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RECOGNIZING APPROXIMATE $(\mathcal{A},\mathcal{B},\mathcal{C})$–TAMENESS

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Abstract. This paper gives some applications of results from the author's article "Forced approximate resolutions" in general topology, dimension theory, and shape theory.

1. Introduction

In the paper [**4**] the author established some general conditions under which a topological space admits an approximate resolution [**11**] with predefined projections, bonding spaces, and/or bonding maps. These conditions are expressed in terms of various properties of maps of topological spaces described by approximately commutative diagrams. Here we shall concentrate on the so called approximate $(\mathcal{A},\mathcal{B},\mathcal{C})$-tameness, where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are classes of maps. We wish to demonstrate that for suitable choices of classes $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ the key results from [**4**] provide interesting consequences in dimension theory, general topology, shape theory, and in the study of spaces which are like a given class of spaces.

It is well known that every space has an approximate resolution into polyhedra [**11**]. An interesting question is to decide what spaces have approximate resolutions with all bonding spaces belonging to some subclass of polyhedra. We give necessary and sufficient conditions on a space X in order to have an approximate resolution into polyhedra of dimension $\leq n$, finite polyhedra, countable polyhedra, locally finite or locally countable polyhedra, and contractible polyhedra. These conditions are that X has dimension $\leq n$ [**13**], X is normally compact, X is normally Lindelöf, X is strongly paracompact, and X has trivial shape. Here, normal compactness and the property to be normally Lindelöf are weaker variants of compactness and the property to be Lindelöf.

Finally, we also tackle the problem whether spaces which are like a class $\mathcal{C}$ of spaces have approximate resolutions into members of $\mathcal{C}$ with onto bonding maps. Our method has the advantage over approaches in [**9**], [**7**], and [**12**] in the fact

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that for the first time we may handle classes of approximate polyhedra rather than just classes of polyhedra.

2. Preliminaries

Let $\mu = (\mathcal{A}, \mathcal{B}, \mathcal{C})$ be a triple of classes of maps. A class of maps $\mathcal{F}$ is called **approximately** μ**-tame** provided for every $f\colon X \to Y$ in $\mathcal{F}$, every $\sigma \in \hat{Y}$, and every $\alpha \in \mathcal{A}$ there are $\beta \in \mathcal{B}$ and $\gamma \in \mathcal{C}$ with $\beta \circ \gamma \overset{\sigma}{=} f \circ \alpha$.

In the above definition we let $\hat{Y}$ denote the collection of all normal covers of a space Y and for maps $f, g\colon X \to Y$ and a $\sigma \in \hat{Y}$ we write $f \overset{\sigma}{=} g$ and say that f and g are σ**-close** provided for every $x \in X$ there is a member of σ which contains both $f(x)$ and $g(x)$.

The tame part of the name for this approximate property of maps comes from its similarity with L. Siebenmann's notion of a tame at infinity space [**14**] (see also [**2**] and [**3**]). In [**4**] we called approximately μ-tame classes of maps simply μ_2 classes. Of course, by identifying a map with a class consisting only of this map, our definition applies also to maps.

We shall also use the properties β_1 and β_2 of pairs of classes of maps from [**4**] that we rename into notions of approximation and right inversion. Let $\mathcal{A}$ and $\mathcal{B}$ be classes of maps. Then $\mathcal{A}$ **approximates** $\mathcal{B}$ if for all $\beta \in \mathcal{B}$ and $\sigma \in \hat{\overline{\beta}}$ there is an $\alpha \in \mathcal{A}$ with $\alpha \overset{\sigma}{=} \beta$ and $\mathcal{A}$ **right inverts** $\mathcal{B}$ if for all $\beta \in \mathcal{B}$ and $\sigma \in \hat{\overline{\beta}}$ there is an $\alpha \in \mathcal{A}$ with $id \overset{\sigma}{=} \beta \circ \alpha$. Let $\mathcal{A}\,\mathcal{B}$ denote compositions of maps from $\mathcal{B}$ and $\mathcal{A}$.

In the above definitions $\underline{f}$ and $\overline{f}$ denote the domain and the codomain of the map f. For a class $\mathcal{A}$ of maps, we use $\underline{\mathcal{A}}$ and $\overline{\mathcal{A}}$ to denote the classes of all domains and all codomains of members of $\mathcal{A}$.

Now we must recall §3 in [**4**] which contains all the basic notions necessary to develop the study of spaces through maps of the space into members of a class of spaces.

Let A be a set, let X be a space, let $\mathcal{P}$ be a class of spaces, and let $\mathcal{F}$ be a class of maps. By an $(A, \mathcal{P})$**-net** we mean a function $x\colon A \to \mathcal{P}$. Whenever possible, we shall replace $x(a)$ with X_a or simply with a.

