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QUASI POLYMATROIDAL FLOW NETWORKS

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Abstract. In this paper we give a flow model on directed multigraphs by introducing reflexions of generalized polymatroids at vertices as constraints for the flow conservation. This model has the essential features of the classical flow model, primarily the max-flow min-cut theorem and the polynomial algorithm for computing the maximal feasible (integral) flow.

1. Introduction

Since the classical network flow model of Ford and Fulkerson appeared in the 1950's, numerous generalizations and variations of this model have been introduced. Very interesting is the polymatroidal network flow model introduced by Lawler and Martel [**12**] and Hassin [**9**] which provides a generalization and unification of both network flow theory and much of the theory of polymatroid optimization (see [**13**]).

Note that (integral) polymatroids are polyhedra of nonnegative vectors bounded by (integral) submodular function. They have been introduced by Edmonds [**2**] as generalizations of matroids. (Integral) polymatroids and the polyhedra arising as intersections of two (integral) polymatroids play a very important role in (integral) optimization. Further generalizations of polymatroids are generalized polymatroids, that are polyhedra bounded by sub- and supermodular functions having an additional property. They were introduced by Frank [**4**]. A comprehensive survey of their properties can be found in Frank and Tardos [**6**].

A **polymatroidal flow network** $\mathcal{F}$ (see Lawler and Martel [**12**]) is a directed multigraph G with a source s, a sink t and for any vertex v of G we have two polymatroids $\mathbb{P}_v^+$, $\mathbb{P}_v^-$ on the set of arcs entering (leaving) v respectively. A flow f is said to be **feasible** in $\mathcal{F}$ if for any vertex v of G, the vector whose coordinates are the values of f on the arcs entering (leaving) v is independent in $\mathbb{P}_v^+$ ($\mathbb{P}_v^-$), (i.e., the vector is an element of the polytope of this polymatroid). Furthermore, $\mathcal{F}$ is called **integral** if $\mathbb{P}_v^+$, $\mathbb{P}_v^-$ are integral for any vertex v of G.

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In [**12**] it was proved that this model has the max-flow min-cut property and a polynomial algorithm that computes a maximal (integral) flow in an (integral) polymatroidal flow network was introduced. Hassin [**9**] dealt with a circulant variant of this model. Another generalization of this model was introduced by Lawler and Martel [**14**] (see also Tardos, Tovey and Trick [**22**]). As pointed out in [**13**], polymatroidal network flow model is a very useful theoretical and practical tool in combinatorial optimization and many polymatroid optimization problems can be easily formulated and solved in the terms of this flow model. Other models of combinatorial optimization equivalent with the polymatroidal network flow model are surveyed in Schrijver [**20**].

An undirected analogue of the polymatroidal network flow model is in fact the Matchoid problem (see Lovász and Plummer [**16**]). This model has its fount in matching theory and was originated by Edmonds (see [**16**] for more details).

The matchoid problem: Let G be an undirected graph such that for any vertex v of G a matroid M_v is given on the set of the edges adjacent with v. Call a set M of edges a matchoid if, for any vertex v of G, the set of the edges of M adjacent with v is independent in M_v. Find a matchoid with the maximum cardinality.

Lovász [**15**] proved that the matchoid problem is NP-hard and that every oracle algorithm for this problem has exponential complexity. This is in contrast with algorithms for computing the integral polymatroidal flow networks.

Let us stress the fact that in the polymatroidal network flow model there are no correlations between the constraints imposed on the set of arcs leaving and the set of arcs entering a vertex (in other words, $\mathbb{P}_v^+$ and $\mathbb{P}_v^-$ are independent of each other for any vertex v). Any feasible flow must satisfy only a natural condition that, for any vertex v different from the source and the sink, the sum of the values of f on the arcs entering v is equal to the sum of the values of f on the arcs leaving v. Flow models with this property we shall call **balanced**.

It is natural to ask the following question in connection with the two above models: Does there exist a reasonable flow model on digraphs (i.e., a model with polynomial algorithm for computing maximal feasible "integral" flow) in which the whole neighbourhood of each vertex (no matter on orientation) will be constrained by a single polymatroid or another relative polyhedra?

In this paper we answer this question affirmatively and introduce a network flow model on digraphs such that a flow f is feasible if for any vertex v the vector f_v is independent in a given generalized polymatroid, where f_v is the direct sum of two vectors f_v^+ and $-f_v^-$ such that the coordinates of f_v^+ (f_v^-) are the values of the flow f on the arcs entering (leaving) v. This model will be called the quasi polymatroidal network flow model. We show that it also has the essential features of the classical flow model (the max-flow min-cut theorem and polynomial algorithm for computing the maximal feasible "integral" flow).

Our flow model has a common feature with the well-known concept of "nowhere-zero flows" on graphs (see, e.g., Jaeger [**10**] for a survey of this topic). It consists in the fact, that if we interchange the orientation of an arc in a quasi polymatroidal flow network, we get a new flow network of the same type and, in a certain sense, equivalent with the original flow network. A more detailed discussion about this matter will be given in the fourth section.

As pointed out earlier the polymatroidal network flow model of Lawler and Martel is balanced. Our flow model is not balanced in general. This is also of some interest with respect to the fact that it is known that there exist unbalanced flow models on digraphs for which it is NP-hard to compute the maximal integral flow (see, e.g., Sahni [**19**] for the integral network flows with multipliers).

Note that our flow model is equivalent with the flow models of Lawler and Martel [**11**], [**13**] and Edmonds and Giles [**3**]. That means that any problem formulated in one model can be formulated in some of the others, though this sometimes requires certain effort. The choice of one model over the others is a matter of aesthetics and ease of applications. Quasi polymatroidal network flow model is the most suitable model for formulation of several results introduced in an accompanied paper [**11**]. If we would formulate these results in other flow models we get very clumsy and unhandy statements. This fact justifies the introduction of the quasi polymatroidal network flow model.

We suppose the reader to be familiar with the theory of matroids, polymatroids, submodular functions and flows. The main literature are the books of Welsh [**23**] and Fujishige [**7**] and the survey articles of Frank and Tardos [**6**] and Schrijver [**20**].

The first two sections are of preliminary character. The main results are introduced in the third part. In the last section are discussed connections of our flow model with other known models.

