Acta Math. Univ. Comenianae
Vol. LX, 2(1991), pp. 257–267

AMALGAMATIONS AND LINK GRAPHS OF CAYLEY GRAPHS

J. TOMANOVÁ

Abstract. The link of a vertex v in a graph G is the subgraph induced by all vertices adjacent to v. If all the links in G are isomorphic to the same graph L, then L is called the link graph of G. We consider the operation of an amalgamation of graphs. Using the construction of the free product of groups with amalgamated subgroups, we give a sufficient condition for a class of link graphs of Cayley graphs to be closed under amalgamations.

1. Introduction

The **link of a vertex** v of a graph G is the subgraph induced by all vertices adjacent to v; we denote it by link (v, G). If all the links in G are isomorphic to the same graph L, then we say that G **has a constant link** L and L is called the **link graph of** G**.** In 1963 Zykov [**6**] posed the problem of characterizing link graphs. It turned out that the problem is algorithmically unsolvable in the class of all (possibly infinite) graphs, see Bulitko [**2**]. However, the solution of Zykov's problem is known for certain classes of graphs (for survey see Hell [**4**] and Blass, Harary and Miller [**1**]). Consequently, it is natural to ask whether the class of link graphs is closed or not under standard binary operations, and how to modify the graph to be a link graph. These problems are treated in Hell [**4**].

In the present paper we shall discuss a similar question, namely, whether an amalgamation of link graphs results in a link graph or not. Using the construction of the free product of groups with amalgamated subgroups, we give a sufficient condition for a class of link graphs of Cayley graphs to be closed under amalgamations. Further, the class of m-treelike graphs is defined and some necessary and sufficient conditions for an m-treelike graph to be a link graph are derived.

Received July 9, 1991; revised September 20, 1991.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 05C25; Secondary 05C75.

2. Preliminaries

2.1 Groups

We follow the standard terminology and notation of Lyndon and Shupp [**5**] and Blass, Harary and Miller [**1**].

All groups considered are finitely generated. Let H be a group. We use 1 to denote the identity element of H. $U \leq H$ means that U is a subgroup of H. $\langle X; R\rangle$ denotes the presentation with generators $x \in X$ and relators $r \in R$. Let

$$H_1 = \langle x_1, \dots, x_n; r_1, \dots, r_n\rangle \text{ and } H_2 = \langle y_1, \dots, y_m; s_1, \dots, s_m\rangle$$

be disjoint groups. Let $U_1 \leq H_1$ and $U_2 \leq H_2$ be subgroups, such that there exists an isomorphism $f\colon U_1 \to U_2$. Then the **free product of H_1 and H_2, amalgamating U_1 and U_2 by the isomorphism** f is the group

$$\langle x_1, \dots, x_n, y_1, \dots, y_m; r_1, \dots, r_n, s_1, \dots, s_m,\ u = f(u),\ u \in U_1\rangle\,.$$

In order to simplify the notation this group will be denoted by

$$\langle H_1 * H_2; u = f(u), u \in U_1\rangle\,.$$

2.2 Graphs

Let H be a group, and let $Z \subseteq H$ be a generating subset closed under inverses and not containing the identity. The **Cayley graph** $[H, Z]$ of H with respect to Z has H as its vertex set, with u and v adjacent if $u^{-1}v \in Z$.

Let G_1 and G_2 be graphs. Let $L_1 \leq G_1$ and $L_2 \leq G_2$ be subgraphs, and let $f\colon L_1 \to L_2$ be an isomorphism. The **graph with amalgamated subgraphs L_1 and L_2 by the isomorphism** f arises from the disjoint union of G_1 and G_2 by identifying every vertex $v \in V(L_1)$ with $f(v) \in V(L_2)$, and every edge of L_1 with the corresponding edge of L_2. The graph just defined will be denoted by (G_1, L_1, f, L_2, G_2).

3. Amalgamations of Link Graphs of Cayley Graphs

Consider the following problem. Let L_1 and L_2 be link graphs. Let $L_1' \leq L_1$ and $L_2' \leq L_2$ be subgraphs, such that there is an isomorphism $f\colon L_1' \to L_2'$. Does there exist a graph G with constant link $(L_1, L_1', f, L_2', L_2)$?

Note that the class of link graphs is not closed under amalgamations. Figure 1 shows the amalgamation of link graphs L_1 and L_2 by the isomorphism f which results in a non-link graph P_3.