Let x be an $(A, \mathcal{P})$-net. An $(x, \mathcal{F})$**-fan on** X is a function $g\colon A \to \mathcal{F}$ such that $X = \underline{g(a)}$ and $a = \overline{g(a)}$ for every $a \in A$. We shall use shorter notation g^a for the map $g(a)$.

Let $A = (A, <)$ be a directed set and let x be an $(A, \mathcal{P})$-net. An $(x, \mathcal{F})$**-rim** is a function $q\colon < \to \mathcal{F}$ such that $a = \underline{q(a, b)}$ and $b = \overline{q(a, b)}$ whenever $b > a$. We shall denote $q(a, b)$ simply by q^a_b.

Many times we shall resort to the simpler notation for the above notions. For example, if the nature of the indexing set A is clear or not important, we talk about a $\mathcal{P}$**-net** and about a **net** if $\mathcal{P}$ is the class of all topological spaces. Also, we shall identify the net x, the rim q and the fan g with the classes $\mathcal{X} = \{X_a \mid a \in A\}$, $\mathcal{Q} = \{q_b^a \mid b \geq a\}$ and $\mathcal{G} = \{g^a \mid a \in A\}$ and use the functional notation $\mathcal{G}\colon X \to \mathcal{X}$ for a fan from X into a net $\mathcal{X}$.

The following notion of domination satisfies Theorems A and B bellow (see [**4**, Theorems 7 and 9]). Let κ denote the 6-tuple $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D},\mathcal{E},\mathcal{F})$ of classes of maps while λ and μ stand for triples $(\mathcal{B},\mathcal{C},\mathcal{D})$ and $(\mathcal{A},\mathcal{B},\mathcal{C})$.

We shall say that the family Δ of fans **κ-tamely dominates** a fan $\mathcal{G}$ from a space X into a net $\mathcal{X}$ on a set A provided for every $a \in A$ and every $\sigma \in \hat{a}$ there is an $\mathcal{H}\colon Y \to \mathcal{Y}$ in Δ, a $u \in U$, and a $\tau \in \hat{u}$ such that for every map $\alpha\colon Z \to X$ in $\mathcal{A}$ there is a map $\beta\colon W \to Y$ in $\mathcal{B}$ so that for every pair of maps $\gamma \in \mathcal{C}$ and $\delta \in \mathcal{D}$ with $\gamma \circ \delta \overset{\tau}{=} h^u \circ \beta$ we can find maps $\varepsilon \in \mathcal{E}$ and $\varphi \in \mathcal{F}$ with $\varepsilon \circ \varphi \overset{\sigma}{=} g^a \circ \alpha$.

Let Π be a property of fans. We say that a class of fans has the property Π if every member of it has property Π.

Theorem A. *A fan $\mathcal{G}$ is approximately μ-tame iff it is κ-tamely dominated by an approximately λ-tame class of fans.*

Let X be a space. Let ν be the pair $(\mathcal{R},\mathcal{S})$ of classes of maps. A class Δ of fans from X is a **ν-expansion** (of X) provided for every map $r\colon X \to Z$ in $\mathcal{R}$ and every $\pi \in \hat{Z}$ there is an $\mathcal{H}\colon X \to (\mathcal{Y},\, U)$ in Δ and a $u \in U$ such that for every $v > u$ there is a map $s\colon Y_v \to Z$ in $\mathcal{S}$ with $r \overset{\pi}{=} s \circ h^v$. When $\mathcal{R}$ and $\mathcal{S}$ are the class $\mathbf{APOL^{TOP}}$ of all maps into approximate polyhedra [**6**], then we talk about an **expansion** (of X).

Theorem B. *Let ν denote a pair $(\mathcal{R},\mathcal{S})$ of classes of maps. Let Δ be a ν-expansion of a space X. Let $\mathcal{G}\colon X \to (\mathcal{X},\, A)$ be a fan. If classes $\mathcal{B}$, $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{R}$ approximate classes $\mathcal{A}$, $\mathcal{G}\,\mathcal{S}$, $\mathcal{D}$, and $\mathcal{G}$, respectively, then $\mathcal{G}$ is κ-tamely dominated by Δ.*

We shall also need the following special cases of Theorems 11 and 12 in [**4**] (stated here as Theorems C and D). Both of these theorems deal with the problem of the existence of special approximate resolutions for spaces. We shall assume that the reader is familiar with the theory of approximate inverse systems and approximate resolutions (see [**8**] and [**11**]).

Let $\mathcal{A}$ and $\mathcal{C}$ be classes of maps and let $\mathcal{B}$ be a class of spaces. An **approximate $(\mathcal{A},\, \mathcal{B},\, \mathcal{C})$-resolution** is an approximate resolution $\mathcal{G} = \{g^a\}\colon X \to \{X_a,\, \varepsilon_a,\, p_b^a,\, A\}$ such that the maps g^a belong to $\mathcal{A}$, spaces X_a are from $\mathcal{B}$, and the maps p_b^a belong to $\mathcal{C}$. Of course, sometimes we shall omit some of the classes from our notation. For example, by an **approximate $\mathcal{B}$-resolution** we mean that all bonding spaces X_a are from the class $\mathcal{B}$.

