

HERMITE-HADAMARD TYPE INEQUALITIES OBTAINED VIA RIEMANN-LIOUVILLE FRACTIONAL CALCULUS

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ABSTRACT. We extend some inequalities obtained by M. A. Latif to the framework of Riemann-Liouville fractional calculus.

1. INTRODUCTION

The Hermite-Hadamard inequality asserts that for every convex function $f: [a, b] \rightarrow \mathbb{R}$, one has

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2},$$

where $a, b \in I$ with $a < b$. One can easily prove that the left term is closer to the integral mean value than the right one. Therefore,

$$(1) \quad \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{1}{2} \left(\frac{f(a) + f(b)}{2} + f\left(\frac{a+b}{2}\right) \right).$$

See [5, p. 52].

A remarkable variety of refinements and generalizations of Hermite-Hadamard inequality have been found; see, for example, [1], [3], [5] and the references cited therein.

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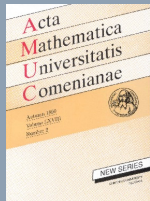


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Our aim is to establish some new inequalities related to (1), using the Riemann-Liouville fractional integration. We deal with functions whose derivatives in absolute value are convex.

Let $f \in L^1[a, b]$, where $a \geq 0$. The Riemann-Liouville integrals $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha > 0$ are defined by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt \quad \text{for } x > a,$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt \quad \text{for } x < b,$$

respectively. Here, $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$ is the Gamma function. We also make the convention

$$J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x).$$

More details about the Riemann-Liouville fractional integrals may be found in [2].

2. MAIN RESULTS

We assume throughout the present paper that $[a, b]$ is a subinterval of $[0, \infty)$ and $f: [a, b] \rightarrow \mathbb{R}$ is a function differentiable on (a, b) such that $f' \in L^1[a, b]$. Throughout this section we define the



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Hermite-Hadamard α -gap by

$$\mathcal{H}_\alpha(x) := \frac{((x-a)^\alpha + (b-x)^\alpha) f(x)}{b-a} + \frac{(x-a)^\alpha f(a) + (b-x)^\alpha f(b)}{b-a} - \frac{2^\alpha \Gamma(\alpha+1)}{b-a} \left[J_{x^-}^\alpha f\left(\frac{x+a}{2}\right) + J_{a^+}^\alpha f\left(\frac{x+a}{2}\right) + J_{b^-}^\alpha f\left(\frac{x+b}{2}\right) + J_{x^+}^\alpha f\left(\frac{x+b}{2}\right) \right].$$

In the particular case where $\alpha = 1$, this reduces to

$$\mathcal{H}(x) = f(x) + \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{2}{b-a} \int_a^b f(t) dt.$$

Thus

$$\mathcal{H}\left(\frac{a+b}{2}\right) = f\left(\frac{a+b}{2}\right) + \frac{f(a) + f(b)}{2} - \frac{2}{b-a} \int_a^b f(t) dt.$$

The value of \mathcal{H} was estimated by M. A. Latif [4] and it is the purpose of the present paper to generalize some of his results. For this we need a preparation.

Lemma 1. *We have*

$$\begin{aligned} \mathcal{H}_\alpha(x) &= \frac{(x-a)^{\alpha+1}}{b-a} \left(\int_0^1 \frac{t^\alpha}{2} f' \left(\frac{1+t}{2} x + \frac{1-t}{2} a \right) dt - \int_0^1 \frac{t^\alpha}{2} f' \left(\frac{1-t}{2} x + \frac{1+t}{2} a \right) dt \right) \\ &\quad - \frac{(b-x)^{\alpha+1}}{b-a} \left(\int_0^1 \frac{t^\alpha}{2} f' \left(\frac{1+t}{2} x + \frac{1-t}{2} b \right) dt + \int_0^1 \frac{t^\alpha}{2} f' \left(\frac{1-t}{2} x + \frac{1+t}{2} b \right) dt \right), \end{aligned}$$

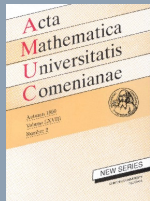


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for all $x \in [a, b]$.

