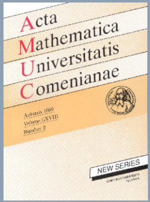

CONCERNING THE CESÀRO MATRIX AND ITS IMMEDIATE OFFSPRING

H. C. RHALY JR.

ABSTRACT. For the Cesàro matrix $C \in B(\ell^2)$ and the unilateral shift U, it is known that C and its immediate offspring U^*CU are both hyponormal and noncompact, and they have the same norm and the same spectrum. Here we investigate similarity and unitary equivalence for C and U^*CU, as well as further generations of offspring.

Necessary conditions are found for a lower triangular factorable matrix to be unitarily equivalent to its immediate offspring. A specialized result is obtained for factorable matrices having a constant main diagonal. Along the way, a more general result is also obtained: necessary conditions are found for two lower triangular factorable matrices to be unitarily equivalent.

1. Introduction

A lower triangular infinite matrix $M = [m_{ij}]$, acting through multiplication to give a bounded linear operator on ℓ^2, is *factorable* if its entries are

$$m_{ij} = \begin{cases} a_i c_j & \text{if } j \le i, \\ 0 & \text{if } j > i, \end{cases}$$

where a_i depends only on i and c_j depends only on j. If $c_j = 1$ for all j, then M is a *terraced* matrix (see [3], [5]). The Cesàro matrix C is the terraced matrix that occurs when $a_i = \frac{1}{i+1}$ for

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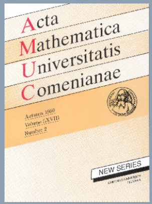


all i. In [1] it is shown that $C \in B(\ell^2)$, the set of all bounded linear operators on ℓ^2, and that C is noncompact and hyponormal. Recall that an operator T on a Hilbert space H is *hyponormal* if it satisfies $\langle (T^*T - TT^*)f, f\rangle \geq 0$ for all $f \in H$.

Recalling the notation and terminology of [2] with an appropriate adjustment, the *immediate offspring* of M, denoted M', is the factorable matrix that occurs when the first row *and* the first column are deleted from M. Note that if U denotes the unilateral shift on ℓ^2, then $M' = U^*MU$.

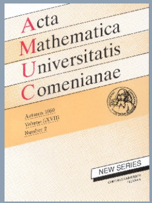
$$M = \begin{bmatrix} c_0a_0 & 0 & 0 & \dots \\ c_0a_1 & c_1a_1 & 0 & \dots \\ c_0a_2 & c_1a_2 & c_2a_2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad M' = \begin{bmatrix} c_1a_1 & 0 & 0 & \dots \\ c_1a_2 & c_2a_2 & 0 & \dots \\ c_1a_3 & c_2a_3 & c_3a_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

We recall that operators T_1, $T_2 \in B(H)$ are *similar* if there exists an invertible operator $Q \in B(H)$ such that $T_2 = Q^{-1}T_1Q$. The operators T_1, T_2 are *unitarily equivalent* if $Q^{-1} = Q^*$.

Proposition 1.1. *C and C' are similar operators.*

Proof. Suppose $Q :\equiv C^* - U$; then Q is invertible and $Q^{-1} = C' - W^*$, where W is the unilateral weighted shift with weights $\{\frac{n+1}{n+2} : n \geq 0\}$. A straightforward calculation reveals that $CQ = C^* = QC'$ and hence $C' = Q^{-1}CQ$. Therefore C and C' are similar. $\square$

In [2] it is demonstrated that the infinite Hilbert matrix A and its immediate offspring A' (obtained by deleting the first row *or* the first column of A) are unitarily equivalent. Note that



$A' = U^*A = AU.$

$$A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & & \\ \frac{1}{3} & \frac{1}{4} & & & \\ \frac{1}{4} & & & & \\ \vdots & & & & \ddots \end{bmatrix} \qquad A' = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & & \\ \frac{1}{4} & \frac{1}{5} & & & \\ \frac{1}{5} & & & & \\ \vdots & & & & \ddots \end{bmatrix}$$

In view of Proposition 1.1, together with the fact that C' is also known to be hyponormal (see [6], [9]), it now seems natural to ask whether C and C' are unitarily equivalent. The next section will provide the answer.

2. Necessary Conditions for Unitary Equivalence of Factorable Matrices

Throughout this section, we continue to assume that $M = [a_i c_j] \in B(\ell^2)$ is a lower triangular factorable matrix and $\{a_i\}$,$\{c_j\}$ are strictly positive sequences.

2.1. For a factorable matrix and its immediate offspring

The following two lemmas are useful in obtaining a necessary condition for M and $M^{'}$ to be unitarily equivalent. They also provide some information about what is required for similarity.