Theorem C. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ be classes of maps. Let $\mu = (\mathcal{A},\mathcal{B},\mathcal{C})$. Let Δ be a class of fans into $\underline{\mathcal{A}}$-nets and an expansion of a space X. If $\underline{\mathcal{B}} \subset$* $\mathbf{APOL}$

and $X \in \overline{\mathcal{A}}$, Δ *is approximately* μ*-tame, and* $\mathcal{D}$ *right inverts* $\mathcal{A}$, *then* X *has an approximate* $\underline{\mathcal{B}}$*-resolution.*

For a class of spaces $\mathcal{A}$, let $\mathcal{I}_{\mathcal{A}}$ denote the class of all identity maps on members of $\mathcal{A}$ and let $\mathcal{I}$ be the class of all identity maps.

Theorem D. *Let* $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, *and* $\mathcal{D}$ *be classes of maps, let* η *denote the triple* $(\mathcal{I}, \mathcal{A}, \mathcal{B})$, *and let* ξ *be a pair* $(\mathcal{B}, \mathcal{C})$. *If a space* X *has a uniform, commutative, and approximately* η*-tame approximate* (**APOL**, $\mathcal{D}$)*-resolution* $\mathcal{G}\colon X \to \mathcal{X}$ *which is also a* ξ*-expansion of* X *and* $\mathcal{E}$ *is a class of maps which includes maps from* $\mathcal{I}_{\underline{\mathcal{A}}}$, $\mathcal{CDA}$, *and* $\mathcal{CG}$, *then* X *has an approximate* ($\mathcal{E}$, $\underline{\mathcal{A}}$, $\mathcal{E}$)*-resolution.*

3. Dimension

We shall say that a space X has dimension $\leq n$ provided every normal cover σ of X has a refinement $\tau \in \hat{X}$ of order $\leq n+1$ (see [**13**]).

Let $\mathcal{M}$, $\mathcal{M}_n$, and $\mathcal{P}_n$ denote classes of all maps, of all maps defined on spaces of dimension $\leq n$, and of all maps defined on polyhedra of dimension $\leq n$. Let $\mu(n)$ and $\pi(n)$ be triples $(\mathcal{M}, \mathcal{M}_n, \mathcal{M})$ and $(\mathcal{M}, \mathcal{P}_n, \mathcal{M})$, respectively.

The equivalence of (iii) and (vi) in the next theorem has been proved in [**16**].

Theorem 1. *For a topological space* X *the following are equivalent.*

(i) *Every fan from* X *to approximate polyhedra is approximately* $\mu(n)$*-tame.*
(ii) X *has an approximately* $\mu(n)$*-tame expansion into approximate polyhedra.*
(iii) X *has an approximate resolution into polyhedra of dimension* $\leq n$.
(iv) X *has an approximately* $\pi(n)$*-tame expansion into approximate polyhedra.*
(v) *Every fan from* X *to approximate polyhedra is approximately* $\pi(n)$*-tame.*
(vi) dim $X \leq n$.

Proof. Implications (v) $\Longrightarrow$ (iv) and (iii) $\Longrightarrow$ (ii) are obvious.

(ii) $\Longrightarrow$ (i). Combine Theorems A and B.

(iv) $\Longrightarrow$ (iii). This follows from Theorem C.

(vi) $\Longrightarrow$ (v). See the proof of Lemma 6 in [**16**].

(i) $\Longrightarrow$ (vi). Suppose that every fan from X to approximate polyhedra is approximately $\mu(n)$-tame. In order to prove that dim $X \leq n$, we will show that every normal cover $\sigma \in \hat{X}$ has a refinement $\tau \in \hat{X}$ of order $\leq n+1$.

Let $\sigma \in \hat{X}$. Let $N = N(\sigma)$ be the nerve of σ and let $|N|$ be its geometric realization (endowed with the CW-topology). Let $f\colon X \to |N|$ be a canonical map so that

$$\text{(a)} \qquad f^{-1}(st(S,\, N)) \subset S \qquad \text{for } S \in \sigma\,.$$

The open stars $st(S,\, N)$ with $S \in \sigma$ form a normal covering κ of $|N|$. Let $\pi \in \kappa^*$, where κ^* denotes the collection of all normal covers whose star refines κ.

Since f could be regarded as a fan from X into an (approximate) polyhedron, there is a space Z of dimension $\leq n$ and maps $g\colon Z \to |N|$ and $h\colon X \to Z$ with

$$\text{(b)} \qquad f \overset{\pi}{=} g \circ h.$$

Let $\varrho = g^{-1}(\pi)$. Since ϱ is a normal cover of Z and Z has dimension $\leq n$, there is a refinement μ of ϱ of order $\leq n+1$. Let $\tau = h^{-1}(\mu)$.