Proof. We use the integration by parts and appropriate substitutions (such as $u = \frac{1+t}{2}x + \frac{1-t}{2}a$, $v = \frac{1-t}{2}x + \frac{1+t}{2}a$ etc.) to show that

$$\int_0^1 \frac{t^\alpha}{2} f' \left(\frac{1+t}{2}x + \frac{1-t}{2}a \right) dt = \frac{f(x)}{x-a} - \frac{2^\alpha \Gamma(\alpha+1)}{(x-a)^{\alpha+1}} J_{x^-}^\alpha f \left(\frac{x+a}{2} \right),$$

$$\int_0^1 \frac{t^\alpha}{2} f' \left(\frac{1-t}{2}x + \frac{1+t}{2}a \right) dt = -\frac{f(a)}{x-a} + \frac{2^\alpha \Gamma(\alpha+1)}{(x-a)^{\alpha+1}} J_{a^+}^\alpha f \left(\frac{x+a}{2} \right),$$

$$\int_0^1 \frac{t^\alpha}{2} f' \left(\frac{1+t}{2}x + \frac{1-t}{2}b \right) dt = -\frac{f(x)}{b-x} + \frac{2^\alpha \Gamma(\alpha+1)}{(b-x)^{\alpha+1}} J_{x^+}^\alpha f \left(\frac{x+b}{2} \right),$$

$$\int_0^1 \frac{t^\alpha}{2} f' \left(\frac{1-t}{2}x + \frac{1+t}{2}b \right) dt = \frac{f(b)}{b-x} - \frac{2^\alpha \Gamma(\alpha+1)}{(b-x)^{\alpha+1}} J_{b^-}^\alpha f \left(\frac{x+b}{2} \right).$$

The proof is complete. □

We are now in a position to state and prove the following theorems.

Theorem 1. Assume $|f'|$ is convex on $[a, b]$. Then

$$|\mathcal{H}_\alpha(x)| \leq \frac{(x-a)^{\alpha+1}}{b-a} \cdot \frac{|f'(x)| + |f'(a)|}{2(\alpha+1)} + \frac{(b-x)^{\alpha+1}}{b-a} \cdot \frac{|f'(x)| + |f'(b)|}{2(\alpha+1)}.$$



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Proof. Using Lemma 1 and taking modulus, we infer from the convexity of $|f'|$ that

$$\begin{aligned}
 |\mathcal{H}_\alpha(x)| &\leq \frac{(x-a)^{\alpha+1}}{b-a} \int_0^1 \frac{t^\alpha}{2} \left[\frac{1+t}{2} |f'(x)| + \frac{1-t}{2} |f'(a)| \right] dt \\
 &\quad + \frac{(x-a)^{\alpha+1}}{b-a} \int_0^1 \frac{t^\alpha}{2} \left[\frac{1-t}{2} |f'(x)| + \frac{1+t}{2} |f'(a)| \right] dt \\
 &\quad + \frac{(b-x)^{\alpha+1}}{b-a} \int_0^1 \frac{t^\alpha}{2} \left[\frac{1+t}{2} |f'(x)| + \frac{1-t}{2} |f'(b)| \right] dt \\
 &\quad + \frac{(b-x)^{\alpha+1}}{b-a} \int_0^1 \frac{t^\alpha}{2} \left[\frac{1-t}{2} |f'(x)| + \frac{1+t}{2} |f'(b)| \right] dt.
 \end{aligned}$$

The result follows after a straightforward computation in the right hand side term. This ends the proof. \square

Our next result reads as

Theorem 2. Assume $|f'|^q$ is convex on $[a, b]$ for some fixed $q > 1$. Then

$$\begin{aligned}
 |\mathcal{H}_\alpha(x)| &\leq \left(\frac{1}{2}\right)^{1+2/q} \times \left(\frac{1}{\alpha p + 1}\right)^{1/p} \times \\
 &\quad \times \left\{ \frac{(x-a)^{\alpha+1}}{b-a} \left[(3|f'(x)|^q + |f'(a)|^q)^{1/q} + (|f'(x)|^q + 3|f'(a)|^q)^{1/q} \right] \right. \\
 &\quad \left. + \frac{(b-x)^{\alpha+1}}{b-a} \left[(3|f'(x)|^q + |f'(b)|^q)^{1/q} + (|f'(x)|^q + 3|f'(b)|^q)^{1/q} \right] \right\}
 \end{aligned}$$

for all $x \in [a, b]$ and $\frac{1}{p} + \frac{1}{q} = 1$.