2. Preliminaries

Throughout this paper let S denote a finite set and $\mathbb{R}^S$ ($\mathbb{Z}^S$) denote the collection of the real (integer) valued vectors indexed by S. For each $\mathbf{x} \in \mathbb{R}^S$ and $s \in S$ denote the s-th coordinate of $\mathbf{x}$ by $\mathbf{x}(s)$. If $\mathbf{x} \in \mathbb{R}^S$ and $A \subseteq S$, $\mathbf{x}(A)$ is defined to be $\sum_{s\in A} \mathbf{x}(s)$, and $\mathbf{x}|A$ denotes the restriction of $\mathbf{x}$ to A. Call the modulus $|\mathbf{x}|$ of $\mathbf{x}$ the quantity

$$|\mathbf{x}| = \mathbf{x}(S) = \sum_{s\in S} \mathbf{x}(s)\,.$$

For two vectors $\mathbf{x} \in \mathbb{R}^S$ and $\mathbf{x}' \in \mathbb{R}^{S'}$ with $S \cap S' = \emptyset$, their **direct sum** $\mathbf{x} \oplus \mathbf{x}' \in \mathbb{R}^{S\cup S'}$ is defined by

$$(\mathbf{x} \oplus \mathbf{x}')(s) = \begin{cases} \mathbf{x}(s) & \text{if } s \in S, \\ \mathbf{x}'(s) & \text{if } s \in S'. \end{cases}$$

If it is clear from the context that we are referring to a set rather than an element we abbreviate $\{x\}$ to x. For example $X \cup x$ means $X \cup \{x\}$ and $\rho(x)$ means $\rho(\{x\})$.

A **generalized polymatroid** (in abbreviation g-**polymatroid**) $\mathbb{P}$ on S is a triple (S, ρ, σ) where S is the **ground set** and ρ, σ are functions $\rho\colon 2^S \to \mathbb{R} \cup \{\infty\}$, $\sigma\colon 2^S \to \mathbb{R} \cup \{-\infty\}$ such that $\rho(\emptyset) = \sigma(\emptyset) = 0$ and, for any $X, Y \subseteq S$,

$$\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y), \tag{1}$$

$$\sigma(X) + \sigma(Y) \leq \sigma(X \cup Y) + \sigma(X \cap Y), \tag{2}$$

$$\rho(X) - \sigma(Y) \geq \rho(X \setminus Y) - \sigma(Y \setminus X). \tag{3}$$

((1) and (2) states that ρ and σ are **submodular** and **supermodular** respectively.) If both ρ and σ are integer valued then $\mathbb{P}$ is called **integral**. A vector $\mathbf{u} \in \mathbb{R}^S$ is called an **independent vector** of $\mathbb{P}$ if $\sigma(X) \leq \mathbf{u}(X) \leq \rho(X)$ for any $X \subseteq S$.

If $\sigma \equiv 0$ then, by (3), ρ is monotone (i.e., $\rho(X) \leq \rho(Y)$ if $X \subseteq Y$) and $\mathbb{P}$ is called a **polymatroid**. Furthermore, if $\rho(x) = 0$, 1, for any $x \in S$, and ρ is integral, then $\mathbb{P}$ is a **matroid**. If $\sigma(X) = -\infty$ for any $\emptyset \neq X \subseteq S$, we get an **extended polymatroid** (see [**8**] or [**21**]). If $\mathbb{P}$ is matroid, polymatroid or extended polymatroid then it is denoted as a couple (S, ρ).

As pointed out in [**6**] the set of (integral) independent vectors of a nontrivial (integral) g-polymatroid is nonempty and, for any $X \subseteq S$,

$$\rho(X) = \max\{\mathbf{u}(X);\ \mathbf{u} \text{ is independent in } \mathbb{P}\}, \tag{4}$$

$$\sigma(X) = \min\{\mathbf{u}(X);\ \mathbf{u} \text{ is independent in } \mathbb{P}\}. \tag{5}$$

If $S' = S \cup s'$, $s' \notin S$ and $\rho'\colon 2^{S'} \to \mathbb{R} \cup \{\infty\}$ such that

$$\rho'(X) = \begin{cases} \rho(X) & \text{if } X \subseteq S, \\ 0 - \sigma(S \setminus X) & \text{if } s' \in X \subseteq S', \end{cases} \tag{6}$$

then, by (1)–(3), ρ' is submodular. The extended polymatroid $\mathbb{P}'' = (S', \rho')$ is called the **primitive** 0**-extension of** $\mathbb{P}$ **to** S'.

Futhermore, if we take $\sigma'\colon 2^{S'} \to \mathbb{R} \cup \{-\infty\}$ such that

$$\sigma'(X) = \begin{cases} \sigma(X) & \text{if } X \subseteq S, \\ 0 - \rho(S \setminus X) & \text{if } s' \in X \subseteq S', \end{cases} \tag{7}$$

then we can check that σ' is supermodular and $\mathbb{P}' = (S, \rho', \sigma')$ is a g-polymatroid. We call it the 0**-extension of** $\mathbb{P}$ **to** S'. Clearly $\mathbf{u} \in \mathbb{R}^{S'}$ is independent in $\mathbb{P}'$ iff $\mathbf{u}|S$ is independent in $\mathbb{P}$ and $\mathbf{u}(s') = -\mathbf{u}(S)$.

The set of independent vectors of $\mathbb{P}$ is in fact the projection on the principal face of the set of independent vectors of its primitive 0-extension (see [**4**], [**20**] for more details).

Let $\rho_\infty(\emptyset) = \sigma_\infty(\emptyset) = 0$ and $\rho_\infty(X) = \infty$, $\sigma_\infty(X) = -\infty$ for any $\emptyset \neq X \subseteq S$. Then the g-polymatroid $(S, \rho_\infty, \sigma_\infty)$ is called the **free g-polymatroid** on S and is denoted by $\mathbb{F}_{\infty,S}$. The set of independent vectors of $\mathbb{F}_{\infty,S}$ is the whole $\mathbb{R}^S$.

By a **principal g-polymatroid** $\mathbb{F}_{0,\{a,b\}}$ on a two-element set $\{a,b\}$ we mean the 0-extension of $\mathbb{F}_{\infty,a}$ (or, equivalently, of $\mathbb{F}_{\infty,b}$) to $\{a,b\}$. Then $\mathbf{x} = (x_a, x_b) \in \mathbb{R}^{\{a,b\}}$ is independent in $\mathbb{F}_{0,\{a,b\}}$ if and only if $x_a = -x_b$, $x_a, x_b \in \mathbb{R}$.