Clearly, τ has order $\leq n+1$. In order to check that τ refines σ, let $T \in \tau$. For every $x \in T$, the relation (b) implies the existence of a $P_x \in \pi$ such that $f(x),\ g(h(x)) \in P_x$. It follows that

$$\text{(c)} \qquad f(T) \subset st(P,\ \pi),$$

where $P = g(h(T)) \in \pi$. Since $\pi \in \kappa^*$, there is an $S \in \sigma$ with

$$\text{(d)} \qquad st(P,\ \pi) \subset st(S,\ N).$$

Relations (a), (c), and (d) together imply $T \subset S$, i.e., that τ refines σ. □

4. Compactness

We turn now to the problem of describing conditions under which a space X admits an approximate resolution into compact polyhedra. This question has been addressed earlier by Mrs. V. Matijević [**12**]. She showed that a Tychonoff space X has an approximate resolution into compact Hausdorff spaces iff X is P-embedded into its Stone-Čech compactification βX. Our improvement uses the following weak form of compactness.

We shall say that a space X is **normally compact** provided every normal cover of X has a finite subcover.

Observe that every normally compact space X is pseudocompact. Indeed, if $f\colon X \to R$ is a continuous real function, then $\{f^{-1}(n,\ n+2)\}$, where n is an integer, is a normal cover of X which has a finite subcover

$$\{f^{-1}(n_1,\ n_1+2), \ldots,\ f^{-1}(n_k,\ n_k+2)\}$$

so that f is bounded by $min\{n_i\}_{i=1}^k$ and $2 + max\{n_i\}_{i=1}^k$.

By Theorem 13 below and remarks in [**12**] it follows that a Tychonoff space of nonmeasurable cardinal is normally compact iff it is pseudocompact.

Let $\mathcal{N}$ and $\mathcal{Q}$ denote classes of all maps defined on normally compact spaces and on compact polyhedra, respectively. Let μ_K and π_K be triples $(\mathcal{M},\mathcal{N},\mathcal{M})$ and $(\mathcal{M},\mathcal{Q},\mathcal{M})$, respectively.

Theorem 2. *For a T_1-space X the following are equivalent.*

(i) *Every fan from X to approximate polyhedra is approximately μ_K-tame.*
(ii) *X has an approximately μ_K-tame expansion into approximate polyhedra.*
(iii) *X has an approximate resolution into compact polyhedra.*
(iv) *X has an approximately π_K-tame expansion into approximate polyhedra.*
(v) *Every fan from X to approximate polyhedra is approximately π_K-tame.*
(vi) *X is normally compact.*
(vii) *X is P-embedded into its Wallman extension wX.*
(viii) *X is P-embeded in some quasi-compact space.*

Proof. That the first six statements are equivalent can be proved in exactly the same way as we proved Theorem 1.

(vi) $\implies$ (vii). Let $\sigma \in \hat{X}$. Since the space X is normally compact, there are sets $S_1, \ldots, S_n$ in σ such that $\tau = \{S_1, \ldots, S_n\}$ is a normal cover of X. Let $\{\lambda_1, \ldots, \lambda_n\}$ be a partition of unity subordinated to τ and let $p\colon X \to N(\tau)$ be a canonical map of X into the nerve $N(\tau)$ of τ defined by

$$p(x) = \sum_{i=1}^{n} \lambda_i(x) S_i.$$

Since $N(\tau)$ is the compact space, there is a map $q\colon wX \to N(\tau)$ with $p = q \circ i$, where $i\colon X \to wX$ is a natural inclusion. Let $st(S_i)$ denote the open star of the vertex S_i in $N(\tau)$ and let $M_i = q^{-1}(st(S_i))$ for $i = 1, \ldots, n$. Then $\mu = \{M_1, \ldots, M_n\}$ is a normal cover of wX whose restriction to X refines the cover σ. Hence, X is P-embedded in wX.

(vii) $\implies$ (viii). Obvious.

(viii) $\implies$ (vi). Let X be P-embedded in a quasi-compact space K. Let $\sigma \in \hat{X}$. Choose a normal cover τ of K such that the restriction of τ to X refines σ. Let $\{T_1, \ldots, T_n\}$ be a finite subcover of τ. Then

$$\{T_1 \cap X, \ldots, T_n \cap X\}$$

is a finite normal cover of X which refines σ. □

5. Lindelöf Property

Our next result gives conditions for a space to have an approximate resolution into countable polyhedra. The property that suffices is related to Lindelöf property.

A space X is **normally Lindelöf** when every normal cover of X has a countable subcover. Clearly, a paracompact space is Lindelöf iff it is normally Lindelöf.

