum_{j_2=0}^{N_2} r_{23j_2j_3}(z_1, \hat{z}_2, \hat{z}_3, z_4, \ldots, z_m) z_2^{j_2}$$

$$(z_1 \in \mathcal{N}_1, z_2 \in \mathcal{N}_2, j_3 = 0, \ldots, N_3).$$

These are equations of type (8). Therefore we can also generalize equation (9) by the same process as in subsection 3.1: for example

$$r_{12j_1j_2}(\hat{z}_1, \hat{z}_2, z_3, z_4, \ldots, z_m) = \sum_{j_3=0}^{N_3} r_{123j_1j_2j_3}(\hat{z}_1, \hat{z}_2, \hat{z}_3, z_4, \ldots, z_m) z_3^{j_3} \quad (z_3 \in \mathcal{N}_3, j_1 = 0, \ldots, N_1, j_2 = 0, \ldots, N_2), \tag{40}$$

where the coefficient $r_{123j_1j_2j_3}(\hat{z}_1, \hat{z}_2, \hat{z}_3, z_4, \ldots, z_m)$ is independent of the choice of $z_1$, $z_2$ and $z_3$. Using (40), equation (35) can be written as

$$Q_{123}^{\hat{z}_1,\hat{z}_2,\hat{z}_3,z_4,\ldots,z_m}(x_1, x_2, z_3, z_4, \ldots, z_m) = P(x_1, x_2, z_3, \ldots, z_m) \sum_{j_1=0}^{N_1} \sum_{j_2=0}^{N_2} \sum_{j_3=0}^{N_3} r_{123j_1j_2j_3}(\hat{z}_1, \hat{z}_2, \hat{z}_3, z_4, \ldots, z_m) x_1^{j_1} x_2^{j_2} z_3^{j_3} \quad (z_3 \in \mathcal{N}_3). \tag{41}$$

Since both sides are polynomials in the third variable for fixed $x_1$, $x_2$ and $\mathcal{N}_3$ is not finite it follows that

$$Q_{123}^{\hat{z}_1,\hat{z}_2,\hat{z}_3,z_4,\ldots,z_m}(x_1, x_2, x_3, z_4, \ldots, z_m) = P(x_1, x_2, x_3, z_4, \ldots, z_m) \sum_{j_1=0}^{N_1} \sum_{j_2=0}^{N_2} \sum_{j_3=0}^{N_3} r_{123j_1j_2j_3}(\hat{z}_1, \hat{z}_2, \hat{z}_3, z_4, \ldots, z_m) x_1^{j_1} x_2^{j_2} x_3^{j_3},$$

i.e. for any $z_4 \in U_4, \ldots, z_m \in U_m$ there is a polynomial $Q_{123}^{\hat{z}_1,\hat{z}_2,\hat{z}_3,z_4,\ldots,z_m} \in \mathbb{Q}[x_1, \ldots, x_m]$ which can be divided by $P(x_1, x_2, x_3, z_4, \ldots, z_m)$:

$$Q_{123}^{\hat{z}_1,\hat{z}_2,\hat{z}_3,z_4,\ldots,z_m}(x_1, x_2, x_3, z_4, \ldots, z_m) = P(x_1, x_2, x_3, z_4, \ldots, z_m) R_{123}(x_1, x_2, x_3, z_4, \ldots, z_m), \tag{42}$$

where $R_{123}(x_1, x_2, x_3, z_4, \ldots, z_m) \in \mathbb{C}[x_1, x_2, x_3]$. It is a generalization of (28) and a polynomial $Q_{ijk}^{z_1,\ldots,\hat{z}_i,\ldots,\hat{z}_j,\ldots,\hat{z}_k,\ldots,z_m} \in \mathbb{Q}[x_1, \ldots, x_m]$ can be choosen for any triplet of different indices in a similar way. Repeating the procedure up to $m$ different indices we can find a polynomial $Q := Q_{1\ldots m}^{\hat{z}_1,\ldots,\hat{z}_m} \in \mathbb{Q}[x_1, \ldots, x_m]$ such that

$$Q(x_1, x_2, \ldots, x_m) = P(x_1, x_2, \ldots, x_m) R(x_1, x_2, \ldots, x_m), \tag{43}$$

where $R(x_1, \ldots, x_m) := R_{1\ldots m}(x_1, \ldots, x_m) \in \mathbb{C}[x_1, \ldots, x_m]$.

4.2. **The comparison of the coefficients.** The next step is to compare the coefficients of the polynomials on different sides of (43). Let $d_1 := \deg_1 P$ be the degree of the polynomial with respect to the first variable. The coefficient of $x_1^{d_1}$ can be written as the polynomial

$$P_{d_1}(x_2, \ldots, x_m) := \sum_{i_2=0}^{d_2} \cdots \sum_{i_m=0}^{d_m} p_{d_1 i_2 \ldots i_m} x_2^{i_2} \cdot \ldots \cdot x_m^{i_m}.$$

of the variables $x_2, \ldots, x_m$ and the substitution of each root $(w_2, \ldots, w_m)$ of $P_{d_1}$ causes a decreasing in the degree of the right hand side of (43) with respect to $x_1$. Therefore the polynomial $Q$ also has to decrease its maximal degree $D_1$ with respect to $x_1$. This means that the polynomial $Q_{D_1}(x_2, \ldots, x_m)$ (the coefficient of $x_1^{D_1}$ in the polynomial $Q$) vanishes at each root of $P_{d_1}(x_2, \ldots, x_m)$, i.e. each root is algebraically dependent. Since the number of the variables is $m-1$ we can use the statement of the main theorem as an inductive hypothesis:

