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ON PERTURBED ITERATIVE LINEAR DIFFERENTIAL EQUATIONS OF ORDER n

J. MORAVČÍK

Abstract. Some sufficient conditions, under which all solutions of a perturbed iterative linear differential equation of the n-th order tend to zero for $x \to \infty$, are established in this paper.

Consider a linear differential equation of second order

$$u'' + p(x)u = 0 \tag{p}$$

on an interval $I = (a, b)$, $-\infty \le a < b \le \infty$, $p \in C^{n-2}(I)$, $n \ge 3$.

Let u_1, u_2 be linearly independent solutions of (p) on I. Let functions $y_i\colon I \to \mathbb{R} = (-\infty, \infty)$, $i = 1, 2, \dots, n$ be determinated by an identity $y_i(x) = u_1^{i-1}(x)\, u_2^{n-i}(x)$ for all $x \in I$. The fact $u_j \in C^n(I)$, $j = 1, 2$ implies $y_i \in C^n(I)$ for all $i = 1, 2, \dots, n$. It is known (see e.g. [**5**]) that the Wronskian of functions $y_1, y_2, \dots, y_n$ is non-vanishing on I. There exists a linear differential equation of the n-th order such that these functions form its fundamental system of solutions. We shall denote this equation

$$[p]_n(y, x; I) = 0\,. \tag{p$_n$}$$

According to [**1**], [**3**] or [**4**] the equation ($\mathrm{p_n}$) is called the iterative equation of the n-th order. The differential equation (p) is called the accompanying equation of the equation ($\mathrm{p_n}$).

Consider now a differential equation

$$[p]_n(y, x; I) + \delta(x)y = 0 \tag{a}$$

where $\delta\colon I \to \mathbb{R}$, $\delta \in C(I)$, which is obtained by a perturbation of the iterative equation ($\mathrm{p_n}$).

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Let

$$[q]_n(z,t;J) + r(t)z = 0 \tag{b}$$

be a perturbation of the iterative differential equation

$$[q]_n(z,t;J) = 0 \tag{q$_n$}$$

with the accompanying equation

$$\ddot{v} + q(t)v = 0$$

on an interval J.

Definition. We say, that the equation (a) is globally equivalent on I to the differential equation (b) on J, if there exists an ordered pair $\{f, h\}$ of functions such that

a) $f\colon I \to \mathbb{R}$, $f \in C^n(I)$, $f(x) \neq 0$ on I, $h \in C^{n+1}(I)$, $h'(x) \neq 0$ on I, $h(I) = J$;

b) the function $y\colon I \to \mathbb{R}$, $y(x) = f(x)z[h(x)]$ is a solution of the equation (a) on I whenever z is a solution of the equation (b) on J.

It is known that the relation of differential equations in our definition has properties of an equivalence. We shall denote this relation by

$$(a)I \underset{g}{\sim} (b)J\{f(x), h(x)\}. \tag{1}$$

We will need the following lemma.

Lemma 1 ([**3**] or [**4**]). *The relation* (1) *is true if and only if for all* $x \in I$

$$f(x) = C|h'(x)|^{(1-n)/2} \tag{2}$$

where $C \neq 0$ *is a real constant, the function* h *is a solution of the next system of the non-linear equations*

$$\{h(x), x\} + q[h(x)]h'^2(x) = p(x), \quad x \in I \tag{3}$$

$$r[h(x)]h'^n(x) = \delta(x), \quad x \in I \tag{4}$$

where $\{h(x), x\} = h'''(x)/2h'(x) - 3/4[h''(x)/h'(x)]^2$ *is the Schwarzian derivative of the function* h *in a number* x.

Further we shall consider the case when the function $q(t) = k \in \mathbb{R}^+ = (0, \infty)$ identically for all $t \in J$. First we shall prove the following lemma.

Lemma 2. *Let $p \in C^{n-2}(I)$, $\delta \in C^n(I)$, $\delta(x) \neq 0$ on I. Then* (1) *is true if and only if* (2) *is true for all $x \in I$, $r \in C^n(J)$, $r(t) \neq 0$ on J, $\operatorname{sgn} r(t) = \operatorname{sgn} \delta(x)$ and the function h fulfills on I the system of equations*

$$|h'(x)| = |\delta(x)|^{1/n} |r[h(x)]|^{-1/n}, \tag{5}$$

$$\begin{aligned} &\left\{ p(x) + \frac{2n+1}{4n^2} \left[\frac{\delta'(x)}{\delta(x)}\right]^2 - \frac{1}{2n} \frac{\delta''(x)}{\delta(x)} \right\} |\delta(x)|^{-2/n} = \\ &= \left\{ k + \frac{2n+1}{4n^2} \left[\frac{\dot r[h(x)]}{r[h(x)]}\right]^2 - \frac{1}{2n} \frac{\ddot r[h(x)]}{r[h(x)]} \right\} |r[h(x)]|^{-2/n}, \end{aligned} \tag{6}$$

where k is a positive constant, $' = \frac{d}{dx}$, $\dot{} = \frac{d}{dt}$.