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Proof. According to Lemma 1 and Hölder's inequality, we have

$$|\mathcal{H}_\alpha(x)| \leq \frac{(x-a)^{\alpha+1}}{b-a} \left(\int_0^1 \left(\frac{t^\alpha}{2} \right)^p dt \right)^{1/p} \left((I_1)^{\frac{1}{q}} + (I_2)^{1/q} \right) + \frac{(b-x)^{\alpha+1}}{b-a} \left(\int_0^1 \left(\frac{t^\alpha}{2} \right)^p dt \right)^{1/p} \left((I_3)^{\frac{1}{q}} + (I_4)^{1/q} \right)$$

for all $x \in [a, b]$. Here

$$\begin{aligned} I_1 &= \int_0^1 \left| f' \left(\frac{1+t}{2}x + \frac{1-t}{2}a \right) \right|^q dt \\ &\leq \int_0^1 \left(\frac{1+t}{2} |f'(x)|^q + \frac{1-t}{2} |f'(a)|^q \right) dt = \frac{3|f'(x)|^q + |f'(a)|^q}{4}, \\ I_2 &= \int_0^1 \left| f' \left(\frac{1-t}{2}x + \frac{1+t}{2}a \right) \right|^q dt \leq \frac{|f'(x)|^q + 3|f'(a)|^q}{4}, \\ I_3 &= \int_0^1 \left| f' \left(\frac{1+t}{2}x + \frac{1-t}{2}b \right) \right|^q dt \leq \frac{3|f'(x)|^q + |f'(b)|^q}{4}, \end{aligned}$$

and

$$I_4 = \int_0^1 \left| f' \left(\frac{1-t}{2}x + \frac{1+t}{2}b \right) \right|^q dt \leq \frac{|f'(x)|^q + 3|f'(b)|^q}{4}.$$

These last inequalities hold due to the convexity of $|f'|^q$ on $[a, b]$. The proof is complete. \square

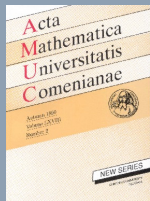


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Theorem 3. Assume $|f'|^q$ is convex on $[a, b]$ for some fixed $q \geq 1$. Then the following inequality

$$\begin{aligned}
 |\mathcal{H}_\alpha(x)| \leq & \left[\frac{1}{2(\alpha+1)} \right]^{1-1/q} \cdot \left[\frac{1}{4(\alpha+1)(\alpha+2)} \right]^{1/q} \\
 & \times \left\{ \frac{(x-a)^{\alpha+1}}{b-a} \left[((2\alpha+3)|f'(x)|^q + |f'(a)|^q)^{1/q} \right. \right. \\
 & \qquad \qquad \qquad \left. \left. + (|f'(x)|^q + (2\alpha+3)|f'(a)|^q)^{1/q} \right] \right. \\
 & \left. + \frac{(b-x)^{\alpha+1}}{b-a} \left[((2\alpha+3)|f'(x)|^q + |f'(b)|^q)^{1/q} \right. \right. \\
 & \qquad \qquad \qquad \left. \left. + (|f'(x)|^q + (2\alpha+3)|f'(b)|^q)^{1/q} \right] \right\}
 \end{aligned}$$

holds for all $x \in [a, b]$.

Proof. Using Lemma 1, the convexity of $|f'|^q$ on $[a, b]$ and the power-mean inequality, we have

$$\begin{aligned}
 |\mathcal{H}_\alpha(x)| \leq & \frac{(x-a)^{\alpha+1}}{b-a} \left(\int_0^1 \frac{t^\alpha}{2} dt \right)^{1-1/q} \left((J_1)^{1/q} + (J_2)^{1/q} \right) \\
 & + \frac{(b-x)^{\alpha+1}}{b-a} \left(\int_0^1 \frac{t^\alpha}{2} dt \right)^{1-1/q} \left((J_3)^{1/q} + (J_4)^{1/q} \right),
 \end{aligned}$$