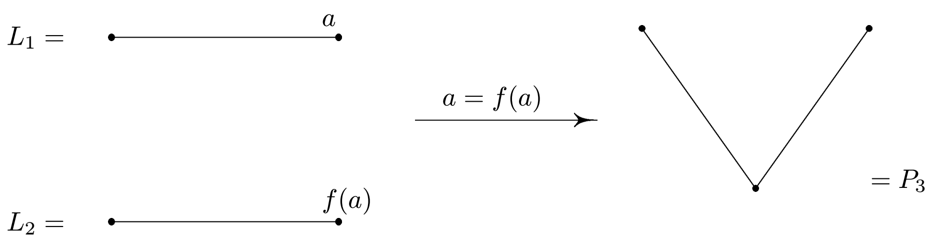

Figure 1.

Thus we are led to restrict the question to the special classes of graphs, namely, to the link graphs of Cayley graphs. (Recall that the Cayley graph $[H, Z]$ is vertex transitive. Hence $L = \text{link}\,(1, [H, Z])$ is the constant link of $[H, Z]$.)

Set $I = \{1, 2\}$. Let H_i be a group with the generating subset Z_i, such that the identity element 1_i of H_i does not belong to Z_i and $Z_i = Z_i^{-1}$, $i \in I$. Consider the Cayley graph $[H_i, Z_i]$ with the constant link L_i, $i \in I$. Let $L'_1 \leq L_1$ and $L'_2 \leq L_2$ be subgraphs, and let $f\colon L'_1 \to L'_2$ be an isomorphism. As $\text{link}\,(1_i, [H_i, Z_i]) \cong L_i$, the vertices in L_i can be considered as elements contained in Z_i, where $i \in I$. Thus the amalgamation of vertices in L'_1 and L'_2 by f induces an amalgamation of elements in Z_1 and Z_2.

Theorem 1. *Let $[H_i, Z_i]$, L_i and L'_1 with $i = 1, 2$ be the graphs defined above. Let $f\colon L'_1 \to L'_2$ be an isomorphism, and let $U_i \leq H_i$ be a subgroup such that $V(L'_i) \subseteq U_i$, for $i = 1, 2$. Suppose that the following properties hold.*
P1. *The isomorphism of graphs L'_1 and L'_2 can be extended to an isomorphism of subgroups U_1 and U_2.*
P2. $U_i \cap Z_i = V(L'_i)$ *for $i = 1, 2$.*
Then there is an infinite graph with constant link $(L_1, L'_1, f, L'_2, L_2)$.

Proof. Denote by $\overline{f}$ the isomorphism of subgroups U_1 and U_2 induced by f. Set

$$H = \left\langle H_1 * H_2;\ u = \overline{f}(u),\ u \in U_1 \right\rangle .$$

Since Z_i is closed under inverses and 1_i does not belong to Z_i for $i \in I$, the generating set of H (say Z) satisfies these conditions, too.

Let L denote the constant link of $[H, Z]$, i.e. $L \cong \text{link}\,(1, [H, Z])$. From the properties P1 and P2 there follows that the elements from Z_1 and Z_2 identified by $\overline{f}$, correspond to the vertices in L_1 and L_2 identified by f, and vice versa. Next, suppose that $u \in Z_1 - V(L'_1)$ and $v \in Z_2 - V(L'_2)$. If $u^{-1}v = c$ and $c \in Z_1$ then $uc \in H_1$. However, it is in contradiction with the fact that $v \in Z_2 - V(L'_2)$. Using similar arguments one can show that $u^{-1}v$ does not belong to Z_2. It implies that L is isomorphic to the graph $(L_1, L'_1, f, L'_2, L_1)$, as required. □

Obviously, the graph constructed by the procedure described above, is infinite. However, in the special cases "finitizing" relations for the group H can be found. For instance, consider a group H_1 generated by the set $Z_1 = \{BC, B, C, CB\}$, and with defining relations $B^2 = C^2 = 1_1$, $BC \neq CB$, $(BC)^3 = CB$. Let H_2 be an Abelian group generated by the set $Z_2 = \{D, A, DA\}$, and with $D^2 = A^2 = 1_2$. Let $V(L_1') = \{B\}$ and $V(L_2') = \{D\}$. Then $U_1 = \{1_1, B\}$ and $U_2 = \{1_2, D\}$. For the group $H = \left\langle H_1 * H_2; u = \overline{f}(u), u \in U_1 \right\rangle$ the finitizing relation can be specified as $AC = CA$. The Cayley graph of the group $\overline{H} = \langle H_1 * H_2;\ u = f(u),\ u \in U_1, AC = CA\rangle$ with generating set $\overline{Z} = \{BC, B, C, CB, A, BA\}$ is shown in Fig. 2.