Lemma 2.1. *If $\{a_i\}$ and $\{c_j\}$ are strictly positive sequences and $T :\equiv [t_{ij}] \in B(\ell^2)$ satisfies $TM = M^{'}T$, then for each i, $t_{i,i+1} = \sum_{k=0}^{i} \frac{c_{k+1}}{c_k}(1 - \frac{c_k a_k}{c_{k+1}a_{k+1}})t_{kk}$ and $t_{ij} = 0$ when $j \geq i+2$.*

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Proof. Assume $X = [x_{ij}] :\equiv TM$ and $Y = [y_{ij}] :\equiv M'T$. Observe that

$$x_{ij} = c_j\Big(\sum_{k=0}^{\infty} t_{i,j+k}a_{j+k}\Big) \quad \text{and} \quad y_{ij} = a_{i+1}\Big(\sum_{k=0}^{i} c_{k+1}t_{k,j}\Big) \quad \text{for all } i, j.$$

Then

$$\frac{c_1}{c_0}x_{00} - x_{01} = \frac{c_1}{c_0}y_{00} - y_{01}$$

yields

$$t_{01} = \frac{c_1}{c_0}\Big(1 - \frac{c_0 a_0}{c_1 a_1}\Big)t_{00}.$$

Similarly,

$$\frac{c_2}{c_1}x_{01} - x_{02} = \frac{c_2}{c_1}y_{01} - y_{02}$$

yields $t_{02} = 0$. By induction on the subscript j,

$$\frac{c_{n+1}}{c_n}x_{0,n} - x_{0,n+1} = \frac{c_{n+1}}{c_n}y_{0,n} - y_{0,n+1}$$

yields $t_{0,n+1} = 0$ for all $n \geq 1$. Next,

$$\frac{c_2}{c_1}x_{11} - x_{12} = \frac{c_2}{c_1}y_{11} - y_{12}$$

yields

$$t_{12} = \sum_{k=0}^{1} \frac{c_{k+1}}{c_k}\Big(1 - \frac{c_k a_k}{c_{k+1}a_{k+1}}\Big)t_{kk}$$

and

$$\frac{c_3}{c_2}x_{12} - x_{13} = \frac{c_3}{c_2}y_{12} - y_{13}$$

yields $t_{13} = 0$. By induction,

$$\frac{c_{n+1}}{c_n} x_{1,n} - x_{1,n+1} = \frac{c_{n+1}}{c_n} y_{1,n} - y_{1,n+1}$$

yields $t_{1,n+1} = 0$ for all $n \geq 2$. Now assume that

$$t_{i,i+1} = \sum_{k=0}^{i} \frac{c_{k+1}}{c_k} \Big(1 - \frac{c_k a_k}{c_{k+1} a_{k+1}}\Big) t_{kk}$$

and $t_{ij} = 0$ when $j \geq i+2$ for $i = 0, 1, \ldots, m$. Then

$$\frac{c_{m+2}}{c_{m+1}} x_{m+1,m+1} - x_{m+1,m+2} = \frac{c_{m+2}}{c_{m+1}} y_{m+1,m+1} - y_{m+1,m+2}$$

yields

$$t_{m+1,m+2} = \sum_{k=0}^{m+1} \frac{c_{k+1}}{c_k} \Big(1 - \frac{c_k a_k}{c_{k+1} a_{k+1}}\Big) t_{kk}$$

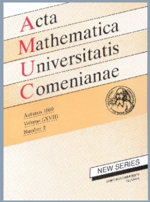
and

$$\frac{c_{m+3}}{c_{m+2}} x_{m+1,m+2} - x_{m+1,m+3} = \frac{c_{m+3}}{c_{m+2}} y_{m+1,m+2} - y_{m+1,m+3}$$

yields $t_{m+1,m+3} = 0$. By strong induction,

$$\frac{c_{m+1+n}}{c_{m+n}} x_{m+1,m+n} - x_{m+1,m+1+n} = \frac{c_{m+1+n}}{c_{m+n}} y_{m+1,m+n} - y_{m+1,m+1+n}$$

yields $t_{m+1,m+1+n} = 0$ for all $n \geq 2$. This completes the proof. $\square$



Lemma 2.2. *If* $\{a_i\}$ *and* $\{c_j\}$ *are strictly positive sequences and* $T :\equiv [t_{ij}] \in B(\ell^2)$ *satisfies* $MT = TM'$, *then* $t_{ii} = \frac{c_{i+1}a_i}{c_1 a_0} t_{00}$ *for all* i *and for all* $j > i$,

$$t_{ij} = \frac{c_{j+1}a_i}{c_1 a_0}\Big[\prod_{m=i+1}^{j}\Big(1 - \frac{c_m a_m}{c_0 a_0}\Big)\Big] t_{00}.$$