In [**4**] is proved:

Theorem 1. *Let $\mathbb{P}_1 = (S, \rho_1, \sigma_1)$ and $\mathbb{P}_2 = (S, \rho_2, \sigma_2)$ be two g-polymatroids. Then the linear system*

$$\sigma_i(A) \leq \mathbf{x}(A) \leq \rho_i(A) \quad \text{for } i = 1, 2, \quad \text{and} \quad A \subseteq S \tag{8}$$

is totally dual integral.

This theorem generalizes the polymatroid intersection theorem of Edmonds [**2**]. More precisely (see also [**4**]):

Corollary 1. *Let $\mathbb{P}_1 = (S, \rho_1, \sigma_1)$ and $\mathbb{P}_2 = (S, \rho_2, \sigma_2)$ be two g-polymatroids and let there exists $\mathbf{x} \in \mathbb{R}^S$ independent in both $\mathbb{P}_1$ and $\mathbb{P}_2$. Then the maximal modulus of a vector $\mathbf{u}$ independent in both $\mathbb{P}_1$ and $\mathbb{P}_2$ is equal to*

$$\min_{X \subseteq S} \big(\rho_1(X) + \rho_2(S \setminus X)\big), \tag{9}$$

and the minimal modulus of a vector $\mathbf{v}$ independent in both $\mathbb{P}_1$ and $\mathbb{P}_2$ is equal to

$$\max_{X \subseteq S} \big(\sigma_1(X) + \sigma_2(S \setminus X)\big). \tag{10}$$

Furthermore, if $\mathbb{P}_1$ and $\mathbb{P}_2$ are both integral we may insist that the vectors $\mathbf{u}$, $\mathbf{v}$ be integral.

Proof. From Theorem 1 it follows that

$$\begin{aligned}
&\max\big\{|\mathbf{u}|;\ \mathbf{u} \text{ is independent in both } \mathbb{P}_1 \text{ and } \mathbb{P}_2\big\} \\
&\quad = \max\big\{|\mathbf{x}|;\ \sigma_i(A) \leq \mathbf{x}(A) \leq \rho_i(A) \quad (i = 1, 2,\ A \subseteq S)\big\} \\
&\quad = \min\left\{ \sum_{i=1,2,\ A \subseteq S} \rho_i(A) y_{i,A} - \sigma_i(A) z_{i,A};\ y_{i,A}, z_{i,A} \geq 0 \ \ (i = 1, 2,\ A \subseteq S) \right. \\
&\qquad\qquad \left. \sum_{i=1,2,\ A \ni s} y_{i,A} - \sum_{i=1,2,\ A \ni s} z_{i,A} = 1 \ \ (s \in S) \right\}.
\end{aligned}$$

In [**4**] it is proved that the last linear programming problem has an optimal solution (if it has one at all) for which the families $\mathcal{L}_i = \{A; y_{i,A} > 0 \text{ or } z_{i,A} > 0\}$ $(i = 1, 2)$ are laminar (i.e., if $A, B \in \mathcal{L}_i$ then $A \subseteq B$ or $B \subseteq A$). From (1), (2) and (3) follows that both $\mathcal{L}_1$, $\mathcal{L}_2$ can be chosen to be singletons, in other words $\mathcal{L}_1 = \{A\}$ and $\mathcal{L}_2 = \{S \setminus A\}$ where $A \subseteq S$ and (9) holds. Similarly can be proved (10). If $\mathbb{P}_1$ and $\mathbb{P}_2$ are integral then $\mathbf{u}$, $\mathbf{v}$ can be chosen to be integer valued. □

Corollary 2. *Let $\mathbb{P}_1 = (S, \rho_1, \sigma_1)$ and $\mathbb{P}_2 = (S, \rho_2, \sigma_2)$ be two (integral) g-polymatroids. Then they have a common (integral) independent vector iff for any $X \subseteq S$,*

$$\rho_1(X) \geq \sigma_2(X) \quad \textit{and} \quad \rho_2(X) \geq \sigma_1(X). \tag{11}$$

Proof. Let $\mathbb{P}_1'' = (S', \rho_1')$ and $\mathbb{P}_2'' = (S', \rho_2')$ be the primitive 0-extensions of $\mathbb{P}_1$ and $\mathbb{P}_2$ to S' $(S' = S \cup s', s' \notin S)$, respectively. Then $\mathbb{P}_1$ and $\mathbb{P}_2$ have a common (integral) independent vector iff $\mathbb{P}_1''$ and $\mathbb{P}_2''$ have a common (integral) independent vector with the modulus equal to 0 and, by Corollary 1, this occurs iff (11) holds for any $X \subseteq S$. □

If $\mathbb{P}_i = (S_i, \rho_i, \sigma_i)$ $(i \in I,\ I$ finite) are g-polymatroids and $S_i \cap S_j = \emptyset$ for any $i \neq j$, then let $S = \bigcup_{i \in I} S_i$ and $\rho\colon 2^S \to \mathbb{R} \cup \{\infty\}$, $\sigma\colon 2^S \to \mathbb{R} \cup \{-\infty\}$ such that for any $X \subseteq S$

$$\rho(X) = \sum_{i \in I} \rho_i(X \cap S_i), \qquad \sigma(X) = \sum_{i \in I} \sigma_i(X \cap S_i).$$

Then $\mathbb{P} = (S, \rho, \sigma)$ is a g-polymatroid. Clearly $\mathbf{u} \in \mathbb{R}^S$ is independent in $\mathbb{P}$ iff $\mathbf{u}|S_i$ is independent in $\mathbb{P}_i$ for any $i \in I$. We call $\mathbb{P}$ the **product** (or **direct sum**) of the g-polymatroids $\mathbb{P}_i$ $(i \in I)$ and denote it by $\bigoplus_{i \in I} \mathbb{P}_i$. If $\mathbb{P}_i$ $(i \in I)$ are integral then also $\mathbb{P}$ is integral.

If $\mathbb{P} = (S, \rho, \sigma)$ is a g-polymatroid, then $-\mathbb{P} = (S, -\sigma, -\rho)$ is also a g-polymatroid. Clearly $\mathbf{u}$ is independent in $\mathbb{P}$ iff $-\mathbf{u}$ is independent in $-\mathbb{P}$.