Let $\mathcal{M}_L$ and $\mathcal{P}_P$ denote classes of all maps defined on Lindelöf spaces and on countable polyhedra, respectively. Let μ_L and π_P be triples $(\mathcal{M}, \mathcal{M}_L, \mathcal{M})$ and $(\mathcal{M}, \mathcal{P}_P, \mathcal{M})$, respectively.

Theorem 3. *For a topological space X the following are equivalent.*

(i) *Every fan from X to approximate polyhedra is approximately μ_L-tame.*
(ii) *X has an approximately μ_L-tame expansion into approximate polyhedra.*
(iii) *X has an approximate resolution into countable polyhedra.*
(iv) *X has an approximately π_P-tame expansion into approximate polyhedra.*
(v) *Every fan from X to approximate polyhedra is approximately π_P-tame.*
(vi) *X is normally Lindelöf.*

Proof. Implications (v) $\Longrightarrow$ (iv) and (iii) $\Longrightarrow$ (ii) are obvious.

(ii) $\Longrightarrow$ (i). Combine Theorems A and B.

(iv) $\Longrightarrow$ (iii). This follows from Theorem C.

(vi) $\Longrightarrow$ (v). Let f be a map of X into a polyhedron P and let $\sigma \in \hat{P}$. Pick a simplicial complex K such that $P = |K|$ and

$$os(K) = \{st(v, K) : v \text{ is a vertex of } K\}$$

refines the cover σ. Since X is normally Lindelöf, there is a countable subcover λ of the normal cover $f^{-1}(os(K))$ of X. Let $g \colon X \to |N(\lambda)|$ be a canonical map of λ. Observe that $|N(\lambda)|$ is a countable polyhedron. We proceed as in the proof of Lemma 6 in [**16**] to construct a map $h \colon |N(\lambda)| \to P$ such that $f \overset{\sigma}{=} h \circ g$.

(i) $\Longrightarrow$ (vi). Suppose that every fan from X to approximate polyhedra is approximately μ_L-tame. We will show that every normal cover of X has a countable subcover.

Let $\sigma \in \hat{X}$. Choose a canonical map $f \colon X \to |N|$ and covers κ and π as in the proof of Theorem 1. Pick a Lindelöf space Z and maps $g \colon Z \to |N|$ and $h \colon X \to Z$ with $f \overset{\pi}{=} g \circ h$. Let $\varrho = g^{-1}(\pi)$. Since ϱ is a normal cover of Z and Z is a Lindelöf space there is a countable subcover μ of ϱ. Let $\tau = h^{-1}(\mu)$.

Clearly, τ is a countable cover of X. One can check that τ refines σ. □

6. Strong Paracompactness

Next we search for conditions on a space to have an approximate resolution into locally countable and locally finite polyhedra. It turns out that strong paracompactness plays role at least in the realm of paracompact spaces. Perhaps, one can try to lift this assumption with the introduction of "normal" versions but at the expense of grave complications.

Recall that a space is strongly paracompact if it is a Hausdorff space and every open cover of it has a star-finite open refinement. For a regular space, by Theorem 5.3.10 in [**5**], this is equivalent to the property that every cover has a star-countable open refinement.

Let $\mathcal{M}_{SP}$, $\mathcal{P}_{LF}$ and $\mathcal{P}_{LC}$ denote classes of all maps defined on strongly paracompact spaces, on locally finite polyhedra, and on locally countable polyhedra, respectively. Let μ_{SP}, π_{LF}, and π_{LC} be triples $(\mathcal{M}, \mathcal{M}_{SP}, \mathcal{M})$ $(\mathcal{M}, \mathcal{P}_{LF}, \mathcal{M})$, and $(\mathcal{M}, \mathcal{P}_{LC}, \mathcal{M})$, respectively.

Theorem 4. *For a paracompact space X the following are equivalent.*

(i) *Every fan from X to approximate polyhedra is approximately μ_{SP}-tame.*
(ii) *X has an approximately μ_{SP}-tame expansion into approximate polyhedra.*
(iii) *X has an approximate resolution into locally countable polyhedra.*
(iv) *X has an approximate resolution into locally finite polyhedra.*
(v) *X has an approximately π_{LC}-tame expansion into approximate polyhedra.*
(vi) *X has an approximately π_{LF}-tame expansion into approximate polyhedra.*
(vii) *Every fan from X to approximate polyhedra is approximately π_{LC}-tame.*
(viii) *Every fan from X to approximate polyhedra is approximately π_{LF}-tame.*
(ix) *X is strongly paracompact.*

Proof. We leave the proof to the reader because it is very similar to the proofs of previous three theorems. □

7. Trivial Shape

As a next example of application of the above technique, we give a new characterization of spaces of trivial shape. Recall that a space has trivial shape if every map of it into a polyhedron is null-homotopic. Of course, this is equivalent to it having the shape of a one point space.