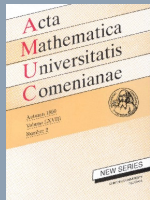


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where

$$J_1 = \int_0^1 \frac{t^\alpha}{2} \left| f' \left(\frac{1+t}{2}x + \frac{1-t}{2}a \right) \right|^q dt \leq \frac{(2\alpha + 3)|f'(x)|^q + |f'(a)|^q}{4(\alpha + 1)(\alpha + 2)},$$

$$J_2 = \int_0^1 \frac{t^\alpha}{2} \left| f' \left(\frac{1-t}{2}x + \frac{1+t}{2}a \right) \right|^q dt \leq \frac{|f'(x)|^q + (2\alpha + 3)|f'(a)|^q}{4(\alpha + 1)(\alpha + 2)},$$

$$J_3 = \int_0^1 \frac{t^\alpha}{2} \left| f' \left(\frac{1+t}{2}x + \frac{1-t}{2}b \right) \right|^q dt \leq \frac{(2\alpha + 3)|f'(x)|^q + |f'(b)|^q}{4(\alpha + 1)(\alpha + 2)},$$

and

$$J_4 = \int_0^1 \frac{t^\alpha}{2} \left| f' \left(\frac{1-t}{2}x + \frac{1+t}{2}b \right) \right|^q dt \leq \frac{|f'(x)|^q + (2\alpha + 3)|f'(b)|^q}{4(\alpha + 1)(\alpha + 2)}.$$

Hence the proof of the theorem is complete. □

Theorem 4. Assume $|f'|^q$ is concave on $[a, b]$ for some fixed $q > 1$. Then

$$\begin{aligned} |\mathcal{H}_\alpha(x)| \leq & \frac{1}{2} \left(\frac{1}{\alpha p + 1} \right)^{1/p} \left\{ \frac{(x-a)^{\alpha+1}}{b-a} \left[\left| f' \left(\frac{3x+a}{4} \right) \right| + \left| f' \left(\frac{x+3a}{4} \right) \right| \right] \right. \\ & \left. + \frac{(b-x)^{\alpha+1}}{b-a} \left[\left| f' \left(\frac{3x+b}{4} \right) \right| + \left| f' \left(\frac{x+3b}{4} \right) \right| \right] \right\} \end{aligned}$$

for all $x \in [a, b]$ and $\frac{1}{p} + \frac{1}{q} = 1$.

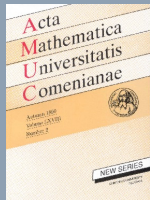


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Proof. From Lemma 1 and Hölder's integral inequality for $q > 1$ and $p = \frac{q}{q-1}$, we have

$$\begin{aligned}
 |\mathcal{H}_\alpha(x)| \leq & \frac{(x-a)^{\alpha+1}}{b-a} \left(\int_0^1 \left(\frac{t^\alpha}{2} \right)^p dt \right)^{1/p} (K_1)^{1/q} + \frac{(x-a)^{\alpha+1}}{b-a} \left(\int_0^1 \left(\frac{t^\alpha}{2} \right)^p dt \right)^{1/p} (K_2)^{1/q} \\
 & + \frac{(b-x)^{\alpha+1}}{b-a} \left(\int_0^1 \left(\frac{t^\alpha}{2} \right)^p dt \right)^{1/p} (K_3)^{1/q} + \frac{(b-x)^{\alpha+1}}{b-a} \left(\int_0^1 \left(\frac{t^\alpha}{2} \right)^p dt \right)^{1/p} (K_4)^{1/q}
 \end{aligned}$$

for all $x \in [a, b]$. Here,

$$\begin{aligned}
 K_1 &= \int_0^1 \left| f' \left(\frac{1+t}{2}x + \frac{1-t}{2}a \right) \right|^q dt \\
 &\leq \left| f' \left(\int_0^1 \left(\frac{1+t}{2}x + \frac{1-t}{2}a \right) dt \right) \right|^q = \left| f' \left(\frac{3x+a}{4} \right) \right|^q.
 \end{aligned}$$

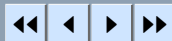
and similarly,

$$K_2 = \int_0^1 \left| f' \left(\frac{1-t}{2}x + \frac{1+t}{2}a \right) \right|^q dt \leq \left| f' \left(\frac{x+3a}{4} \right) \right|^q,$$

$$K_3 = \int_0^1 \left| f' \left(\frac{1+t}{2}x + \frac{1-t}{2}b \right) \right|^q dt \leq \left| f' \left(\frac{3x+b}{4} \right) \right|^q,$$

$$K_4 = \int_0^1 \left| f' \left(\frac{1-t}{2}x + \frac{1+t}{2}b \right) \right|^q dt \leq \left| f' \left(\frac{x+3b}{4} \right) \right|^q.$$

We used the concavity of $|f'|^q$ on $[a, b]$ and Jensen's integral inequality in order to obtain the last four inequalities. This completes the proof of the theorem. \square



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Our final result is the following theorem.