A **quasi polymatroid** (in abbreviation ***q*-polymatroid**) $\mathbb{Q}$ on the sets S_1 and S_2 is an ordered quadruple (S_1, S_2, ρ, σ) such that S_1 and S_2 are finite and disjoint sets and $\mathbb{P} = (S_1 \cup S_2, \rho, \sigma)$ is a g-polymatroid. If $\mathbf{u}_1 \in \mathbb{R}^{S_1}$, $\mathbf{u}_2 \in \mathbb{R}^{S_2}$, then the vector $\mathbf{u}_1 \oplus \mathbf{u}_2$ is said to be an **independent vector** of $\mathbb{Q}$ if $\mathbf{u}_1 \oplus (-\mathbf{u}_2)$ is independent in $\mathbb{P}$. $\mathbb{P}$ is called the **underlying** g-polymatroid of $\mathbb{Q}$. Furthermore, $\mathbb{Q}$ is called **integral** if ρ and σ are integer valued. The set of independent vectors of $\mathbb{Q}$ is the image of the set of independent vectors of $\mathbb{P}$ under the reflexion of coordinates of S_2. Clearly, if $S_2 = \emptyset$ then $\mathbb{Q} = \mathbb{P}$ and if $S_1 = \emptyset$ then $\mathbb{Q} = -\mathbb{P}$. Thus any g-polymatroid is a q-polymatroid. On the other hand we can check that the set $\{(x, y); x = y\} \subseteq \mathbb{R}^2$ is a q-polymatroid but not a g-polymatroid.

Note that until now we have not found the concept of q-polymatroids in the literature. But there are well studied polyhedra known as **universal polymatroids** (see Nakamura [**17**]), which can be characterized as the convex sets such that the greedy algorithm always works on them. Further details about universal polymatroids or similar concepts can be found in [**17**], [**7**] or [**1**].

From the results of Nakamura [**17**] follows that any q-polymatroid is in fact a universal polymatroid. But the opposite implication does not hold. For instance we can check that the convex hull of $\{\pm\mathbf{e}_1, \pm\mathbf{e}_2\}$ is a universal polymatroid in $\mathbb{R}^2$ but not a q-polymatroid. Thus q-polymatroids form a proper subclass of the class of universal polymatroids and g-polymatroids form a proper subclass of the class of q-polymatroids.

Note that, by Grötschel, Lovász and Schrijver [**8**], there exists a polynomial algorithm that finds the maximum of any objective function over the polytope arising as intersection of finite number of q-polymatroids (or universal polymatroids). But to find optimal integral vector independent in a g-polymatroid and a q-polymatroid is NP-hard, because in [**1**] is in fact proved that it can be reduced to the matchoid problem. On the other hand, if $\mathbb{Q}_1$ and $\mathbb{Q}_2$ are integral q-polymatroids on the same couple of sets S_1 and S_2, then, by Theorem 1, this problem becomes polynomially solvable.

3. Quasi Polymatroidal Flow Networks

By a digraph $G = (V, E)$ we mean a connected oriented graph with multiple arcs and without oriented loops. If $U \subseteq V$, then by Δ_U^+ (Δ_U^-) we mean the set of arcs oriented from $V \setminus U$ into U (from U into $V \setminus U$, respectively). Further $\Delta_U = \Delta_U^+ \cup \Delta_U^-$. We write Δ_v^+, Δ_v^- and Δ_v if $U = \{v\}$. Otherwise we use standard graph theoretic terms.

A **q-polymatroidal flow network** $\mathcal{F}$ is a digraph $G = (V, E)$ with a **source** s, a **sink** t and a collection of q-polymatroids $\mathbb{Q}_v = (\Delta_v^+, \Delta_v^-, \rho_v, \sigma_v)$. We call $\mathcal{F}$ **integral** if any $\mathbb{Q}_v$ is integral. A **flow** in the network $\mathcal{F}$ is a function $f\colon E \to \mathbb{R}$. Since f is in fact a vector of $\mathbb{R}^E$ we can use the notation introduced for vectors also for flows (e.g. $f(X)$, $f(e)$ for any $X \subseteq E$, $e \in E$, and similarly). If a flow f in $\mathcal{F}$ is integer valued we call it **integral**. A flow f in $\mathcal{F}$ is said to be **feasible** in $\mathcal{F}$ if, for any $v \in V$, $f|\Delta_v$ is independent in $\mathbb{Q}_v$, i.e., for any $X \subseteq \Delta_v$,

$$\sigma_v(X) \le f(X \cap \Delta_v^+) - f(X \cap \Delta_v^-) \le \rho_v(X).$$

We did not allow oriented loops in G. This restriction was given because we want to have $\Delta_v^+ \cap \Delta_v^- = \emptyset$ for any vertex v of G. But this restriction is not substantial and can be avoided such that an oriented loop $e = (v, v)$ is replaced by two arcs (v, v_e), (v_e, v) where v_e is a vertex with out- and indegree one and the underlying g-polymatroid of $\mathbb{Q}_{v_e}$ is the principal g-polymatroid (i.e., $f(v, v_e) = f(v_e, v)$ for any feasible flow f).

It is usual that the flow model satisfy $f(\Delta_v^+) = f(\Delta_v^-)$ for any $v \in V$, $v \neq s, t$. But this condition does not need to be satisfied for any q-polymatroidal flow network. Therefore we introduce another additional definition.

A q-polymatroidal flow network $\mathcal{F}$ is called **balanced**, if $\rho_v(\Delta_v) = \sigma_v(\Delta_v) = 0$ for any $v \in V$, $v \neq s, t$. In other case $\mathcal{F}$ is called **unbalanced**. Clearly if $\mathcal{F}$ is balanced, then any feasible flow has the property that $f(\Delta_v^+) = f(\Delta_v^-)$ for any $v \in V$, $v \neq s, t$.

If U, W is a partition of the set of vertices V into two parts and f is a feasible flow in $\mathcal{F}$ then denote the (U, W)**-value** of f the quantity

$$v_{f,U,W} = f(\Delta_U^-) - f(\Delta_U^+).$$

If $\mathcal{F}$ is not balanced, then $v_{f,U,W}$ can have different values for different partitions U, W. If $\mathcal{F}$ is balanced, then $v_{f,U,W}$ has the same value for any partition U, W such that $s \in U$, $t \in W$. Then we denote this common value by v_f and call it the **value** of f.