Proof. Let (1) be true. When is $\delta(x) \neq 0$ on I, then it follows from (4) that $r[h(x)] \neq 0$ on I, $\operatorname{sgn} r[h(x)] = \operatorname{sgn} \delta(x)$. Since $h(I) = J$, it implies that $r(t) \neq 0$ on J, $\operatorname{sgn} r(t) = \operatorname{sgn} \delta(x)$. The equations (4) and (5) are evidently equivalent on I. From (5) we get by the derivation, step by step (in the case when $\operatorname{sgn} h' = \operatorname{sgn} \delta = \operatorname{sgn} r = 1$; in the another case we can proceed by analogy),

$$\begin{aligned} h''(x) = &\frac{1}{n}[\delta(x)]^{-(n-1)/n} \delta'(x) [r(h(x)]^{-1/n} \\ &- \frac{1}{n}[\delta(x)]^{2/n} [r(h(x))]^{-(n+2)/n} \dot r(h(x)), \end{aligned} \tag{7}$$

$$\begin{aligned} h'''(x) = &\frac{1-n}{n^2}[\delta(x)]^{(1-2n)/n} \delta'^2(x) [r(h(x)]^{-1/n} \\ &+ \frac{1}{n}[\delta(x)]^{(1-n)/n} \delta''(x) [r(h(x))]^{-1/n} \\ &- \frac{3}{n^2}[\delta(x)]^{-(n-2)/n} \delta'(x) [r(h(x))]^{-(n+2)/n} \dot r(h(x)) \\ &+ \frac{n+2}{n^2}[\delta(x)]^{3/n} [r(h(x))]^{-(2n+3)/n} \dot r^2(h(x)) \\ &- \frac{1}{n}[\delta(x)]^{3/n} [r(h(x))]^{-(n+3)/n} \ddot r(h(x)) \,. \end{aligned} \tag{8}$$

Substituting derivatives from (5), (7), (8) in the equation (3), where $q[h(x)] \equiv k$, and making some transformations we obtain (6).

Let inversely $r \in C^n(J)$, $r(t) \neq 0$ on J, $\operatorname{sgn} \delta(x) = \operatorname{sgn} r(t)$ and the function h fulfills the system of equations (5) and (6) on I. In view of these assumptions each solution h of the equation (5) has the next properties:

1) $h \in C^n(I)$, $h'(x) \neq 0$ and h' fulfills the equation (4) on I;

2) when the function h fulfills the equation (6) on I, then this function fulfills also the equation (3), because both equations are equivalent.

Hence all assumptions of the Lemma 1 are fulfilled. That means that the relation (1) holds.

This completes the proof. □

We can now establish the main result.

Theorem. *Let $J = (\alpha, \infty)$, $\alpha \geq -\infty$; and let $r \in C^n(J)$ be a real function with the next properties:*

1) $r(t) \neq 0$ *on* J *and* $\lim_{t\to\infty} r(t) = 0$;

2) $\int\limits_{t_0}^{\infty} |r(s)|\, ds < \infty$, $t_0 > \alpha$;

3) $\int\limits_{t_0}^{\infty} |r(s)|^{1/n}\, ds = \infty$ *and*

$$R(t) = \int_{t_0}^{t} |r(s)|^{1/n}\, ds, \quad t \in J\,. \tag{9}$$

In addition let the coefficients of differential equation (a) *satisfy the following conditions:*

(i) $p \in C^{n-2}(I)$, $\delta \in C^n(I)$, *where* $I = (a, \infty)$, $a \geq -\infty$, $\operatorname{sgn} \delta(x) = \operatorname{sgn} r(t)$ *and either*

(ii) $\delta(x) > 0$ *on* I *and* $\liminf_{x\to\infty} \delta(x) > 0$ *or*

(iii) $\delta(x) < 0$ *on* I *and* $\limsup_{x\to\infty} \delta(x) < 0$.