Proof. Assume $X = [x_{ij}] :\equiv MT$ and $Y = [y_{ij}] :\equiv TM'$. Observe that

$$x_{ij} = a_i\Big(\sum_{k=0}^{i} c_k t_{k,j}\Big) \quad \text{and} \quad y_{ij} = c_{j+1}\Big(\sum_{k=0}^{\infty} t_{i,j+k}a_{j+k+1}\Big) \quad \text{for all } \ i, j.$$

Then

$$\frac{c_2}{c_1}x_{00} - x_{01} = \frac{c_2}{c_1}y_{00} - y_{01}$$

yields

$$t_{01} = \frac{c_2}{c_1}(1 - \frac{c_1 a_1}{c_0 a_0})t_{00}.$$

Similarly,

$$\frac{c_3}{c_2}x_{01} - x_{02} = \frac{c_3}{c_2}y_{01} - y_{02}$$

yields

$$t_{02} = \frac{c_3}{c_2}\Big(1 - \frac{c_2 a_2}{c_0 a_0}\Big)t_{01} = \frac{c_3}{c_1}\Big(1 - \frac{c_1 a_1}{c_0 a_0}\Big)\Big(1 - \frac{c_2 a_2}{c_0 a_0}\Big)t_{00}.$$

By induction on the subscript j,

$$\frac{c_{n+2}}{c_{n+1}}x_{0,n} - x_{0,n+1} = \frac{c_{n+2}}{c_{n+1}}y_{0,n} - y_{0,n+1}$$

yields

$$t_{0,n+1} = \frac{c_{n+2}}{c_1}\Big[\prod_{m=1}^{n+1}\Big(1 - \frac{c_m a_m}{c_0 a_0}\Big)\Big] t_{00}$$

for all $n \geq 0$. Next,

$$\frac{c_2}{c_1} x_{10} - x_{11} = \frac{c_2}{c_1} y_{10} - y_{11}$$

yields

$$t_{11} = \frac{c_2 a_1}{c_1 a_0} t_{00}.$$

Then

$$\frac{c_3}{c_2} x_{11} - x_{12} = \frac{c_3}{c_2} y_{11} - y_{12}$$

yields

$$t_{12} = \frac{c_3 a_1}{c_1 a_0}\Big(1 - \frac{c_2 a_2}{c_0 a_0}\Big) t_{00}.$$

Again by induction,

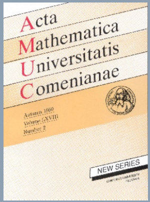
$$\frac{c_{n+2}}{c_{n+1}} x_{1,n} - x_{1,n+1} = \frac{c_{n+2}}{c_{n+1}} y_{1,n} - y_{1,n+1}$$

yields

$$t_{1,n+1} = \frac{c_{n+2} a_1}{c_1 a_0}\Big[\prod_{m=2}^{n+1}\Big(1 - \frac{c_m a_m}{c_0 a_0}\Big)\Big] t_{00}$$

for all $n \geq 1$. Now assume that $t_{ii} = \frac{c_{i+1} a_i}{c_1 a_0} t_{00}$ for $i = 0, 1, 2, \ldots, n$, and

$$t_{ij} = \frac{c_{j+1} a_i}{c_1 a_0}\Big[\prod_{m=i+1}^{j}\Big(1 - \frac{c_m a_m}{c_0 a_0}\Big)\Big] t_{00}$$



for all $j \geq i+1$. Then

$$\frac{c_{n+2}}{c_{n+1}} x_{n+1,n} - x_{n+1,n+1} = \frac{c_{n+2}}{c_{n+1}} y_{n+1,n} - y_{n+1,n+1}$$

gives

$$\begin{aligned} a_{n+1}\Big[\sum_{l=0}^{n-1} \Big(\frac{c_{n+2}}{c_{n+1}} c_l t_{l,n} - c_l t_{l,n+1} \Big) \\ + \frac{c_{n+2}}{c_{n+1}} c_n t_{n,n} - c_n t_{n,n+1} + c_{n+2} t_{n+1,n} - c_{n+1} t_{n+1,n+1} \Big] \\ = c_{n+2} a_{n+1} t_{n+1,n}; \end{aligned}$$

then

$$\begin{aligned} \Big[\frac{c_{n+2} c_{n+1} a_{n+1}}{c_0 c_1 a_0^2} \sum_{l=0}^{n-1} \Big(c_l a_l \prod_{m=l+1}^{n} \Big(1 - \frac{c_m a_m}{c_0 a_0} \Big) \Big) \\ + \frac{c_{n+2} c_n a_n}{c_1 a_0} - \frac{c_n c_{n+2} a_n}{c_1 a_0} \Big(1 - \frac{c_{n+1} a_{n+1}}{c_0 a_0} \Big) \Big] t_{00} \\ = c_{n+1} t_{n+1,n+1}, \end{aligned}$$