An ordered quadruple $\mathcal{C} = (U, W, A, B)$ is defined to be a **complete cut** of G if the couple U, W is a partition of the set of vertices into two parts and the couple A, B is a partition of the set of arcs into two parts. The **upper capacity** of the complete cut (U, W, A, B) is defined as

$$c_{\mathrm{up}}(U, W, A, B) = \sum_{v \in U} -\sigma_v(\Delta_v \cap A) + \sum_{v \in W} \rho_v(\Delta_v \cap B).$$

The **lower capacity** of the complete cut (U, W, A, B) is defined as

$$c_{\mathrm{low}}(U, W, A, B) = \sum_{v \in U} -\rho_v(\Delta_v \cap A) + \sum_{v \in W} \sigma_v(\Delta_v \cap B).$$

We can check that

$$c_{\mathrm{up}}(U, W, A, B) = -c_{\mathrm{low}}(W, U, B, A). \tag{12}$$

Note that this definition makes sense if some of the sets U, W, A, B are empty.

A complete cut (U, W, A, B) we shall also call a (U, W)**-cut** if we want to stress that it contains the partition of the set of vertices U, W.

Similarly as in the classical flow model the minimal upper capacity of a complete cut will be equal to the maximal value of a feasible flow and, by the symmetry, the maximal lower capacity will be equal to the minimal value of a feasible flow. That is why we have introduced the unusual lower capacity. In the classical model the analogue of the lower capacity was equal to 0, and this is trivially a lower bound for the value of a feasible flow. But from symmetry it follows that the problem to find the minimal value of a flow in a q-polymatroidal flow network is as difficult problem as to find the maximal value.

We will formulate our results in the general case (i.e. for unbalanced networks) and separately for balanced networks (these results will be presented as corollaries of the general case). For the balanced network flows we shall use the following lemma.

Lemma 1. *Let $\mathcal{F}$ be a balanced q-polymatroidal flow network on digraph $G = (V, E)$ with a source s, a sink t and a collection of q-polymatroids $\mathbb{Q}_v$, $v \in V$. Let $\{U, W\}$ and $\{A, B\}$ be partitions of V and E, respectively. Then*

$$c_{\text{up}}(U, W, A, B) = c_{\text{up}}(U \setminus u, W \cup u, A, B),$$
$$c_{\text{low}}(U, W, A, B) = c_{\text{low}}(U \setminus u, W \cup u, A, B)$$

for any $u \in U$, $u \neq s, t$.

Proof. Since $\mathcal{F}$ is balanced and $u \neq s, t$ then $-\sigma_u(\varDelta_u \cap A) = \rho_u(\varDelta_u \setminus A) = \rho_u(\varDelta_u \cap B)$. Therefore,

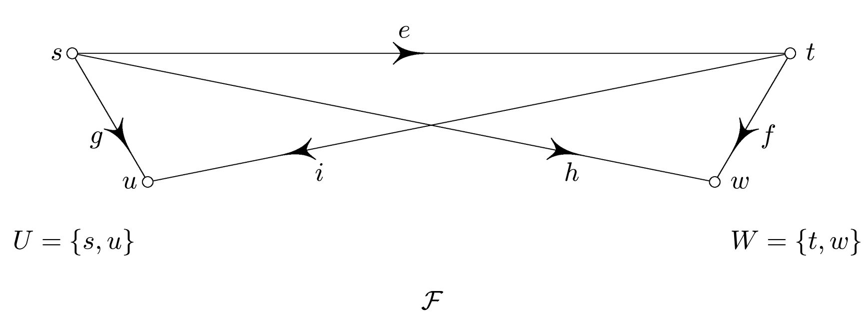
$$\begin{aligned} c_{\text{up}}(U, W, A, B) &= -\sigma_u(\varDelta_u \cap A) + \sum_{v \in U \setminus u} -\sigma_v(\varDelta_v \cap A) + \sum_{v \in W} \rho_v(\varDelta_v \cap B) \\ &= \rho_u(\varDelta_u \cap B) + \sum_{v \in U \setminus u} -\sigma_v(\varDelta_v \cap A) + \sum_{v \in W} \rho_v(\varDelta_v \cap B) \\ &= c_{\text{up}}(U \setminus u, W \cup u, A, B). \end{aligned}$$

The analogous equality for lower capacities follows from (12). □

From Lemma 1 follows, that if we deal with balanced networks, then the upper (lower) capacity of a complete cut (U, W, A, B) does not depend on the partition of vertices, but on the partition of edges. This, of course, seems to be in contrast with the classical case. But note, that this contrast became smaller in the polymatroidal network flow model of Martel and Lawler [**12**], [**13**] (which is in fact a weaker version of our model), where the complete cuts are replaced by "arc partitioned cuts" that depends equally on the partitions of arcs and vertices (see [**12**], [**13**] for more details). Finally in the classical flow model (which is a weaker version of the model of Lawler and Martel) the arc partitioned cuts are transformed to the classical cuts.

Now we introduce an auxiliary construction.

Construction 1. Let $\mathcal{F}$ be a q-polymatroidal flow network on a digraph $G = (V, E)$ with a source s, a sink t and the collection of q-polymatroids $\mathbb{Q}_v$, $v \in V$. Let $\{U, W\}$ be a partition of the vertex set V. Then define a new q-polymatroidal flow network $\mathcal{F}_{U,W}$ such that (see Fig. 1):



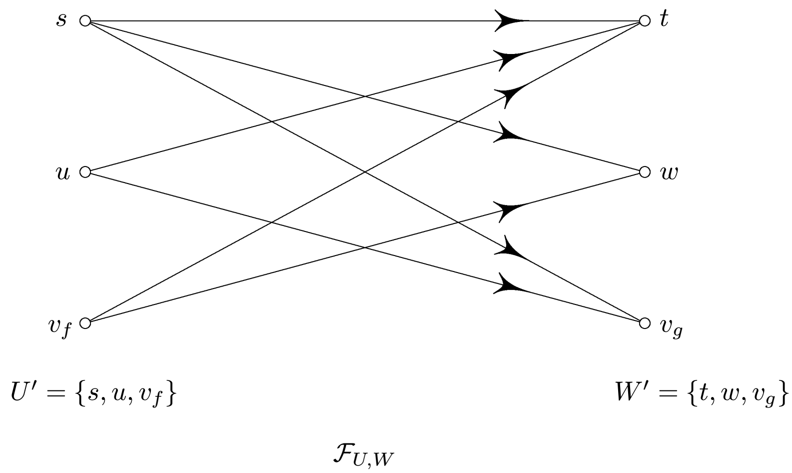


Figure 1.