If the function $h\colon I \to \mathbb{R}$,

$$h(x) = R_{-1}\left(\int_{x_0}^{x} |\delta(s)|^{1/n}\, ds\right), \quad x_0 \in I \tag{10}$$

where R_{-1} means the inverse function to R, fulfills the equation (6) *on I, then all solutions of the differential equation* (a) *tend to zero for $x \to \infty$.*

Proof. The function R defined by (9) is obviously increasing on J and according to the property 3) it transforms the interval $\langle t_0, \infty)$ on the interval $\langle x_0, \infty)$. The function h given by (10) is therefore defined on I and transforms the interval $\langle x_0, \infty)$ on the interval $\langle t_0, \infty)$ because the condition (ii) or (iii) implies that the integral $\int_{x_0}^{\infty} |\delta(s)|^{1/n}\, ds$ is divergent. Further for the function h we obtain:

$$\begin{aligned} h'(x) &= R'_{-1}\left(\int_{x_0}^{x} |\delta(s)|^{1/n} ds\right) |\delta(x)|^{1/n} = \\ &= \dot{R}(t)^{-1}\Big|_{t=h(x)} |\delta(x)|^{1/n} = |\delta(x)|^{1/n} |r(h(x))|^{-1/n}\,. \end{aligned}$$

This means, that $h'(x) > 0$ for all $x \in I$ and the function h fulfills the equation (5) on I. When we put for any $x \in I$

$$f(x) = C|h'(x)|^{(1-n)/2} = C|\delta(x)|^{(1-n)/2n}|r[h(x)]|^{(n-1)/2n}, \tag{11}$$

where $C \neq 0$ is a real constant; then $f \in C^n(I)$, $f(x) \neq 0$ on I and the relation (2) holds, because $r \in C^n(J)$, $r(t) \neq 0$ on J, $\delta \in C^n(I)$, $\delta(x) \neq 0$ on I and therefore $h \in C^{n+1}(I)$. Since according to the assumption the function h satisfies also (6) on I, all assumptions of the Lemma 2 are fulfilled. It means that (1) holds.

Let y be now an arbitrary non-trivial solution of the differential equation (a) on I. Then there exists the solution z of the differential equation (b), where $q(t) \equiv k$ on J, such that $y(x) = f(x)z[h(x)]$ holds and according to (11) for all $x \in I$ is

$$y(x) = C|\delta(x)|^{(1-n)/2n}|r[h(x)]|^{(n-1)/n} z[h(x)], \tag{12}$$

where $C \neq 0$ is a real constant.

In this case the equation

$$\ddot{v} + kv = 0, \quad k \in \mathbb{R}^+. \tag{13}$$

is the accompanying differential equation of the differential equation ($\mathrm{q_n}$). The general solution of (13) is

$$v(t) = k_1 \cos\sqrt{k}t + k_2 \sin\sqrt{k}t\,,$$

where k_1, k_2 are real constants. Hence all non-trivial solutions of the differential equation (13) are bounded and oscillatory on $(-\infty, \infty)$ and also on any J. According to the property 2) of the function r and by the Theorem 1 of the paper [**2**] all solutions of the differential equation (b) are bounded on the interval $\langle t_0, \infty)$. With regard to the property 1) it is $\lim_{t\to\infty} |r(t)|^{1/n} = 0$ too. That means, of course, that also $\lim_{x\to\infty} |r(h(x))|^{1/n} = 0$. In view of assumption (ii) or (iii) the function $|\delta(x)|^{(1-n)/2n}$ is also bounded for sufficiently large x and therefore, according to the mentioned above, from (12) we get

$$\lim_{x\to\infty} y(x) = 0\,.$$

This completes the proof. □

We can further prove that this theorem implies the next corollary.

Corollary 1. *Let $\delta\colon I \to \mathbb{R}$, $I = (a, \infty)$, $a \geq -\infty$ be a function such that the following conditions are fulfilled:*

(i) $\delta \in C^n(I)$ *and either*
(ii) $\delta(x) > 0$ *for all* $x \in I$ *and* $\liminf_{x\to\infty} \delta(x) > 0$ *or*
(iii) $\delta(x) < 0$ *for all* $x \in I$ *and* $\limsup_{x\to\infty} \delta(x) < 0$.

Let further $p\colon I \to \mathbb{R}$ *be a function such that for any* $x \in I$

$$p(x) = \frac{1}{2n}\frac{\delta''(x)}{\delta(x)} - \frac{2n+1}{4n^2}\left[\frac{\delta'(x)}{\delta(x)}\right]^2 + \left[k\exp(2\int_{x_0}^{x}|\delta(s)|^{1/n}ds) - \frac{1}{4}\right]|\delta(x)|^{2/n}, \quad x_0 \in I, \tag{14}$$

where k *is a positive constant.*

Then all solutions of the differential equation (a) *tend to zero for* $x \to \infty$.