so

$$\begin{aligned} \Big[\frac{c_{n+2} c_{n+1} a_{n+1}}{c_1 a_0} \Big(1 - \frac{c_n a_n}{c_0 a_0} \Big) + \Big(\frac{c_n c_{n+2} a_n}{c_1 a_0} \Big) \Big(\frac{c_{n+1} a_{n+1}}{c_0 a_0} \Big) \Big] t_{00} \\ = c_{n+1} t_{n+1,n+1}, \end{aligned}$$

or

$$t_{n+1,n+1} = \frac{c_{n+2} a_{n+1}}{c_1 a_0} t_{00}.$$

Next,

$$\frac{c_{n+3}}{c_{n+2}} x_{n+1,n+1} - x_{n+1,n+2} = \frac{c_{n+3}}{c_{n+2}} y_{n+1,n+1} - y_{n+1,n+2}$$

leads to

$$
\begin{aligned}
a_{n+1}\Big[\sum_{l=0}^{n}\Big(\frac{c_{n+3}}{c_{n+2}}c_l t_{l,n+1} - c_l t_{l,n+2}\Big)
&+ \frac{c_{n+3}}{c_{n+2}}c_{n+1}t_{n+1,n+1} - c_{n+1}t_{n+1,n+2}\Big] \\
&= c_{n+3}a_{n+2}t_{n+1,n+1}.
\end{aligned}
$$

Then

$$
\begin{aligned}
c_{n+1}t_{n+1,n+2} &= \frac{c_{n+3}c_{n+2}a_{n+2}}{c_0c_1a_0^2}\Big[\sum_{l=0}^{n}c_la_l\prod_{m=l+1}^{n+1}\Big(1-\frac{c_ma_m}{c_0a_0}\Big)t_{00}\Big] \\
&\quad + \frac{c_{n+3}}{c_1a_0}(c_{n+1}a_{n+1} - c_{n+2}a_{n+2})t_{00} \\
&= \Big[\frac{c_{n+3}c_{n+2}a_{n+2}}{c_1a_0}\Big(1-\frac{c_{n+1}a_{n+1}}{c_0a_0}\Big) \\
&\quad + \frac{c_{n+3}}{c_1a_0}(c_{n+1}a_{n+1} - c_{n+2}a_{n+2})\Big]t_{00},
\end{aligned}
$$

so

$$
t_{n+1,n+2} = \frac{c_{n+3}a_{n+1}}{c_1a_0}\Big(1-\frac{c_{n+2}a_{n+2}}{c_0a_0}\Big)t_{00}.
$$

Now assume that

$$
t_{n+1,n+k} = \frac{c_{n+k+1}a_{n+1}}{c_1a_0}\prod_{m=n+2}^{n+k}\Big(1-\frac{c_ma_m}{c_0a_0}\Big)t_{00}
$$

for some $k \geq 2$. Then

$$
\frac{c_{n+k+2}}{c_{n+k+1}}x_{n+1,n+k} - x_{n+1,n+k+1} = \frac{c_{n+k+2}}{c_{n+k+1}}y_{n+1,n+k} - y_{n+1,n+k+1}
$$

yields

$$
\begin{aligned}
a_{n+1}\Big[\sum_{l=0}^{n}\Big(\frac{c_{n+k+2}}{c_{n+k+1}}c_l t_{l,n+k} - c_l t_{l,n+k+1}\Big)& \\
+\frac{c_{n+k+2}}{c_{n+k+1}}c_{n+1}t_{n+1,n+k} - c_{n+1}t_{n+1,n+k+1}\Big]& \\
= c_{n+k+2}a_{n+k+1}t_{n+1,n+k}.&
\end{aligned}
$$

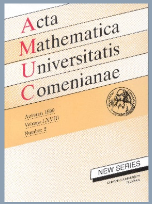
Therefore,

$$
\begin{aligned}
&c_{n+1}t_{n+1,n+k+1} \\
&\quad = \frac{c_{n+k+2}c_{n+k+1}a_{n+k+1}}{c_1 a_0}\prod_{m=n+1}^{n+k}\Big(1-\frac{c_m a_m}{c_0 a_0}\Big)t_{00} \\
&\qquad + \frac{c_{n+k+2}}{c_1 a_0}\Big[\prod_{m=n+2}^{n+k}(1-\frac{c_m a_m}{c_0 a_0})\Big](c_{n+1}a_{n+1} - c_{n+k+1}a_{n+k+1})t_{00} \\
&\quad = \frac{c_{n+k+2}}{c_1 a_0}\Big[\prod_{m=n+2}^{n+k}(1-\frac{c_m a_m}{c_0 a_0})\Big] \\
&\qquad \times \Big[c_{n+k+1}a_{n+k+1}\Big(1-\frac{c_{n+1}a_{n+1}}{c_0 a_0}\Big) + c_{n+1}a_{n+1} - c_{n+k+1}a_{n+k+1}\Big]t_{00},
\end{aligned}
$$