- delete each arc $e = (u, v)$ with both endpoints in U and replace it by two new arcs (u, v_e), (v, v_e) where v_e is a new vertex;
- delete each arc $e = (u, v)$ with both endpoints in W and replace it by two new arcs (v_e, u), (v_e, v) where v_e is a new vertex;
- delete each arc (u, v) where $u \in W$, $v \in U$ and replace it by a new arc (v, u).

After these changes we obtain a bipartite graph $G' = (V', E')$ where the arcs are directed from one partition to the other. Clearly $V \subseteq V'$ and U, W are subsets of different partitions of V'. Let us denote U', W' the two partitions of V' such that $U \subseteq U'$, $W \subseteq W'$. If $X \subseteq E$, then $\varphi_{U,W}(X)$ denotes the set of the arcs in E' that erase from the arcs of X in the above construction. (For instance if $X = \{i, g\}$ in Fig. 1, then $\varphi_{U,W}(X) = \{(u, t), (s, v_g), (u, v_g)\}$.)

Endow each $v \in V$ by a g-polymatroid $\mathbb{P}_v$ such that (for simplicity any arc $e \in E$ from the neighbourhood of a vertex v is identified with the new arc $e' \in E'$ replacing e in the above construction of G', or, to be more precise, with $e' \in \varphi_{U,W}(e) \cap \Delta_v$):

- if $v \in U$ then $\mathbb{P}_v = -\mathbb{P}'_v$, where $\mathbb{P}'_v$ is the underlying polymatroid of $\mathbb{Q}_v$;
- if $v \in W$ then $\mathbb{P}_v$ is the underlying polymatroid of $\mathbb{Q}_v$;
- if $v \in V' \setminus V$, then $\mathbb{P}_v$ is the principal g-polymatroid on Δ_v, $\mathbb{P}_v = \mathbb{F}_{0,\Delta_v}$ (note that $|\Delta_v| = 2$ in this case).

Then $\mathcal{F}_{U,W}$ is the q-polymatroidal flow network on the digraph G' with the source s, the sink t and the collection of q-polymatroids $\mathbb{P}_v$, $v \in V'$ (where each $\mathbb{P}_v$ is in fact a g-polymatroid). Clearly $\mathcal{F}_{U,V}$ is balanced if and only if $\mathcal{F}$ is. $\mathcal{F}_{U,W}$ is called a (U, W)-**splitting of** $\mathcal{F}$.

Let f be a feasible flow in $\mathcal{F}$. Then it determines a new flow $f_{U,W}$, feasible in $\mathcal{F}_{U,W}$ such that:

- if $e = (u, v) \in E$ and $u, v \in U$, then $f_{U,W}(u, v_e) = -f_{U,W}(v, v_e) = f(u, v)$;

- if $e = (u, v) \in E$ and $u, v \in W$, then $f_{U,W}(v_e, v) = -f_{U,W}(v_e, u) = f(u, v)$;
- if $e = (u, v) \in E$ and $u \in W$, $v \in U$, then $f_{U,W}(v, u) = -f(u, v)$;
- if $e = (u, v) \in E$ and $u \in U$, $v \in W$, then $f_{U,W}(u, v) = f(u, v)$.

Then the mapping $f \mapsto f_{U,W}$ is a bijection from the set of feasible flows in $\mathcal{F}$ to the set of feasible flows in $\mathcal{F}_{U,W}$. Moreover, $v_{f,U,W} = v_{f_{U,W},S,T}$ for any partition S, T of V' such that $U \subseteq S$, $W \subseteq T$. Furthermore, if $\mathcal{F}$ is balanced, then $v_f = v_{f_{U,W}}$.

Primarily we solve the problem whether there exists a feasible flow in a q-polymatroidal flow network.

Theorem 2. *Let $\mathcal{F}$ be an (integral) q-polymatroidal flow network on digraph $G = (V, E)$. Then the following conditions are equivalent:*

(a) *$\mathcal{F}$ has an (integral) feasible flow.*
(b) *Every complete $(V, \emptyset)$- and $(\emptyset, V)$-cuts of $\mathcal{F}$ have nonnegative upper capacities.*
(c) *Every complete $(V, \emptyset)$- and $(\emptyset, V)$-cuts of $\mathcal{F}$ have nonpositive lower capacities.*

Proof. Let $\{U, W\}$ be a partition of V. Take $\mathcal{F}_{U,W}$ with the parameters denoted as in Construction 1. Let $\mathbb{P}_1 = (E', \rho_1, \sigma_1) = \bigoplus_{v \in U'} \mathbb{P}_v$ and $\mathbb{P}_2 = (E', \rho_2, \sigma_2) = \bigoplus_{v \in W'} \mathbb{P}_v$. Then $\mathcal{F}_{U,W}$ has an (integral) feasible flow iff $\mathbb{P}_1$ and $\mathbb{P}_2$ have a common (integral) independent vector. By Corollary 2, it holds iff for any $X' \subseteq E'$,

$$\begin{aligned} \rho_1(X') &\geq \sigma_2(X'), \\ \rho_2(X') &\geq \sigma_1(X'). \end{aligned} \tag{13}$$