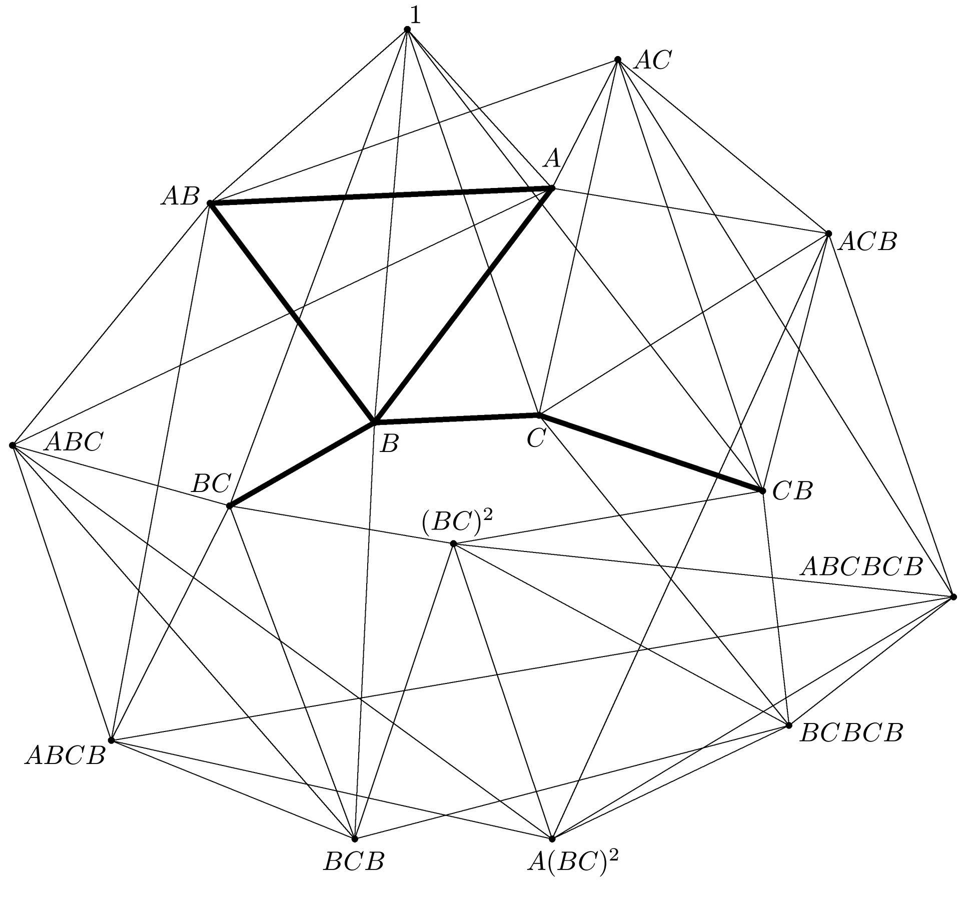

Figure 2.

Theorem 2. *Let $[H_i, Z_i]$ be a Cayley graph of an Abelian group H_i, such that* $\operatorname{link}(1_i, [H_i, Z_i]) \cong L_i$ *for $i = 1, 2$. Let $L_i' \leq L_i$, $i = 1, 2$ be a subgraph, and let $f\colon L_1' \to L_2'$ be an isomorphism. Consider the subgroups $U_1 \leq H_1$ and $U_2 \leq H_2$*

satisfying the assumptions of Theorem 1. *Then there exists a finite Cayley graph with constant link* $(L_1, L_1', f, L_2', L_2)$.

Proof. Let $\overline{f}$ be the isomorphism of U_1 and U_2 induced by f. Consider the group $H = <H_1 * H_2;\ u = \overline{f}(u),\ u \in U_1>$ and the relation

$$\text{(A)} \qquad uv = vu \quad \text{if} \quad u \in Z_1 \quad \text{and} \quad v \in Z_2.$$

We shall show that the Cayley graph of the group

$$\overline{H} = \langle H_1 * H_2;\ u = \overline{f}(u),\ u \in U_1, (A) \rangle$$

has the constant link $(L_1, L_1', f, L_2', L_2)$. In fact, we shall prove that
1. The generating set of $\overline{H}$ (say $\overline{Z}$) coincides with Z.
2. (A) preserves the edges and non-edges in $(L_1, L_1', f, L_2', L_2)$.