so

$$
t_{n+1,n+k+1} = \frac{c_{n+k+2}a_{n+1}}{c_1 a_0}\Big[\prod_{m=n+2}^{n+k+1}\Big(1-\frac{c_m a_m}{c_0 a_0}\Big)\Big]t_{00}.
$$

This completes the proof. $\square$



Suppose that we wish to determine an invertible operator $Q \in B(\ell^2)$ such that $M' = Q^{-1}MQ$; that is, we wish to show that M and M' are similar operators. Since it is required that $M'Q^{-1} = Q^{-1}M$, Lemma 2.1 specifies some of the entries of Q^{-1}, and since it is required that $MQ = QM'$, Lemma 2.2 specifies some of the entries of Q. That still leaves infinitely many entries of the two matrices Q and Q^{-1} undetermined. Consequently, we see that the somewhat serendipitous success of Proposition 1.1 may not be that easy to duplicate in other examples. However, if Q is unitary, then all of its entries are determined by Lemmas 2.1 and 2.2 once q_{00} is specified. This observation leads to the following result.

Proposition 2.3. *Suppose $M = [a_i c_j] \in B(\ell^2)$ is a lower triangular factorable matrix. If $V \in B(\ell^2)$ is a unitary operator such that $M' = V^*MV$, then V must have the form $V =$*

$$v_{00}\begin{bmatrix} 1 & \frac{c_2}{c_1}\left(1-\frac{c_1a_1}{c_0a_0}\right) & \frac{c_3}{c_1}\prod_{m=1}^{2}\left(1-\frac{c_ma_m}{c_0a_0}\right) & \frac{c_4}{c_1}\prod_{m=1}^{3}\left(1-\frac{c_ma_m}{c_0a_0}\right) & \cdots \\ \frac{c_1a_1-c_0a_0}{c_0a_1} & \frac{c_2a_1}{c_1a_0} & \frac{c_3a_1}{c_1a_0}\left(1-\frac{c_2a_2}{c_0a_0}\right) & \frac{c_4a_1}{c_1a_0}\prod_{m=2}^{3}\left(1-\frac{c_ma_m}{c_0a_0}\right) & \cdots \\ 0 & s_{21} & \frac{c_3a_2}{c_1a_0} & \frac{c_4a_2}{c_1a_0}\left(1-\frac{c_3a_3}{c_0a_0}\right) & \cdots \\ 0 & 0 & s_{32} & \frac{c_4a_3}{c_1a_0} & \cdots \\ 0 & 0 & 0 & s_{43} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where the entries on the first subdiagonal satisfy $s_{i+1,i} = \sum_{k=0}^{i} \frac{c_{k+1}}{c_k}\left(1-\frac{c_ka_k}{c_{k+1}a_{k+1}}\right)\frac{c_{k+1}a_k}{c_1a_0}$ for each i.

Proof. This result is an immediate consequence of two facts:
(1) $T = V^*$ satisfies Lemma 2.1 and
(2) $T = V$ satisfies Lemma 2.2. □

Theorem 2.4. *Suppose $M = [a_i c_j] \in B(\ell^2)$ is a lower triangular factorable matrix with a constant main diagonal. Then M and M' are unitarily equivalent if and only if $M' = M$.*

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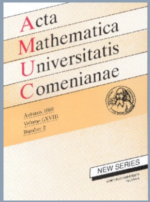

Proof. Suppose $V \in B(\ell^2)$ is a unitary operator such that $M' = V^*MV$. Then V must have the form specified in Proposition 2.3. Since $\{c_n a_n\}$ is a constant sequence, the non-diagonal entries of V are all 0. For M and M' to be unitarily equivalent in this case, it is necessary that $V = v_{00}I$ where $|v_{00}| = 1$. $\square$

Remark 2.5. We note that in the above proof it can be verified that $V = \frac{c_0}{c_1} v_{00} \operatorname{diag}\{\frac{c_{n+1}}{c_n} : n \geq 0\}$; so, for M and M' to be unitarily equivalent, it is necessary that $c_{n+1} = \frac{c_1}{c_0} c_n$ for all n.

All of the matrices that satisfy the condition in Remark 2.5 in reference to Theorem 2.4 are scalar multiples of the following example.