Let $\mathcal{M}_C$ and $\mathcal{P}_C$ denote classes of all maps defined on contractible spaces and on contractible polyhedra, respectively. Let μ_C and π_C be triples $(\mathcal{M}, \mathcal{M}_C, \mathcal{M})$ and $(\mathcal{M}, \mathcal{P}_C, \mathcal{M})$, respectively.

Theorem 5. *For a topological space X the following are equivalent.*

(i) *Every fan from X to approximate polyhedra is approximately μ_C-tame.*
(ii) *X has an approximately μ_C-tame expansion into approximate polyhedra.*
(iii) *X has an approximate resolution into contractible polyhedra.*
(iv) *X has an approximately π_C-tame expansion into approximate polyhedra.*
(v) *Every fan from X to approximate polyhedra is approximately π_C-tame.*
(vi) *X has trivial shape.*

Proof. The scheme of proof is analogous to the proof of previous theorems. Hence, we must show that (vi) $\Longrightarrow$ (v) and (i) $\Longrightarrow$ (vi).

(vi) $\Longrightarrow$ (v). Let $f\colon X \to P$ be a map of a space of trivial shape X into a polyhedron P and let $\sigma \in \hat{P}$. Let $\tau \in \sigma^*$. Let $\mathbf{p} = \{p^a\}\colon X \to \mathbf{X} = \{X_a, \varepsilon_a, p_b^a, A\}$ be an approximate resolution of X into polyhedra. By the condition (R1) for $\mathbf{p}$, there is an $a \in A$ and a map $f_a\colon X_a \to P$ with $f \overset{\tau}{=} f_a \circ p^a$.

Let $\varrho \in \hat{a}$ be a refinement of $(f_a)^{-1}(\tau)$ with the property that ϱ-close maps into X_a are homotopic. From the property (A2) for $\mathbf{X}$, we get an index $b > a$ such that $p_c^a \overset{\varrho}{=} p_b^a \circ p_c^b$, for every $c > b$. Since X has trivial shape and an approximate resolution is an expansion in the sense of Morita [**13**], there is a $c > b$ such that the map p_c^b is null-homotopic. It follows that the composition $p_b^a \circ p_c^b$ is also null-homotopic so that p_c^a is null-homotopic because it is homotopic to this composition. Let $C(X_c)$ be a cone on X_c and let $i\colon X_c \to C(X_c)$ be a natural embedding. Let $h_t\colon X_c \to X_a$ $(0 \geq t \leq 1)$ be a homotopy between p_c^a and a constant map.

Define a map $h\colon C(X_c) \to P$ by $h([x,\, t]) = f_a(h_t(x))$, for x in X_c and t in [0, 1]. Let $g\colon X \to C(X_c)$ be the composition $i \circ p^c$. Clearly, $C(X_c)$ is a contractible polyhedron while our selections imply that $f \overset{\sigma}{=} h \circ g$.

(i) $\Longrightarrow$ (vi). Let $f\colon X \to P$ be a map of a space X into a polyhedron P. Choose a cover $\sigma \in \hat{P}$ with the property that σ-close maps into P are homotopic. Since f can be regarded as a fan from X into (approximate) polyhedra, by assumption, there is a contractible space Z and maps $g\colon X \to Z$ and $h\colon Z \to P$ with $f \overset{\sigma}{=} h \circ g$. Since $h \circ g$ is null-homotopic, our selections imply that f is also null-homotopic. ☐

8. $\mathcal{C}$-like Spaces

Now we shall turn our attention to the question if spaces that are like a class $\mathcal{C}$ of spaces have approximate resolutions with bonding spaces belonging to $\mathcal{C}$ and with bonding maps onto. Previously, this problem was considered in [**9**], [**7**], and [**12**]. In all three cases only classes of polyhedra were allowed.

Let $f\colon X \to Y$ be a map and let $\sigma \in \hat{Y}$. We shall say that f is **dense** if the image of X under f is dense in Y and σ-**dense** if the σ-star $st(f(X),\, \sigma)$ of $f(X)$ with respect to the cover σ is all of Y.

Let $\mathcal{C}$ be a class of spaces. Recall [**12**] that a space X is $\mathcal{C}$-**like** provided for every $\sigma \in \hat{X}$ there is a member Y of $\mathcal{C}$, a cover $\tau \in \hat{Y}$, and a dense map $f\colon X \to Y$ such that $f^{-1}(\tau)$ refines σ.

Lemma. *Let $\mathcal{B}$ be a class of approximate polyhedra. If $f\colon X \to Y$ is a map of a $\mathcal{B}$-like space X into an approximate polyhedron Y, then for every $\sigma \in \hat{Y}$ there is a member Z of $\mathcal{B}$, a dense map $g\colon X \to Z$, and a map $h\colon Z \to Y$ such that $f \overset{\sigma}{=} h \circ g$.*

Proof. Let $\pi \in \sigma^*$. Since Y is an approximate polyhedron, there is a polyhedron Q and maps $v\colon Y \to Q$ and $e\colon Q \to Y$ with

$$e \circ v \overset{\pi}{=} id_Y. \tag{1}$$

Let $\lambda = e^{-1}(\pi)$. Choose a simplicial complex K such that $Q = |\,K\,|$ and the cover $os(K)$ refines λ.