Take the elements $u \in Z_1 - U_1$ and $v \in Z_2 - U_2$. Set $uv = c$ and $vu = d$. If $c^{-1}d \in H_i$ for $i \in \{1, 2\}$ and $c^{-1}d \neq 1$ with respect to the defining relations in H_i, then (A) produces relations which are not valid in H_i, and consequently $Z \neq \overline{Z}$. Now, we shall show, it is not the case.

Since both H_1 and H_2 are Abelian groups, u and v can be written as $u = X_1Y_1$ and $v = X_2Y_2$ where $Y_i \in U_i$, $X_i \in Z_i - U_i$, and there is no element from U_i contained in X_i, $i = 1, 2$. As $Y_1X_2Y_2 \in H_2$ and $Y_2X_1Y_1 \in H_1$, we have $uv = X_1X_2Y_1Y_2$ and $vu = X_2X_1Y_2Y_1$. It implies that the equation $c^{-1}d = 1$ holds in H_i, $i = 1, 2$. Hence $Z = \overline{Z}$, and the applying of (A) does not result in the new edges in L_i, $i = 1, 2$. Similarly as in the proof of Theorem 1 one can derive that if $u \in Z_1 - U_1$ and $v \in Z_2 - U_2$ then $u^{-1}v \notin Z_i$ for $i = 1, 2$. This completes the proof. □

4. m-Treelike Graphs

In this section the operation of the amalgamation of groups will be used to construct graphs with constant link isomorphic to the so-called m-**treelike** graphs.

Definition 1. Let n and m be integers such that $n \geq 3$ and $m \geq 1$. A connected graph T is said to be m-treelike if
A. T does not contain any cycle of length greater than three as an induced subgraph.
B. The maximal cliques in T have the same size n. The intersection of any two maximal cliques is empty or is the complete graph on m vertices.

Note that the concept of m-**treelike** graph generalizes that of **treelike** graph introduced in Harary and Palmer [**3**].

An m-treelike graph is called m-**starlike** if all its maximal cliques have exactly m vertices in common. An m-starlike graph in which the number of maximal cliques is $k \geq 2$ will be denoted by $S(n, m, k)$, see Fig. 3.

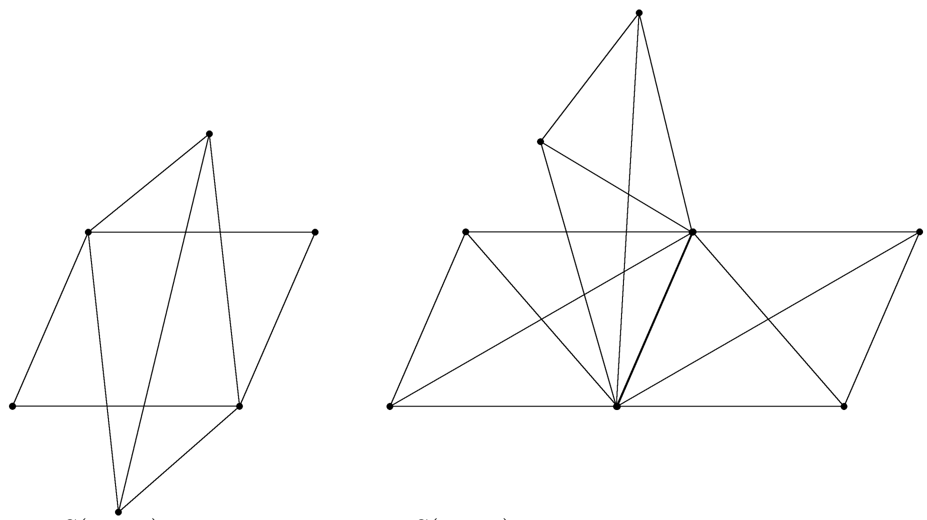

Figure 3.

The next proposition gives a necessary and sufficient condition for an m-starlike graph to be a link graph. As an m-starlike graph has exactly m universal vertices, the assertion of our proposition follows also from Theorem 1 in Hell [**4**] (for the definition of an universal vertex see the same paper, [**4**]). However, the method we shall use to prove it allows us to construct Cayley graphs with constant link isomorphic to the prescribed m-starlike graphs.

Proposition 1. *An m-starlike graph $S(n,m,k)$ is the link graph if and only if $n+1=c(m+1)$ for an integer $c>1$.*

Proof of Proposition 1.