Example 2.6 (Toeplitz matrix)**.** Suppose $M = [a_i c_j] \in B(\ell^2)$ is the lower triangular factorable matrix given by $c_j = \lambda^{-j}$ and $a_i = \lambda^i$ for $0 \leq j \leq i$ and $0 < \lambda < 1$. Since $M' = M$, we know that M and M' must be unitarily equivalent. Clearly the condition in Remark 2.5 is satisfied.

For similarity, it turns out that with appropriate modifications, the serendipitous success of Proposition 1.1 can be repeated in the following situation.

Theorem 2.7. *Suppose $M = [a_i c_j] \in B(\ell^2)$ is a lower triangular factorable matrix such that $\{\frac{c_{n+1}}{c_n}\}$ is a bounded sequence and $c_n a_n = \alpha$ (constant) for all n. Then M and M' are similar operators.*

Proof. Take $Q :\equiv \operatorname{diag}\{\frac{c_{n+1}}{c_n} : n \geq 0\} - U$. Suppose that the entries of $T = [t_{ij}]$ are given by

$$t_{ij} = \begin{cases} \frac{c_j}{c_{i+1}} & \text{if } \ i \geq j; \\ 0 & \text{if } \ i < j. \end{cases}$$

Note that $T \in B(\ell^2)$ since $T = \frac{1}{\alpha} U^*(M - \alpha I)$. It can be verified that

$$MQ = \operatorname{diag}\{c_{n+1} a_n : n \geq 0\} = QM'$$

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and $QT = I = TQ$; so $Q^{-1} = T$, and M and M' are similar operators. $\square$

Example 2.8. Suppose $M = [a_i c_j] \in B(\ell^2)$ is the lower triangular factorable matrix given by $c_j = \sum_{k=0}^{j} 2^k$ and $a_i = \frac{1}{\sum_{k=0}^{i} 2^k}$ for all i, j. Then $c_2 = 7$ but $\frac{c_1}{c_0} c_1 = 9$, so $c_2 \neq \frac{c_1}{c_0} c_1$. So by Remark 2.5, M' cannot be unitarily equivalent to M. However, M and M' are similar operators by Thereom 2.7. Moreover, we note that it was proved in [8] that M is hyponormal, so M' is also hyponormal (by [6]).

Let $\{e_n : n = 0, 1, 2, \dots\}$ denote the standard orthonormal basis for ℓ^2.

Theorem 2.9. *Suppose $M = [a_i c_j] \in B(\ell^2)$ is a lower triangular factorable matrix. In order for M and M' to be unitarily equivalent, it is necessary that*

$$\sum_{n=1}^{\infty} \left| \frac{c_{n+1}}{c_1} \prod_{k=1}^{n} \left(1 - \frac{c_k a_k}{c_0 a_0}\right) \right|^2 = \left| \frac{c_1 a_1 - c_0 a_0}{c_0 a_1} \right|^2 .$$

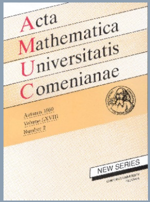
Proof. If $V \in B(\ell^2)$ is a unitary operator such that $M' = V^* M V$, then V must have the form specified in Proposition 2.3. Since $\|V e_0\|^2 = 1 = \|V^* e_0\|^2$, the result is immediate. $\square$

Remark 2.10. To see that the necessary condition of Theorem 2.9 is not sufficient for unitary equivalence, note that the condition is satisfied by all lower triangular factorable matrices M having a constant main diagonal. However, Example 2.8 presents such a matrix M for which it was shown that M and its immediate offspring M' are not unitarily equivalent.

In the following proposition, $C'' :\equiv U^* C' U = (U^*)^2 C U^2$. Recall that it was shown in the introduction that C and C' are similar operators.

Proposition 2.11.

(a) *C and C' are not unitarily equivalent.*



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(b) C' *and* C'' *are similar operators, but they are not unitarily equivalent.*

Proof. (a) The necessary condition in Theorem 2.9 is not satisfied since $\frac{\pi^2}{6}-1\neq 1$, so C and C' are not unitarily equivalent.

(b) Suppose the entries of $Q = [q_{ij}]$ are given by

$$q_{ij} = \begin{cases} \frac{2(i+1)}{(j+1)(j+2)} & \text{if } \ i \leq j; \\ -1 & \text{if } \ i = j+1; \\ 0 & \text{if } \ i > j+1. \end{cases}$$

A direct calculation shows that Q is invertible and $Q^{-1} = 2C'' - W^*$, where W is the unilateral weighted shift with weights $\{\frac{n+1}{n+3} : n \geq 0\}$. If $Y = [y_{ij}]$ is defined by

$$y_{ij} = \begin{cases} \frac{i+1}{(j+1)(j+2)} & \text{if } \ i \leq j; \\ 0 & \text{if } \ i > j, \end{cases}$$

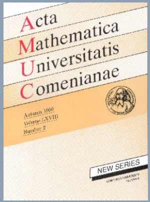
then it can be verified that $C'Q = Y = QC''$, so C' and C'' are similar.