Let $\tau = (v \circ f)^{-1}(os(K))$. We use now the assumption that X is $\mathcal{B}$-like to obtain a member Z of $\mathcal{B}$, a dense map g of X into Z, and a cover $\chi \in \hat{Z}$ such that $g^{-1}(\chi)$ refines τ.

Since Z is an approximate polyhedron, there is a polyhedron P and maps $u\colon Z \to P$ and $d\colon P \to Z$ with

$$d \circ u \overset{\chi}{=} id_P. \tag{2}$$

Let $\nu = d^{-1}(\chi)$. Choose a simplicial complex L such that $P = |L|$ and the cover $os(L)$ refines ν.

We now proceed to construct a simplicial map $s\colon L \to K$. Let ℓ be a vertex of L. Select members N of ν, E of χ, and T of τ and a vertex k of K such that $st(\ell, L) \subset N$, $N = d^{-1}(E)$, $g^{-1}(E) \subset T$, and $T = (v \circ f)^{-1}(st(k, K))$. Put $k = s(\ell)$.

In this paragraph we shall check that s is indeed a simplicial map. Suppose that vertices $\ell_1, \dots, \ell_n$ of L span a simplex in L. Then

$$st(\ell_1, L) \cap \cdots \cap st(\ell_n, L) \neq \emptyset,$$

$$N_1 \cap \cdots \cap N_n \neq \emptyset,$$

and

$$E_1 \cap \cdots \cap E_n \neq \emptyset.$$

Since the map g is dense, the last relation implies

$$g^{-1}(E_1) \cap \cdots \cap g^{-1}(E_n) \neq \emptyset.$$

From here we conclude that

$$T_1 \cap \cdots \cap T_n \neq \emptyset,$$

and

$$st(k_1, L) \cap \cdots \cap st(k_n, L) \neq \emptyset.$$

Hence, the vertices $s(k_1), \dots, s(k_n)$ span a simplex of K. Here, E_n denotes the set "E" associated to ℓ_n in our selections. Other indexed sets are described similarly.

Let h denote the composition $e \circ |s| \circ u$, where $|s|$ is a map of P into Q induced by s. Then h is the required map.

Indeed, let $x \in X$. Choose open sets $T, U \in \tau$, $E, F \in \chi$, $N \in \nu$, $A, C \in \lambda$, and $B, D, R \in \pi$, and a vertex ℓ of L and vertices j and k of K such that

$$E \quad \text{contains both } g(x) \text{ and } d \circ u \circ g(x), \tag{3}$$

$$u \circ g(x) \in st(\ell, L) \subset N \text{ and } N = d^{-1}(F), \tag{4}$$

$$T = (v \circ f)^{-1}(st(j, K)) \supset g^{-1}(E), \tag{5}$$

$$U = (v \circ f)^{-1}(st(k, K)) \supset g^{-1}(F), \tag{6}$$

$$R \text{ contains both } f(x) \text{ and } e \circ v \circ f(x), \tag{7}$$

$$A = e^{-1}(B) \supset st(j, K), \tag{8}$$

and

$$C = e^{-1}(D) \supset st(k,\, K). \tag{9}$$

The existence of E and R which satisfy (3) and (7) is the consequence of (2) and (1), respectively.

Now, since s is simplicial, $|s| \circ u \circ g(x)$ is in the set C, so that $e \circ |s| \circ u \circ g(x)$ is an element of D. On the other hand, the point $v \circ f(x)$ lies in the set A, so that $e \circ v \circ f(x)$, by (7), is in the intersection of sets R and B. But, B and D have nonempty intersection because E and F also are not disjoint by (4) and (5), (6), (8), and (9) are true. Hence, both $f(x)$ and $h \circ g(x)$ belong to $st(B,\, \pi)$. Since π is a star-refinement of σ, we get $f \overset{\sigma}{=} h \circ g$. □

Let $\mathcal{A}$ and $\mathcal{B}$ be classes of spaces. We say that $\mathcal{B}$ is $\mathcal{A}$-**densic** provided for every $Y \in \mathcal{B}$ and every $\sigma \in \hat{Y}$ there is a $\tau \in \hat{Y}$ such that for every $X \in \mathcal{A}$ and every τ-dense map $f\colon X \to Y$ there is a dense map $g\colon X \to Y$ with $f \overset{\sigma}{=} g$. Classes $\mathcal{A}$ and $\mathcal{B}$ are **densequivalent** provided $\mathcal{B}$ is $\mathcal{A}$-densic and $\mathcal{A}$ is $\mathcal{B}$-densic. If we require that the map g is an onto map we shall get notions "$\mathcal{B}$ is $\mathcal{A}$-**epic**" and "$\mathcal{A}$ and $\mathcal{B}$ are **epiequivalent**".