Sufficiency. Let S be an m-starlike graph with $k\geq 2$, and let $I=\{1,\dots,k\}$ be an index set. Since $n+1=c(m+1)$, each maximal clique in S (say C_i with $i\in I$) can be represented as the link graph of a Cayley graph defined in the following way. Let H_i be an Abelian group with the generating set

$$Z_i=\{x_i^h a_i^r\colon h\in\{0,\dots,m\},\ r\in\{0,\dots,c-1\},\ (h,r)\neq(0,0)\}$$

and with defining relations $x_i^{m+1}=a_i^c=1_i$ for $i\in I$.
Obviously, $\operatorname{link}(1_i,[H_i,Z_i])\cong C_i$ for $i\in I$.

Consider the subgraph $C_i'\leq C_i$ induced by the vertices $x_i,\dots,x_i^m$, $i\in I$. Let U_i be a subgroup of H_i with $U_i=\{1_i,x_i,\dots,x_i^m\}$, and let $f_1\colon C_1'\to C_2'$ be the mapping defined as $f_1(x_1^t)=x_2^t$ for $t=1,\dots,m$. Then f_1 is an isomorphism and moreover, it can be naturally extended to the isomorphism of U_1 and U_2. Set

$\overline{f}_1(x_1^t) = x_2^t$ for $t = 0, \dots, m$. Then by Theorem 2, the Cayley graph of the group

$$P_1 = \big\langle H_1 * H_2;\ x_1 = \overline{f}_1(x_1),\ x_1 \in U_1,\ uv = vu,\ u \in Z_1 \text{ and } v \in Z_2 \big\rangle$$

has the constant link (say L_1) isomorphic to $S(n, m, 2)$.

If $k = 2$ then $L_1 \cong S$; otherwise consider the subgraph $L_1' \le L_1$ induced by the vertices $x_1, \dots, x_1^m$, and the subgraph $C_3' \le C_3$ induced by the vertices $x_3, \dots, x_3^m$. Let Z denote the generating set of P_1 and let 1 be its identity element. An isomorphism $f_2\colon L_1' \to C_3'$ can be extended to the isomorphism (say $\overline{f}_2$) of groups $A_1 = \{1, x_1, \dots, x_1^m\}$ and $U_3 = \{1_3, x_3, \dots, x_3^m\}$. As $U_3 \cap Z_3 = C_3'$ and $A_1 \cap Z = L_1'$, the Cayley graph of the group

$$P_2 = \big\langle P_1 * H_3;\ x_1 = \overline{f}_2(x_1),\ x_1 \in A_1,\ uv = vu,\ u \in Z \text{ and } v \in Z_3 \big\rangle$$

has constant link (say L_2) isomorphic to $S(n, m, 3)$.

If $k = 3$ then $L_2 \cong S$; otherwise the construction described above will be used repeatedly (exactly $k - 2$-times) to derive the Cayley graph with constant link S.

Necessity. In order to prove the necessary condition we shall need the next definition, given in [**1**]. Let u and v be adjacent vertices in a graph G. The number of vertices adjacent to both u and v is called the **relative degree,** and is denoted by $\alpha(u, v)$. If $\alpha(u, v) = q$ then we say that the edge (u, v) is **marked** q.

Suppose that an m-starlike graph S is the link graph of a graph G. If X denotes the centre of S then by the definition of the relative degree we obtain

$$\alpha(x, y) = k(n - m) + (m - 1) \qquad \text{for any } x, y \in X \text{ and}$$
$$\alpha(a, x) = n - 1 \qquad \text{for any } x \in X \text{ and } a \in S - X.$$

Let a be a vertex contained in $S - X$. Then there are m vertices in $S - X$ which belong to the centre of the link (a, G), say $a_1, \dots, a_m$. Clearly, the vertices $a, a_1, \dots, a_m$ belong to the same maximal clique in S, i.e.

$$\alpha(a, a_i) = \alpha(a_i, a_j) = k(n - m) + (m - 1) \qquad \text{for } i, j = 1, \dots, m.$$

Thus the edges **marked** $k(n - m) + (n - 1)$ indicate a K_{m+1} factor of $S - X$, and the assertion follows. □

Let T be a graph of type $S(n, m, k)$. The set of all vertices in T with degree $k(n - m) + (m - 1)$ will be denoted by X. Now, we shall introduce a new class of m-treelike graphs derived from a given graph $S(n, m, k)$.

