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ON A TAYLOR REMAINDER

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Abstract. In this note we derive a new Taylor remainder, which extends the well known Lagrange remainder as well as the obscure Gonçalves remainder.

1. Introduction

The Taylor formula is one of the most well-known results in analysis. The Lagrange, Cauchy and Peano remainders are well-known but there are other less known Taylor remainders e.g. Schlömilch, Bourget, Blumenthal and Gonçalves among others, see [2] for a detailed review and historical remarks on these remainders.

2. New Taylor remainder

Since our main result depends crucially on the Cauchy generalized mean value theorem, we give its exact formulation taken from [3, Theorem 5.9]:

Cauchy generalized mean value theorem: *If f and g are continuous real functions on $[a,b]$ which are differentiable in (a,b), then there is a point $x \in (a,b)$ at which*

$$[f(b)-f(a)]\,g'(x) = [g(b)-g(a)]f'(x).$$

Note that differentiability is not required at the end points. Moreover, if $g'(\tau) \neq 0$ for all $\tau \in (a,b)$ then we have the equality

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(x)}{g'(x)}. \tag{1}$$

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We highlight the fact that $g'(\xi)$ can be zero in $\xi = a$ or $\xi = b$, since the conditions in the Cauchy generalized mean value theorem only require that $g'(\xi) \neq 0$ in (a, b). For a recent work on mean value theorem we refer the reader to [1].

The following observation will be useful. Let φ be continuous in $[a, b]$ and differentiable in (a, b) such that $\varphi(x) \neq 0$ for all $x \in (a, b)$. If $\varphi'(\xi)R(\xi)$ has a primitive $\psi(\xi)$, then by the Cauchy generalized mean value theorem we get

$$\frac{\psi(b) - \psi(a)}{\varphi(b) - \varphi(a)} = \frac{\psi'(\xi)}{\varphi'(\xi)} = R(\xi), \tag{2}$$

where ξ lies between b and a.

Our main results reads:

Theorem 1. *Let $f^{(j)}$ and $\varphi^{(j)}$ be real-valued continuous functions in $[\alpha, \beta]$ and differentiable in (α, β) for $j = 0, \ldots, n-1$. Then, for $a \in [\alpha, \beta]$ and all $x \in [\alpha, \beta] \setminus \{a\}$ and $\varphi = \varphi(x)$ such that $\varphi^{(s)}(\xi) \neq 0$ for all ξ lying between a and x and for $s = 1, \ldots, n$ we have*

$$f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + r_n(x; a), \tag{3}$$

with

$$r_n(x; a) = \left(\varphi(x) - \sum_{k=0}^{n-1} \frac{(x-a)^k}{k!}\varphi^{(k)}(a)\right)\left(\frac{f^{(n)}(\xi) - f^{(n)}(a)}{\varphi^{(n)}(\xi)}\right), \tag{4}$$

where $x \neq a$ and ξ lies between a and x.

Proof. Let $x > a$ and define $R(x)$ as

$$f^{(n)}(x) - f^{(n)}(a) = \varphi^{(n)}(x)R(x). \tag{5}$$

Applying (2) to (5) we obtain

$$f^{(n-1)}(x) - f^{(n-1)}(a) - (x-a)f^{(n)}(a) = [\varphi^{(n-1)}(x) - \varphi^{(n-1)}(a)](R \circ \vartheta_1)(x), \tag{6}$$

where $\vartheta_1(x) = a + \theta_1(x)(x-a)$ and $\theta_1(x) \in (0, 1)$. Applying again (2) to (6) we obtain

$$\begin{aligned} f^{(n-2)}(x) - f^{(n-2)}(a) - (x-a)f^{(n-1)}(a) - \frac{(x-a)^2}{2}f^{(n)}(a) \\ = \left[\varphi^{(n-2)}(x) - \varphi^{(n-2)}(a) - (x-a)\varphi^{(n-1)}(a)\right](R \circ \vartheta_1 \circ \vartheta 2)(x) \end{aligned}$$

where $\vartheta_2(x) = a + \theta_2(x)(x-a)$ and $\theta_2(x) \in (0, 1)$.

Now iterating the previous reasoning we finally obtain

$$f(x) = f(a) + (x-a)f'(a) + \cdots + \frac{(x-a)^n f^{(n)}(a)}{n!} + \left[\varphi(x) - \sum_{k=0}^{n-1} \frac{(x-a)^k}{k!}\varphi^{(k)}(a)\right](R \circ \vartheta_1 \circ \cdots \circ \vartheta_n)(x)$$

from which we get the desired result since $R(x) = \frac{f^{(n)}(x) - f^{(n)}(a)}{\varphi^{(n)}(x)}$. The case of $x < a$ is similar. □

Remark 1. Taking $\varphi(\kappa) = (\kappa - a)^{n+1}$ we obtain the Gonçalves remainder (see [2] for an historical account on this remainder)

$$r_n(x; a) = \frac{(x-a)^{n+1}}{(n+1)!}\left(\frac{f^{(n)}(\xi) - f^{(n)}(a)}{\xi - a}\right), \tag{7}$$

and if f has $n+1$ derivatives it follows from (7) and the Lagrange mean value theorem the well-known Lagrange remainder.

Since in the nth remainder (4) we only ask for the existence of the derivative of f up to order n instead of up to order $n+1$ as in the case of Lagrange, Cauchy, Schlömilch, etc., the remainder (4) gives a theoretical improvement. Let us give an example.

Example 1. Taking $f(x) = \frac{1}{2}x^2 \operatorname{sign}(x)$ and remembering that $f'(x) = |x|$, we now calculate the Taylor formula with two different remainders.

For $x < 0$ and $a = 1$, we have the Taylor formula with Lagrange remainder of f given by

$$f(x) = \frac{1}{2} + (x-1)|\xi|, \quad \xi \in (x, 1),$$

whereas taking the Taylor formula with Gonçalves remainder (7) we obtain

$$f(x) = \frac{1}{2} + (x-1) + \frac{(x-1)^2}{2}\left(\frac{|\xi| - 1}{\xi - 1}\right), \quad \xi \in (x, 1).$$

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