Let $\mathcal{D}$ and $\mathcal{O}$ denote classes of all dense and all onto maps, respectively.

Theorem 6. *Let $\mathcal{A}$ and $\mathcal{B}$ be classes of approximate polyhedra and let X be a $\mathcal{B}$-like space.*

- (i) *If the classes $\mathcal{A}$ and $\mathcal{B}$ are densequivalent and X has an approximate $(\mathcal{D},\, \mathcal{A},\, \mathcal{D})$-resolution, then X has an approximate $(\mathcal{D},\, \mathcal{B},\, \mathcal{D})$-resolution.*
- (ii) *If the classes $\mathcal{A}$ and $\mathcal{B}$ are epiequivalent and X has an approximate $(\mathcal{O},\, \mathcal{A},\, \mathcal{O})$-resolution, then X has an approximate $(\mathcal{O},\, \mathcal{B},\, \mathcal{O})$-resolution.*

Proof. (2). Let $\mathcal{G}$ be an approximate $(\mathcal{O},\, \mathcal{A},\, \mathcal{O})$-resolution of X. We shall prove that $\mathcal{G}$ is approximately $(\mathcal{I},\, \mathcal{A}^{\mathcal{B}} \cap \mathcal{O},\, \mathcal{B}^{X} \cap \mathcal{D})$-tame $(\mathcal{B}^{X} \cap \mathcal{D},\, \mathcal{B}^{\mathcal{A}} \cap \mathcal{O})$-expansion and apply Theorem D to reach the conclusion.

In order to see that $\mathcal{G}$ is approximately $(\mathcal{I},\, \mathcal{A}^{\mathcal{B}} \cap \mathcal{O},\, \mathcal{B}^{X} \cap \mathcal{D})$-tame, let an index $a \in A$ and a $\sigma \in \hat{a}$ be given. Let $\tau \in \sigma^{*}$. Since $\mathcal{A}$ is $\mathcal{B}$-epic, there is a refinement π of τ with the property that every π-dense map from a member of $\mathcal{B}$ into X_a is τ-close to an onto map. By the Lemma, there is a $Y \in \mathcal{B}$, a dense map $\gamma\colon X \to Y$, and a map $\delta\colon Y \to X_a$ with

$$g^{a} \overset{\pi}{=} \delta \circ \gamma. \tag{1}$$

Since g^a is onto, it follows that δ is a π-dense map of Y into X_a. Hence, there is an onto map $\beta\colon Y \to X_a$ with

$$\beta \overset{\tau}{=} \delta. \tag{2}$$

From (1) and (2) we get the required relation

$$g^a \overset{\sigma}{=} \beta \circ \gamma.$$

In order to prove that $\mathcal{G}$ is a $(\mathcal{B}^X \cap \mathcal{D},\ \mathcal{B}^{\mathcal{A}} \cap \mathcal{O})$-expansion, let $f\colon X \to Y$ be a dense map of X into a member Y of the class $\mathcal{B}$ and let $\sigma \in \hat{Y}$. Let $\tau \in \sigma^*$. Since $\mathcal{B}$ is $\mathcal{A}$-epic, there is a refinement π of τ with the property that every π-dense map of a member of $\mathcal{A}$ into Y is τ-close to an onto map.

Let $\lambda \in \pi^*$ and $\mu = f^{-1}(\lambda)$. By the property (B1) for $\mathcal{G}$, there is an index $a \in A$ such that the cover $(g^b)^{-1}(\varepsilon_b)$ refines μ for every $b > a$.

With only minor modifications to the proof of the Lemma, we can prove that for every $b > a$ there is a map $k_b\colon X_b \to Y$ with

$$f \overset{\lambda}{=} k_b \circ g^b. \tag{3}$$

Since f is a dense map, k_b is π-dense. It follows that there is an onto map $h_b\colon X_b \to Y$ with

$$h_b \overset{\tau}{=} k_b. \tag{4}$$

From (3) and (4), we get the desired conclusion that $f \overset{\sigma}{=} h_b \circ g^b$ for every $b > a$. □

Corollary. *Let $\mathcal{A}$ be a class of locally finite polyhedra without isolated points and let $\mathcal{B}$ be a class of approximate polyhedra. If $\mathcal{A}$ and $\mathcal{B}$ are epiequivalent, then every space X which is both $\mathcal{A}$-like and $\mathcal{B}$-like has an approximate $(\mathcal{O},\ \mathcal{B},\ \mathcal{O})$-resolution.*

Proof. By [**12**, §4, Theorem 3], X has an approximate $(\mathcal{O},\ \mathcal{A},\ \mathcal{O})$-resolution. Now, we may use Theorem 6 to get the conclusion. □

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