Definition 2. Let k and l be integers such that $k \ge 2$ and $l \ge 1$. Define the class $S(n, m, k, l)$ of m-treelike graphs with $n \ge 2m$ as follows.
1. $S(n, m, k, 1) = \{S(n, m, k)\}$.
2. A. Let $k \ge 3$ and $l \ge 2$. Consider a graph $T \in S(n, m, k, l - 1)$, and all its

maximal cliques such that each of them contains a vertex at distance $l-1$ from X; the maximal cliques with the above property will be denoted by $C_1, \dots, C_j$, where $j \le k$. Let $C'_i \le C_i$ be complete subgraph on m vertices, such that if u and v belong to $V(C'_i)$ then $\deg(u,T) = \deg(v,T) = n-1$, $i = 1, \dots, j$. Further, let $L_1, \dots, L_t$ be complete graphs on n vertices with $t \le j$, and $L'_i \le L_i$ be the complete subgraph on m vertices, $i = 1, \dots, t$. Consider an isomorphism $f_i \colon C'_i \to L'_i$ where $i = 1, \dots, t$. We define $H^j_t(T)$ to be the graph derived from the disjoint union of $T, L_1, \dots, L_t$ by identifying every vertex $v \in V(C'_i)$ with $f_i(v) \in V(L'_i)$, and every edge of C'_i with the corresponding edge of L'_i, $i = 1, \dots, t$.

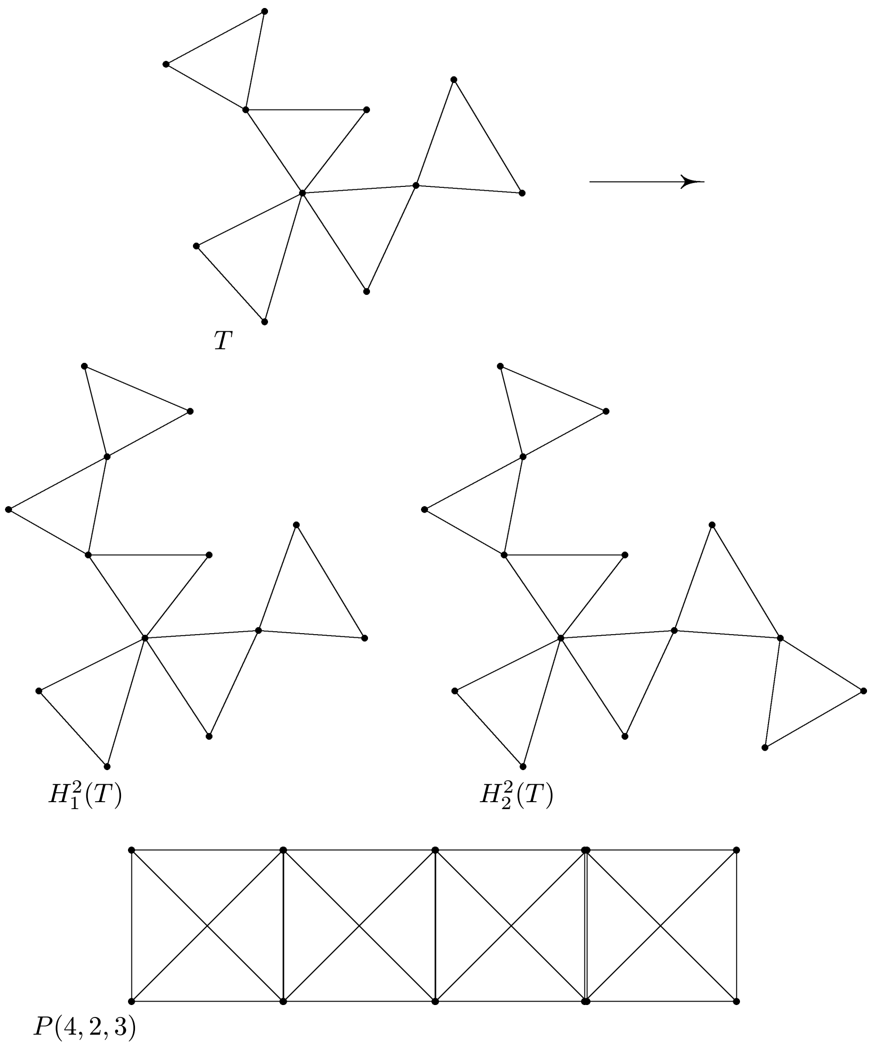

Figure 4.

Let G be an m-treelike graph. If there is a graph $T \in S(n,m,k,l-1)$ such that $H_t^j(T)$ is isomorphic to G for some t, then G belongs to the class $S(n,m,k,l)$.