$$p_{d_1 i_2 \ldots i_m} = \lambda \omega_{i_2 \ldots i_m}, \tag{44}$$

where $\lambda \in \mathbb{C}$ is a common proportional term and $\omega_{i_2 \ldots i_m}$ are algebraic numbers over the rationals. Let us choose algebraic numbers $a_2, \ldots, a_m$ such that $Q(x_1, a_2, \ldots, a_m) \in \mathbb{C}[x_1]$ is not identically zero. Then the solutions of equation

$$P(x_1, a_2, \ldots, a_m) = 0$$

are algebraic because of (43) and the coefficients of the polynomial

$$P(x_1, a_2, \ldots, a_m) = \sum_{i_1=0}^{d_1} \left( \sum_{i_2=0}^{d_2} \cdots \sum_{i_m=0}^{d_m} p_{i_1 i_2 \ldots i_m} a_2^{i_2} \cdot \ldots \cdot a_m^{i_m} \right) x_1^{i_1} \tag{45}$$

can be written as

$$\sum_{i_2=0}^{d_2} \cdots \sum_{i_m=0}^{d_m} p_{i_1 i_2 \ldots i_m} a_2^{i_2} \cdot \ldots \cdot a_m^{i_m} = c_{i_1}(a_2, \ldots, a_m) \underbrace{\sum_{i_2=0}^{d_2} \cdots \sum_{i_m=0}^{d_m} p_{d_1 i_2 \ldots i_m} a_2^{i_2} \cdot \ldots \cdot a_m^{i_m}}_{\text{the main coeff. of (45).}},$$

where $i_1 = 0, \ldots, d_1$ and $c_{i_1}(a_2, \ldots, a_m)$ is an algebraic number depending on $a_2, \ldots, a_m$; especially $c_{d_1}(a_2, \ldots, a_m) = 1$. According to (44)

$$\sum_{i_2=0}^{d_2} \cdots \sum_{i_m=0}^{d_m} p_{i_1 i_2 \ldots i_m} a_2^{i_2} \cdot \ldots \cdot a_m^{i_m} = \lambda \omega_{i_1}(a_2, \ldots, a_m), \tag{46}$$

$$\omega_{i_1}(a_2, \ldots, a_m) = c_{i_1}(a_2, \ldots, a_m) \sum_{i_2=0}^{d_2} \cdots \sum_{i_m=0}^{d_m} \omega_{i_2 \ldots i_m} a_2^{i_2} \cdot \ldots \cdot a_m^{i_m},$$

where $i_1 = 0, \ldots, d_1$ and $\omega_{i_1}(a_2, \ldots, a_m)$'s are algebraic numbers depending on $a_2, \ldots, a_m$. Since the cardinality of the algebraic numbers is not finite we can choose different values $a_{20},\dots,a_{2N}$ to satisfy (46). In terms of a linear system of equations

$$V(a_{20},\dots,a_{2N})\begin{pmatrix} X_{00} & \dots & X_{N0}\\ X_{01} & \dots & X_{N1}\\ . & . & .\\ X_{0N} & \dots & X_{NN}\end{pmatrix}=$$

$$\lambda\begin{pmatrix} \omega_0(a_{20},a_3,\dots a_m) & \dots & \omega_N(a_{20},a_3,\dots,a_N)\\ \omega_0(a_{21},a_3,\dots a_m) & \dots & \omega_N(a_{21},a_3,\dots,a_N)\\ . & . & .\\ \omega_0(a_{2N},a_3,\dots a_m) & \dots & \omega_N(a_{2N},a_3,\dots,a_N)\end{pmatrix},$$

where

$$V(a_{20},\dots,a_{2N}):=\begin{pmatrix} 1 & a_{20} & \dots & a_{20}^N\\ 1 & a_{21} & \dots & a_{21}^N\\ . & . & . & .\\ 1 & a_{2N} & \dots & a_{2N}^N\end{pmatrix}$$

is the usual Vandermonde matrix,

$$X_{kl}:=\sum_{i_3=0}^{d_3}\dots\sum_{i_m=0}^{d_m} p_{kli_3\dots i_m}a_3^{i_3}\cdot\ldots\cdot a_m^{i_m},$$

where $k,l=0,\dots,N$ and $N:=\max\{d_1,\dots,d_m\}$; note that we allow zero elements too, i.e. if the monomial term $x_1^kx_2^lx_3^{i_3}\cdot\ldots\cdot x_m^{i_m}$ is missing for some values of $k$ and $l$ in the polynomial $P$ then $p_{kli_3\dots i_m}:=0$. Since the algebraic numbers form a field the matrix $V^{-1}(a_{20},\dots,a_{2N})$ contains algebraic numbers and we have that $X_{kl}$'s must be also algebraic, i.e.

$$\sum_{i_3=0}^{d_3}\dots\sum_{i_m=0}^{d_m} p_{i_1i_2i_3\dots i_m}a_3^{i_3}\cdot\ldots\cdot a_m^{i_m}=\lambda\omega_{i_1i_2}(a_3,\dots,a_m), \tag{47}$$

where $\omega_{i_1i_2}(a_3,\dots,a_m)$'s are algebraic numbers depending on $a_3,\dots,a_m$, $i_1=0,\dots,d_1$ and $i_2=0,\dots,d_2$. Equation (47) is of the same type as (46). The process can be repeated by choosing different values $a_{30},\dots,a_{3N}$ to satisfy (47) and so on. After finitely many steps we can conclude that

$$p_{i_1\dots i_m}=\lambda\omega_{i_1\dots i_m},$$

where $\omega_{i_1\dots i_m}$ are algebraic numbers, $\lambda\in\mathbb{C}$ and $0\le i_1\le d_1,\dots,0\le i_m\le d_m$.

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