Theorem 5. *Suppose $|f'|$ is concave on $[a, b]$. Then*

$$|\mathcal{H}_\alpha(x)| \leq \frac{(x-a)^{\alpha+1}}{2(\alpha+1)(b-a)} \left[\left| f' \left(\frac{(2\alpha+3)x+a}{2(\alpha+2)} \right) \right| + \left| f' \left(\frac{x+(2\alpha+3)a}{2(\alpha+2)} \right) \right| \right] \\ + \frac{(b-x)^{\alpha+1}}{2(\alpha+1)(b-a)} \left[\left| f' \left(\frac{(2\alpha+3)x+b}{2(\alpha+2)} \right) \right| + \left| f' \left(\frac{x+(2\alpha+3)b}{2(\alpha+2)} \right) \right| \right]$$

for all $x \in [a, b]$.

Proof. Using Lemma 1 and taking modulus, we infer from the concavity of $|f'|$ that

$$|\mathcal{H}_\alpha(x)| \leq \frac{(x-a)^{\alpha+1}}{b-a} \int_0^1 \left| f' \left(\frac{1+t}{2}x + \frac{1-t}{2}a \right) \right| \frac{t^\alpha}{2} dt \\ + \frac{(x-a)^{\alpha+1}}{b-a} \int_0^1 \left| f' \left(\frac{1-t}{2}x + \frac{1+t}{2}a \right) \right| \frac{t^\alpha}{2} dt \\ + \frac{(b-x)^{\alpha+1}}{b-a} \int_0^1 \left| f' \left(\frac{1+t}{2}x + \frac{1-t}{2}b \right) \right| \frac{t^\alpha}{2} dt \\ + \frac{(b-x)^{\alpha+1}}{b-a} \int_0^1 \left| f' \left(\frac{1-t}{2}x + \frac{1+t}{2}b \right) \right| \frac{t^\alpha}{2} dt$$



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$$\begin{aligned}
 &\leq \frac{(x-a)^{\alpha+1}}{b-a} \left(\int_0^1 \frac{t^\alpha}{2} dt \right) \left| f' \left(\frac{\int_0^1 \left(\frac{1+t}{2}x + \frac{1-t}{2}a \right) \frac{t^\alpha}{2} dt}{\int_0^1 \frac{t^\alpha}{2} dt} \right) \right| \\
 &+ \frac{(x-a)^{\alpha+1}}{b-a} \left(\int_0^1 \frac{t^\alpha}{2} dt \right) \left| f' \left(\frac{\int_0^1 \frac{t^\alpha}{2} \left(\frac{1-t}{2}x + \frac{1+t}{2}a \right) dt}{\int_0^1 \frac{t^\alpha}{2} dt} \right) \right| \\
 &+ \frac{(b-x)^{\alpha+1}}{b-a} \left(\int_0^1 \frac{t^\alpha}{2} dt \right) \left| f' \left(\frac{\int_0^1 \frac{t^\alpha}{2} \left(\frac{1+t}{2}x + \frac{1-t}{2}b \right) dt}{\int_0^1 \frac{t^\alpha}{2} dt} \right) \right| \\
 &+ \frac{(b-x)^{\alpha+1}}{b-a} \left(\int_0^1 \frac{t^\alpha}{2} dt \right) \left| f' \left(\frac{\int_0^1 \frac{t^\alpha}{2} \left(\frac{1-t}{2}x + \frac{1+t}{2}b \right) dt}{\int_0^1 \frac{t^\alpha}{2} dt} \right) \right|
 \end{aligned}$$

for all $x \in [a, b]$, which is equivalent to the inequality in the statement of Theorem 5. □

The case where $\alpha = 1$ in our Theorems 2–5 was previously noted by M. A. Latif [4].

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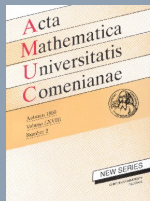


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