B. If $k=2$ then the class $S(n,m,2,l)$ contains a single element, and we denote it by $P(n,m,l)$, see Fig. 4.

Figure 4 shows the graph $T \in S(3,1,3,2)$, and graphs $H_1^2(T)$ and $H_2^2(T)$ derived from T.

To simplify the statement of the next proposition we give the following definition. Let T be a graph in $S(n,m,k,l)$ with $k \geq 3$. We say that the **branch** B_i **of** T **at** X **has length** l_i if there is a maximal clique C in B_i, such that C contains a vertex v at distance l_i from X, $i \in \{1,\dots,k\}$.
The branches of T at X will be simply called the branches of T.
Note that if $T \in S(n,m,k,l)$ then $1 \leq l_i \leq l$ for $i=1,\dots,k$, and there is at least one branch in T of length equal to l.

Proposition 2. *Let T be a graph in $S(n,m,k,l)$ with $k \geq 3$. Suppose that the length of each branch in T is greater than or equal to* 2. *If T is a* **link graph then** $n+1 \geq (m+1)^2$**.**

Proof. Let G be a graph with constant link T. Take a vertex $v \in G$, and consider the link (v,G) and the corresponding set $X = \{x\colon x \in \operatorname{link}(v,G)$ such that $\deg(x, \operatorname{link}(v,G)) = k(n-m)+(m-1)\}$.
By the definition of the relative degree we have

$$\alpha(x,y) = k(n-m)+(m-1) \qquad \text{for any } x,\, y \in X.$$

Let $I=\{1,\dots,k\}$. To each $i \in I$ there corresponds a branch B_i in T containing a maximal clique C_i so that $X \leq C_i$. As $n \geq 2m$ there are m vertices in C_i (say $s_{i,1},\dots,s_{i,m}$) $i \in I$, with degrees equal to $r = 2(n-m)+(m-1)$.
Since

$$\alpha(v, s_{i,j}) = r \qquad \text{for } j=1,\dots,m,$$

we obtain

$$\alpha(s_{i,p}, s_{i,j}) = r \qquad \text{for } p,\, j = 1,\dots,m.$$

The last equation follows from the following fact: if

$$\alpha(s_{i,p}, s_{i,j}) = k(n-n)+(m-1) \qquad \text{for } p, j \in \{1,\dots,m\}$$

then by the definition of m-treelike graph we have

$$\deg(v, \operatorname{link}(s_{i,j}, G)) = k(n-m)+(m-1), \qquad \text{for } j=1,\dots,m.$$

However, it is a contradiction with

$$\alpha(v, s_{i,j}) = r, \qquad \text{for } j=1,\dots,m.$$

Using similar arguments one can derive the inequality

$$\alpha(x, s_{i,j}) \neq r \qquad \text{for any } x \in X, \text{ and } j = 1, \dots, m.$$

Next, consider link (x, T) where $x \in X$, and the maximal clique $C_i' = C_i - \{x\} \cup \{v\}$ where $i \in I$. As $\alpha(x, s_{i,j}) \neq r$ for $j = 1, \dots, m$, there are m vertices in C_i', say $s_{i,1}', \dots, s_{i,m}'$, such that $\alpha(x, s_{i,j}') = \alpha(s_{i,j}', s_{i,p}')_p = r$ for $p, j = 1, \dots, m$, and $i \in I$. Hence, $n - 2m \geq m^2$. □

Theorem 3. *Let T be a graph from the class $S(n, m, k, l)$ with $k \geq 2$ and $l \geq 2$. If $n + 1 = c(m+1)^2$ for an integer $c \geq 1$ then T is the link graph.*

Proof. First we construct the Cayley graph with constant link $S(n, m, k)$. Let $I = \{1, \dots, k\}$ be the index set.

Consider an Abelian group H with the generating set

$$Z = \{x^h y_i^q a_i^p \colon h, q \in \{0, \dots, m\},\ p \in \{0, \dots, c-1\},\ (h, q, p) \neq (0,0,0),\ i \in I\}$$

and with $x^{m+1} = y_i^{m+1} = a_i^c = 1$ for $i \in I$.
For each $i \in I$ the elements from the set

$$Z_i = \{x^h y_i^q a_i^p \colon h, q \in \{0, \dots, m\},\ p \in \{0, \dots, c-1\},\ (h, q, p) \neq (0,0,0)\}$$

correspond to the vertices of the complete subgraph on n vertices in L, where L denotes the constant link of $[H, Z]$. Since the elements $a_i^p a_j^r$, $y_i^q y_j^s$, $y_i^q a_j^p$ do not belong to Z if $i \neq j$, p, $r \in \{1, \dots, c-1\}$, and q, $s \in \{1, \dots, m\}$, L is isomorphic to $S(n, m, k)$.