Since $\frac{1}{9} + \frac{1}{36} + 4(\sum_{k=4}^{\infty} \frac{1}{k^2(k+1)^2}) < \frac{1}{4}$, the necessary condition in Theorem 2.9 is not satisfied, so C' and C'' are not unitarily equivalent. □

2.2. A more general result

Since some of the most useful information in the previous subsection emerged from considering the first row and first column of the unitary operator V (see Theorem 2.9), we employ the same approach here in a more general setting.

Theorem 2.12. *Suppose* $M_1 :\equiv [a_i c_j] \in B(\ell^2)$ *and* $M_2 :\equiv [b_i d_j] \in B(\ell^2)$ *are lower triangular factorable matrices associated with strictly positive sequences* $\{a_i\}$, $\{c_j\}$, $\{b_i\}$, $\{d_j\}$. *In order for*



M_1 and M_2 to be unitarily equivalent, it is necessary that

$$\sum_{n=0}^{\infty}\Big|\frac{c_{n+1}}{c_0}\prod_{k=0}^{n}(1-\frac{c_k a_k}{d_0 b_0})\Big|^2=\sum_{n=0}^{\infty}\Big|\frac{d_{n+1}}{d_0}\prod_{k=0}^{n}(1-\frac{d_k b_k}{c_0 a_0})\Big|^2.$$

Proof. Suppose that $V\in B(\ell^2)$ is a unitary operator satisfying $M_2=V^*M_1V$. We note that $VM_2=M_1V$. Assume $X=[x_{ij}]:\equiv VM_2$ and $Y=[y_{ij}]:\equiv M_1V$. Observe that $x_{0j}=d_j\sum_{n=j}^{\infty}b_n v_{0n}$ and $y_{0j}=c_0a_0v_{0j}$ for all j. Then

$$x_{00}-\frac{d_0}{d_1}x_{01}=y_{00}-\frac{d_0}{d_1}y_{01}$$

yields

$$v_{01}=\frac{d_1}{d_0}(1-\frac{d_0b_0}{c_0a_0})v_{00}.$$

Similarly,

$$x_{01}-\frac{d_1}{d_2}x_{02}=y_{01}-\frac{d_1}{d_2}y_{02}$$

yields

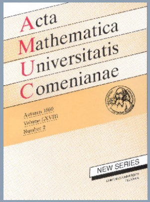
$$v_{02}=\frac{d_2}{d_0}\Big(1-\frac{d_0b_0}{c_0a_0}\Big)\Big(1-\frac{d_1b_1}{c_0a_0}\Big)v_{00}$$

By induction on the second subscript,

$$x_{0,n}-\frac{d_n}{d_{n+1}}x_{0,n+1}=y_{0,n}-\frac{d_n}{d_{n+1}}y_{0,n+1}$$

yields

$$v_{0,n+1}=\frac{d_{n+1}}{d_0}\prod_{k=0}^{n}\Big(1-\frac{d_kb_k}{c_0a_0}\Big)v_{00}$$



for all $n > 0$. By using $V^*M_1 = M_2V^*$ and similar reasoning, one obtains

$$\overline{v_{n+1,0}} = \frac{c_{n+1}}{c_0}\prod_{k=0}^{n}\Big(1 - \frac{c_k a_k}{d_0 b_0}\Big)\overline{v_{00}}$$

for all $n > 0$. Since $\|Ve_0\|^2 = 1 = \|V^*e_0\|^2$, the result is now immediate. $\square$

Note that Theorem 2.9 is the special case of Theorem 2.12 that occurs when $b_i = a_{i+1}$ and $d_j = c_{j+1}$ for all i, j.

Corollary 2.13. *In order for terraced matrices $M_1 :\equiv [a_i \cdot 1]$ and $M_2 :\equiv [b_i \cdot 1]$ to be unitarily equivalent, it is necessary that*

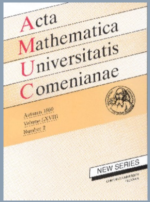
$$\sum_{n=0}^{\infty}\Big|\prod_{k=0}^{n}(1 - \frac{a_k}{b_0})\Big|^2 = \sum_{n=0}^{\infty}\Big|\prod_{k=0}^{n}(1 - \frac{b_k}{a_0})\Big|^2.$$

Remark 2.14. It is worth noting that the condition in Theorem 2.12 is satisfied whenever $c_0a_0 = d_0b_0$, but that is not sufficient to guarantee unitary equivalence. To see this, consider the terraced matrices determined by $a_i = \frac{1}{(i+1)^2}$ and $b_i = \frac{1}{i+1}$ for all i. These matrices cannot be unitarily equivalent since the first matrix is not hyponormal (see [4]), but the second matrix is the Cesàro matrix, which is known to be hyponormal.