If $u \in U' \setminus U$ and $|\Delta_u \cap X'| = 1$, then $\rho_1(X') = \infty$, $\sigma_1(X') = -\infty$ and (13) holds. Similarly if $w \in W' \setminus W$ and $|\Delta_w \cap X'| = 1$. Then it remains to deal with $X' \subseteq E'$ satisfying $|\Delta_v \cap X'| = 0, 2$ for any $v \in V' \setminus V$. Thus $\rho_v(\Delta_v \cap X') = \sigma_v(\Delta_v \cap X') = 0$ for any $v \in V' \setminus V$. Furthermore, we can check that $X' = \varphi_{U,W}(X)$ for a subset X of E in this case. Therefore (13) can be rephrased as

$$\begin{aligned} \sum_{v \in U} -\sigma_v(\Delta_v \cap X) &\geq \sum_{v \in W} \sigma_v(\Delta_v \cap X), \\ \sum_{v \in W} \rho_v(\Delta_v \cap X) &\geq \sum_{v \in U} -\rho_v(\Delta_v \cap X), \end{aligned}$$

what is equivalent with

$$\begin{aligned} c_{\mathrm{up}}(V, \emptyset, X, E \setminus X) &\geq 0, \\ c_{\mathrm{up}}(\emptyset, V, E \setminus X, X) &\geq 0. \end{aligned}$$

Since this holds for any $X \subseteq E$ then (a), (b) are equivalent. By (12), (b) and (c) are equivalent too. $\square$

Corollary 3. *Let $\mathcal{F}$ be a balanced (integral) q-polymatroidal flow network on digraph $G = (V, E)$ with a source s and a sink t. Then the following conditions are equivalent:*

(a) *$\mathcal{F}$ has an (integral) feasible flow.*
(b) *Every complete (U, W)-cut of $\mathcal{F}$ such that $U \cap \{s, t\} \neq 1$ has nonnegative upper capacity.*
(c) *Every complete (U, W)-cut of $\mathcal{F}$ such that $U \cap \{s, t\} \neq 1$ has nonpositive lower capacity.*

Proof. Follows from Theorem 2 and Lemma 1. □

Now we can state the max-flow min-cut theorem.

Theorem 3. *Let $\mathcal{F}$ be an (integral) q-polymatroidal flow network on digraph $G = (V, E)$ with a source s, a sink t and $\{U, W\}$ be a partition of V. Then the maximal (U, W)-value of a flow feasible in $\mathcal{F}$ is equal to the minimal upper capacity of a (U, W)-cut. Furthermore, is $\mathcal{F}$ is integral, then there exists an integral flow feasible in $\mathcal{F}$ with the maximal (U, W)-value.*

Proof. Take $\mathcal{F}_{U,W}$ with the parameters denoted as in Construction 1. Let $\mathbb{P}_1 = (E', \rho_1, \sigma_1) = \bigoplus_{v \in U'} \mathbb{P}_v$ and $\mathbb{P}_2 = (E', \rho_2, \sigma_2) = \bigoplus_{v \in W'} \mathbb{P}_v$. Let $\overline{v}_{U,W}$ denote the maximal (U, W)-value of a flow feasible in $\mathcal{F}$. Then $\overline{v}_{U,W}$ is equal to the maximal modulus of a vector independent in both $\mathbb{P}_1$, $\mathbb{P}_2$. Thus, by Corollary 1,

$$\overline{v}_{U,W} = \min_{X' \subseteq E'} \left(\rho_1(X') + \rho_2(E' \setminus X')\right).$$

Clearly, if $v \in V' \setminus V$ and $|\Delta_v \cap X'| = 1$ then $\rho_1(X') + \rho_2(E' \setminus X') = \infty$. Therefore, using the same arguments as in the proof of Theorem 2 we can check that

$$\begin{aligned} \overline{v}_{U,W} &= \min_{X \subseteq E} \Big(\sum_{v \in U} -\sigma_v(\Delta_v \cap X) + \sum_{v \in W} \rho_v(\Delta_v \setminus X) \Big) \\ &= \min_{X \subseteq E} (c_{\mathrm{up}}(U, W, X, E \setminus X)). \end{aligned}$$

Then the maximal (U, W)-value of a flow feasible in $\mathcal{F}$ is equal to the minimal upper capacity of a (U, W)-cut.

The conditions for integrality follows from Corollary 2. □

Corollary 4. *Let $\mathcal{F}$ be a balanced (integral) q-polymatroidal flow network on digraph $G = (V, E)$ with a source s and a sink t. Then the maximal value of a flow feasible in $\mathcal{F}$ is equal to the minimal capacity of a (U, W)-cut such that $s \in U$, $t \in W$. Furthermore, if $\mathcal{F}$ is integral, then there exists an integral flow feasible in $\mathcal{F}$ with the maximal value.*

Proof. Follows from Theorem 3 and Lemma 1. □

The existence of a polynomial algorithm for computing an (integral) flow with maximal (U, W)-value feasible in an (integral) q-polymatroidal flow network follows from the fact that this flow can be understood as an (integral) independent vector of two (integral) g-polymatroids. Thus we can apply the general algorithms from Grötschel, Lovász and Schrijver [**8**] based on the ellipsoid method or the algorithm of Frank [**5**].

4. Relations with Other Concepts

Let $\mathcal{F}$ be a q-polymatroidal flow network on a digraph $G = (V, E)$ with a source s, a sink t and the collection of q-polymatroids $\mathbb{Q}_v$, $v \in V$. Let $e = (u, w)$ be an arc of G. Then define a new q-polymatroidal flow network $\mathcal{F}_e$ as follows: Change the orientation of the arc e, i.e., delete e and replace it by a new arc $e' = (w, u)$. Endow u, w by $\mathbb{Q}'_u$ and $\mathbb{Q}'_w$ such that $\mathbb{Q}'_u$ and $\mathbb{Q}'_w$ have the same underlying g-polymatroids as $\mathbb{Q}_u$ and $\mathbb{Q}_w$ respectively. Otherwise let $\mathbb{Q}_v$ remains unchanged. Then a flow f' is feasible in $\mathcal{F}_e$ iff the flow f satisfying $f(e) = -f'(e')$ and $f(x) = f'(x)$ for any $x \in E \setminus e$ is feasible in $\mathcal{F}$.

It is natural to consider the flow networks $\mathcal{F}$ and $\mathcal{F}_e$ to be equivalent because their feasible flows have opposite values on the opposite oriented arc. Thus $\mathcal{F}$ is uniquely determined by the graph G' we get from the digraph G after forgetting the orientations of the arcs, and the collection of g-polymatroids $\mathbb{P}_v$, $v \in V$, where each $\mathbb{P}_v$ is the underlying g-polymatroid of $\mathbb{Q}_v$. If each edge of G' is endowed with an orientation such that G' turns to G, then each $\mathbb{P}_v$ will turn to the q-polymatroid $\mathbb{Q}_v$, receiving the parameters of $\mathcal{F}$. This situation is apparently similar to the concept of nowhere-zero flows used in graph theory (see, e.g., Jaeger [**10**]).