Let G be a graph which belongs to the class $S(n, m, k, 2)$. Now, the graph with constant link isomorphic to G will be derived from $[H, Z]$. Let H_1' be an Abelian group generated by the set

$$Z_1' = \{r_1^h s_1^q b_1^p \colon h, q \in \{0, \dots, m\},\ p \in \{0, \dots, c-1\},\ (h, q, p) \neq (0,0,0)\}$$

with $r_1^{m+1} = s_1^{n+1} = b_1^c = 1_1$.
Then link $(1_1, [H_1', Z_1'])$ is isomorphic to the complete graph on n vertices, say L_1. Let $L_1' \leq L_1$ be the subgraph induced by the vertices $r_1, \dots, r_1^m$, and let $L' \leq L$ be the subgraph induced by the vertices $y_1, \dots, y_1^m$. Then the mapping $f \colon L_1' \to L'$ defined as $f(r_1^h) = y_1^h$ for $h = 1, \dots, m$ is an isomorphism, and it can be extended to the isomorphism of the groups $U_1' = \{1_1, r_1, \dots, r_1^m\}$ and $U = \{1, y_1, \dots, y_1^m\}$, say $\overline{f}$. According to Theorem 2 we obtain, that the Cayley graph of the group

$$H' = \left\langle H * H_1';\ r_1 = \overline{f}(r_1),\ r_1 \in U_1',\ uv = vu,\ u \in Z \text{ and } v \in Z_1' \right\rangle$$

has constant link isomorphic to a graph (say G_1) from $S(n, m, k, 2)$. Let Z' denote the generating set of H'.

If $k = 2$ then $G \cong G_1$, and the above construction gives the graph $[H', Z']$ with the constant link from $S(n, m, 2, 2)$. Since link $(1, [H', Z'])$ contains the subgraph induced by the vertices $s_1, \dots, s_1^m$ and H' contains the subgroup $\{1, s_1, \dots, s_1^m\}$, the operation of the amalgamation can be used repeatedly. In such a way we can construct the Cayley graph with the constant link isomorphic to a graph from $S(n, m, k, l)$ with $k = 2$ and $l \geq 2$.

Suppose that $k \geq 3$. Then G has j $(1 \leq j \leq k)$ branches of length two, and link $(1, [H', Z'])$ has exactly one branch of length 2. However, link $(1, [H', Z'])$ contains the subgraph induced by the set $Y_i = \{y_i, \dots, y_i^m\}$, such that $\{1\} \cup Y_i$ is the subgroup of H' for $i = 1, \dots, j$. It means that each of j branches in link $(1, [H', Z'])$ can be prolonged by the analogous procedure as we have prolonged the branch containing the subgraph Y_1. Hence, there exists a Cayley graph with the constant link isomorphic to G. As link $(1, [H', Z'])$ contains the subgraph induced by the set $s_1, \dots, s_1^m$, and H' contains the subgroup $\{1, s_1, \dots, s_1^m\}$, the operation of the amalgamation can be used to determine a Cayley graph with the constant link isomorphic to a graph from $S(n, m, k, 3)$. Hence, the proof of theorem follows by induction on l. □

References

1. Blass A., Harary F. and Miller Z., *Which trees are link graphs?*, J. Com. Theory B **29** (1980), 277–292.

2. Bulitko V. K., *On graphs with given vertex neighborhoods*, Abstract, Proc. Sixth All-Soviet Topological Conf. Tbilisi (1972), 23–24.

3. Harary F. and Palmer E. M., *Graphical Enumeration*, Academic Press, New York and London, 1973.

4. Hell P., *Graphs with given neighborhoods I.*, In Proc. Colloque, Inter. C. N. R. S., Orsay, 1976, pp. 219–223.

5. Lyndon R. C. and Schupp P. E., *Combinatorial Graph Theory*, Springer-Verlag, 1977.

6. Zykov A. A., *Problem 30*, Theory of Graphs and Applications (M. Fiedler, ed.), Academia, Praha, 1964, pp. 164.

J. Tomanová, Department of Algebra and Number Theory, Faculty of Mathematics and Physics, Comenius University, 842 15 Bratislava, Czechoslovakia