We already know that C and C' are not unitarily equivalent. Corollary 2.13 will allow us to settle the question of unitary equivalence for C and its non-immediate offspring C''.

Proposition 2.15. *C and C'' are similar operators, but they are not unitarily equivalent.*

Proof. Similarity can be justified by pairing Propositions 1.1 and 2.11(b) and using transitivity. Next, suppose that $a_i = \frac{1}{i+3}$ and $b_i = \frac{1}{i+1}$ for all i. Note that $C'' = [a_i \cdot 1]$, $C = [b_i \cdot 1]$ and



$b_2 = a_0$. Since $\frac{2}{3}\pi^2 - 5 \neq 5$, the necessary condition for unitary equivalence in Corollary 2.13 is not satisfied. $\square$

In investigating further generations of offspring of C, we find it convenient to depart from the traditional usage of the prime symbol and introduce alternative notation. For a fixed positive integer m, consider

$$C_m :\equiv (U^*)^{m-1} C U^{m-1}.$$

Note that $C_1 = C$, $C_2 = C'$ and $C_3 = C''$. It is known that all of these operators have the same norm and the same spectrum and are hyponormal.

Proposition 2.16. *If $m > 1$ is a positive integer, then C_m and C_{m+1} are similar operators.*

Proof. If $Y = [y_{ij}] \in B(\ell^2)$ is defined by

$$y_{ij} = \begin{cases} \frac{\prod_{k=1}^{m-1}(i+k)}{\prod_{k=1}^{m}(j+k)} & \text{if } \; i \leq j; \\ 0 & \text{if } \; i > j, \end{cases}$$

and $Q :\equiv mY - U$, then Q is invertible and $Q^{-1} = mC_{m+1} - W^*$ where W is the unilateral weighed shift with weights $\{\frac{n+1}{n+m+1} : n \geq 0\}$. It can be verified that

$$C_m Q = Y = Q C_{m+1},$$

so C_m and C_{m+1} are similar operators. $\square$

Proposition 2.17. *If $m > 1$ is a fixed positive integer, then C and C_m are similar operators, but they are not unitarily equivalent.*

Proof. Similarity is a consequence of Propositions 1.1 and 2.16 (and induction), so our attention turns to the question of unitary equivalence. In preparation for an application of Corollary 2.13, consider $C_m = M_1 = [a_i \cdot 1]$, where $a_i = \frac{1}{i+m}$ and $C = M_2 = [b_i \cdot 1]$, where $b_i = \frac{1}{i+1}$ for each nonnegative integer i. Note that $b_{m-1} = a_0$. The necessary condition in the corollary requires that

$$(m-1)^2\Big(\frac{\pi^2}{6} - \sum_{n=1}^{m-1}\frac{1}{n^2}\Big) = \sum_{n=0}^{m-2}\prod_{k=0}^{n}\Big(\frac{m-k-1}{k+1}\Big)^2,$$

but this is clearly impossible since the right side is a rational number while the left side is irrational. □

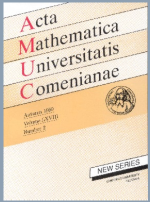
We close with a proposition that presents a non-terraced factorable matrix M with *all entries nonnegative* that *is* unitarily equivalent to C. A double dose of serendipity seems to be required here since (1) there is no general procedure available for identifying a good candidate M and (2) there is no analogue of Proposition 2.3 available to help supply the associated unitary operator V.

Regarding the choice for M here, it should be noted that (1) the nonzero entries of M are strictly smaller than the corresponding entries of C, (2) the main diagonal of M is exactly the same as the main diagonal of U^*CU, and (3) M is known to be hyponormal (see [7]).

Proposition 2.18. *If $a_i = \frac{1}{\sqrt{(i+1)(i+2)}}$ and $c_j = \sqrt{\frac{j+1}{j+2}}$ for all i, j, then the lower triangular factorable matrix $M = [a_i c_j] \in B(\ell^2)$ is unitarily equivalent to C.*

Proof. Suppose $V :\equiv Z^* - W$ where Z is the terraced matrix $Z :\equiv [a_i \cdot 1]$ and W is the unilateral weighted shift with weights $\{\sqrt{\frac{n+1}{n+2}} : n \geq 0\}$. Straightforward computations demonstrate that V is unitary and $M = V^*CV$. □

One may easily verify that the operators M and C from Proposition 2.18 satisfy Theorem 2.12.



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