We show that the models of polymatroidal flow networks introduced by Lawler and Martel [**12**], [**14**] and Hassin [**9**] can be understood as balanced q-polymatroidal flow networks. The models from [**14**] and [**9**] can be formulated as follows: A **g-polymatroidal flow network** $\mathcal{F}$ is a digraph $G = (V, E)$ with a **source** s, a **sink** t and a collection of g-polymatroids $\mathbb{P}_v^+ = (\Delta_v^+, \rho_v^+, \sigma_v^+)$, $\mathbb{P}_v^- = (\Delta_v^-, \rho_v^-, \sigma_v^-)$. We call $\mathcal{F}$ **integral** if all $\mathbb{P}_v^+$, $\mathbb{P}_v^-$ are integral. A **flow** in the network $\mathcal{F}$ is a function $f\colon E \to \mathbb{R}$. If a flow f in $\mathcal{F}$ is integer valued we call it **integral**. A flow f in $\mathcal{F}$ is said to be **feasible** in $\mathcal{F}$ if

$$\begin{aligned} f(\Delta_v^+) = f(\Delta_v^-) &\quad \text{for any } v \in V,\ v \neq s, t, \\ \sigma_v^+(X) \le f(X) \le \rho_v^+(X) &\quad \text{for any } v \in V \text{ and } X \subseteq \Delta_v^+, \\ \sigma_v^-(X) \le f(X) \le \rho_v^-(X) &\quad \text{for any } v \in V \text{ and } X \subseteq \Delta_v^-. \end{aligned}$$

If f is a feasible flow in $\mathcal{F}$ then $v_f = f(\Delta_s^-) - f(\Delta_s^+) = f(\Delta_t^+) - f(\Delta_t^-)$ is called the **value** of f.

If each $\mathbb{P}_v^+$ and $\mathbb{P}_v^-$ are polymatroids we get a **polymatroidal flow network** introduced by Lawler and Martel [**12**].

Let $\mathcal{F}$ be a g-polymatroidal flow network. We transform $\mathcal{F}$ into a new g-polymatroidal flow network $\mathcal{F}''$ as follows: Let $G'' = (V'', E'')$ be the digraph erasing from G by splitting each vertex $v \in V$ into two vertices v_1 and v_2 in such a way that the arcs that have left v are adjacent with v_1 and the arcs that have entered v are adjacent with v_2. Finally, add a new arc $e_v = (v_1, v_2)$ for any $v \in V$. In other words $\Delta^+_{v_1} = \emptyset$, $\Delta^-_{v_1} = \Delta^-_v \cup e_v$, $\Delta^-_{v_2} = \emptyset$ and $\Delta^+_{v_2} = \Delta^+_v \cup e_v$. G'' is a bipartite digraph with arcs directed from one partition of the vertices to the other.

For any $v \in V$, $v \neq s$, let $\mathbb{P}''_{v_1} = (\Delta^-_{v_1}, \rho''_{v_1}, \sigma''_{v_1})$ be the 0-extension of $\mathbb{P}^-_v = (\Delta^-_v, \rho^-_v, \sigma^-_v)$ to $\Delta^-_{v_1}$, and, for any $v \in V$, $v \neq t$, let $\mathbb{P}''_{v_2} = (\Delta^+_{v_2}, \rho''_{v_2}, \sigma''_{v_2})$ be the 0-extension of $\mathbb{P}^+_v = (\Delta^+_v, \rho^+_v, \sigma^+_v)$ to $\Delta^+_{v_2}$. Let $\mathbb{P}''_{s_1} = (\Delta^-_{s_1}, \rho''_{s_1}, \sigma''_{s_1}) = \mathbb{P}^-_s \oplus \mathbb{F}_{\infty, e_s}$ and $\mathbb{P}''_{t_2} = (\Delta^+_{t_2}, \rho''_{t_2}, \sigma''_{t_2}) = \mathbb{P}^+_t \oplus \mathbb{F}_{\infty, e_t}$. Then let $\mathcal{F}''$ be the g-polymatroidal flow network on the digraph G'' with the source s_1, the sink t_2 and the collection of g-polymatroids $\mathbb{P}''_{v_1} = (\Delta^-_{v_1}, \rho''_{v_1}, \sigma''_{v_1})$, $\mathbb{P}''_{v_2} = (\Delta^+_{v_2}, \rho''_{v_2}, \sigma''_{v_2})$ $(v_1, v_2 \in V'')$.

If f is a feasible flow in $\mathcal{F}$ then f can be extended to a feasible flow f'' in $\mathcal{F}''$ in such a way that $f''|E = f$, $f''(e_s) = -f(\Delta^+_s)$, $f''(e_t) = -f(\Delta^-_t)$ and $f''(e_v) = -f(\Delta^+_v) = -f(\Delta^-_v)$ for any $v \neq s, t$. Similarly if a f'' is feasible flow in $\mathcal{F}''$ then $f''|E$ is a feasible flow in $\mathcal{F}$.

But $\mathcal{F}''$ is a g-polymatroidal flow network on a bipartite digraph with arcs directed from one partition of the vertices to the other. Thus $\mathcal{F}''$ can be considered as a balanced q-polymatroidal flow network. In other words, the flow models of Lawler and Martel and Hassin can be considered as balanced q-polymatroidal flow networks.

Similarly we can check that q-polymatroidal network flow model can be formulated in framework of the g-polymatroidal flow networks, in other words, these two models are equivalent. As pointed out in [**14**], another equivalent flow model was introduced by Edmonds and Giles [**3**]. In Schrijver [**21**] is shown that these flow models are equivalent with other important models of combinatorial optimization, especially with the Edmonds' intersection theorem [**2**].

Note that the paper [**14**] is devoted to the analysis of the augmenting path algorithm on g-polymatroidal flow networks. There is also introduced an analogue of Theorem 3, but no analogue of Theorem 2. That is why we have transformed Theorems 2 and 3 to Corollary 2 and Theorem 1 and not directly to the results from [**14**]. But we can check that Theorem 3 is equivalent with [**14**, Theorem 8.1]. On the other hand the existence of an (integral) feasible flow in an (integral) g-polymatroidal flow network can be checked similarly as in Theorem 2.

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