Proof. Consider the function $r(t) = t^{-n}$ or $r(t) = -t^{-n}$. Obviously this function satisfies the assumptions of Theorem. Then from (9) we obtain

$$R(t) = \int_{t_0}^{t} s^{-1}\,ds = \ln\frac{t}{t_0}\,.$$

The function $u = R(t) = \ln\frac{t}{t_0}$ transforms the interval $\langle t_0, \infty)$ on the interval $\langle 0, \infty)$ and $t = R_{-1}(u) = t_0 \exp u$. Then from (10) we get

$$h(x) = t_0 \exp\left(\int_{x_0}^{x}|\delta(s)|^{1/n}\,ds\right), \quad x \in \langle x_0, \infty) \tag{15}$$

and in view of $r(t) = t^{-n}$ or $r(t) = -t^{-n}$ we have

$$|r(h(x))| = t_0^{-n}\exp\left(-n\int_{x_0}^{x}|\delta(s)|^{1/n}ds\right). \tag{16}$$

When we put for any $t \in \langle t_0, \infty)$

$$V[r(t)] = \left\{k + \frac{2n+1}{4n^2}\left[\frac{\dot r(t)}{r(t)}\right]^2 - \frac{1}{2n}\frac{\ddot r(t)}{r(t)}\right\}|r(t)|^{-2/n},$$

then for $r(t) = t^{-n}$ or $r(t) = -t^{-n}$ we get

$$V[r(t)] = kt^2 - \frac{1}{4}\,.$$

Hence for the function h defined by (15) we obtain

$$V[r(h(x))] = kt_0^2\exp\left(2\int_{x_0}^{x}|\delta(s)|^{1/n}\,ds\right) - \frac{1}{4}\,. \tag{17}$$

Choosing in the special way $t_0 = 1$ and substituting (17) in the second side of (6) we get (14). This means that the function (15) satisfies the equation (6). With regard to (15) and (16) we can easy find out that the function h fulfills the equation (5) on I too. Because $\delta \in C^n(I)$, according to (14), the function $p \in C^{n-2}(I)$. Hence all assumptions of the Theorem are fulfilled. This completes the proof. □

Further we can deduce from the proof of Theorem and from the known criterion of the global equivalence of the linear differential equations of the n-th order ([**4,** Th. 5.2.1]) the following result for the equation (p).

Corollary 2. *Let $n \geq 3$ be a integer and $r\colon J \to \mathbb{R}$, $J = (\alpha, \infty)$, $\alpha \geq -\infty$ be a function with the next properties:*

1) $r(t) \neq 0$ *on* J; $r \in C^n(J)$;
2) $\int_{t_0}^{\infty} |r(s)|\, ds < \infty$, $t_0 > \alpha$;
3) $\int_{t_0}^{\infty} |r(s)|^{1/n}\, ds = \infty$.

Let $R\colon J \to \mathbb{R}$ be the function defined by the relation (9).

Let further $\delta\colon I \to \mathbb{R}$, $I = (a, \infty)$, $a \geq -\infty$ be a function with the following properties:

(i) $\delta(x) \neq 0$ *on* I, $\delta \in C^n(I)$, $\operatorname{sgn} \delta(x) = sgn\ r(t)$ *and either*
(ii) $\delta(x) > 0$ *on* I *and* $\liminf_{x\to\infty} \delta(x) > 0$ *or*
(iii) $\delta(x) < 0$ *on* I *and* $\limsup_{x\to\infty} \delta(x) < 0$,

$h\colon I \to \mathbb{R}$ be the function given by the relation (10) *and $p\colon I \to \mathbb{R}$ be defined by this relation*

$$p(x) = \left\{ \frac{1}{2n} \frac{\delta''(x)}{\delta(x)} - \frac{2n+1}{4n^2} \left[\frac{\delta'(x)}{\delta(x)} \right]^2 \right\} |\delta(x)|^{-2/n} + \left\{ k + \frac{2n+1}{4n^2} \left[\frac{\dot r(h(x))}{r(h(x))} \right]^2 - \frac{1}{2n} \frac{\ddot r(h(x))}{r(h(x))} \right\} |r(h(x))|^{-2/n},$$

where k is a positive constant, $' = \frac{d}{dx}$, $\dot{} = \frac{d}{dt}$.

Then the equation (p) *is oscillatory on the interval I.*

Proof. Let $f\colon I \to \mathbb{R}$ be given by the relation (11). It follows from the proof of Theorem that at our assumptions the relation (1) holds for the equation (b), where $q(t) \equiv k$. By the Theorem 5.2.1 of the monograph [**4**] we obtain from this, that for any non-trivial solution u of the equation (p) on I holds the relation

$$u(x) = A[h'(x)]^{-1/2} v[h(x)], \tag{18}$$

where $A \neq 0$ is constant, v is a non-trivial solution of the equation (13). But all non-trivial solutions of the equation (13) are oscillatory on J. With regard to (18) the equation (p) has the same property on I. This completes the proof. □

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J. Moravčík, University of Transport and Communications, 010 26 Žilina, Slovakia