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# SUPER–GEOMETRIC QUANTIZATION

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Abstract. Let $K$ be the complex line bundle where the Kostant-Souriau geometric quantization operators are defined. We discuss possible prolongations of these operators to the linear superspace of the $K$-valued differential forms, such that the Poisson bracket is represented by the supercommutator of the corresponding operators. We also discuss the possibility to obtain such super-geometric quantizations by (anti)Hermitian operators on a Hilbert superspace. We apply our general considerations to Kähler manifolds and to cotangent bundles of Riemannian manifolds.

## 1. Recalling Geometric Quantization

In differential geometry, the problem of geometric quantization is a two stage problem which can be stated in the following terms (e.g., [**10**], [**9**]).

**Stage 1 — Prequantization**. Let $M$ be a Poisson manifold with the Poisson bracket

$$\{f, g\} = P(df, dg) \qquad (f, g \in C^\infty(M)). \tag{1.1}$$

Find linear representations of the Lie algebra (1.1) on the space $\Gamma(K)$ of cross sections of a complex line bundle $K$ over $M$ by differential operators of order one and symbol equal to the Hamiltonian vector field $X_f^P$.

**Stage 2 — Quantization**. Restrict prequantization in such a way as to obtain irreducible anti-Hermitian[1] representations of a subalgebra of $C^\infty(M)$ with bracket (1.1) on a Hilbert space derived from $\Gamma(K)$.

In this paper, we define the problem of **super-geometric quantization** as the problem of prolonging the representations mentioned above to linear and Hilbert superspaces.

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[1] The fact that we use anti-Hermitian operators here is just a technicality. If these operators are multiplied by a purely imaginary constant they become Hermitian operators.

Now, let us be more precise. While more general prequantization representations may exist [**5**], [**9**], we consider only the fundamental **Kostant-Souriau representation**. The latter is given by the operators

$$\hat{f}\sigma = \nabla_{X_f}\sigma + 2\pi\sqrt{-1}f\sigma \qquad (f \in C^\infty(M),\ \sigma \in \Gamma(K)), \tag{1.2}$$

where $\nabla$ is a connection on $K$ which preserves a Hermitian metric $h$ of $K$.

The condition that (1.2) is a representation means

$$\widehat{\{f,g\}} = \hat{f} \circ \hat{g} - \hat{g} \circ \hat{f}, \tag{1.3}$$

and this condition is equivalent to

$$\Omega(X_f, X_g) = -2\pi\sqrt{-1}\{f,g\}, \tag{1.4}$$

where $\Omega$ is the curvature of $\nabla$. In particular, (1.4) shows that $K$, $h$ and $\nabla$ exist iff $P$ defines an integral Poisson cohomology class (namely, the image of the integral first Chern class of $K$) [**9**], and, then, we say that $(M,P)$ is a **quantizable Poisson manifold**. In the symplectic case, the integrality condition is just that the symplectic form represents an integral cohomology class [**10**].

Furthermore, let $\mathcal{D}$ be the bundle of complex valued **halfdensities** of $M$ (e.g., [**6**], [**7**]). Then (1.2) extends to $\Gamma(K \otimes \mathcal{D})$ by

$$\hat{f}(\sigma \otimes \rho) = (\hat{f}\sigma) \otimes \rho + \sigma \otimes L_{X_f}\rho \qquad (\sigma \in \Gamma(K),\ \rho \in \gamma(\mathcal{D})) \tag{1.5}$$

where $L$ denotes the Lie derivative, and Stokes' theorem shows that the operators (1.5) are anti-Hermitian on $\Gamma_c(K \otimes \mathcal{D})$ ($c$ means "with compact support") endowed with the scalar product

$$\langle \sigma_1 \otimes \rho_1, \sigma_2 \otimes \rho_2 \rangle = \int_M h(\sigma_1, \sigma_2)\rho_1\bar{\rho}_2 \tag{1.6}$$

(the bar means complex conjugation) i.e., we have

$$\left\langle \hat{f}\alpha, \beta \right\rangle + \left\langle \alpha, \hat{f}\beta \right\rangle = 0 \qquad (\alpha, \beta \in \Gamma_c(K \otimes \mathcal{D})). \tag{1.7}$$

Of course, we may complete $\Gamma_c(K \otimes \mathcal{D})$ to a Hilbert space but, we still remain in the prequantization stage since we do not have irreducibility.

Now, the stage of quantization is based on the notion of a **polarization**, for which we adopt here a new definition that includes the classical definition as a particular case. Let $\mathcal{F}$ denote the sheaf of Poisson algebras of germs of complex valued $C^\infty$ functions of $M$ with the bracket (1.1). Then, a **polarization** $\mathcal{P}$ of $(M,P)$ is a subsheaf $\mathcal{P}$ of $\mathcal{F}$ whose stalks are abelian subalgebras of the stalks of $\mathcal{F}$.

If $\mathcal{P}$ is given, we may look at the linear space

$$\Gamma_0(K) = \{\sigma \in \Gamma(K) \ / \ \nabla_{X_\varphi}\sigma = 0,\ \forall \varphi \in \mathcal{P}\}, \tag{1.8}$$

and we may apply the operators (1.2) to $\Gamma_0(K)$ if $\Gamma_0(K) \neq \{0\}$. It is easy to see that, $\forall f \in C^\infty(M)$ such that $\{\varphi, f\} \in \mathcal{P}$ whenever $\varphi \in \mathcal{P}$, $\hat{f}(\Gamma_0(K)) \subseteq \Gamma_0(K)$. The set $\mathcal{Q}(M,\mathcal{P})$ of such functions $f$ is a Lie subalgebra of $(C^\infty(M), \{\ ,\ \})$ which includes all the real global sections $\psi$ of $\mathcal{P}$, and for these $\psi$ one has $\hat{\psi}\sigma = 2\pi\sqrt{-1}\psi\sigma,\ \forall \sigma \in \Gamma_0(K)$, as needed for irreducibility [**10**].

Furthermore, if $\Gamma_0(K)$ has nonzero elements with compact support, it may be possible to adapt conveniently the scalar product (1.6), and obtain a Hilbert space where (1.7) holds $\forall f \in \mathcal{Q}(M,\mathcal{P})$. Otherwise, the idea is to project the whole configuration onto a lower dimensional quotient manifold $N$, if possible, and get a similar scalar product by integration over $N$ [**10**], [**7**], [**9**].

The basic types of polarizations encountered in applications are as follows (e.g., [**10**]).

1) Let $(M^{2n}, \omega)$ $(d\omega = 0)$ be a quantizable symplectic manifold, with the Poisson brackets defined by $\omega$, and assume that $M$ has a real Lagrangian foliation $\mathcal{L}$. Then, the sheaf $\mathcal{P}$ of germs of functions which are constant along the leaves of $\mathcal{L}$ is a polarization called a **real Lagrangian polarization**. An important particular case is that of a cotangent bundle $M = T^*N$, where $\omega = d\theta,\ \theta :=$ the Liouville 1-form of $T^*N$, and $\mathcal{L}$ is the foliation by the fibers of $T^*N$. In this case, if $\sigma \in \Gamma_0(K)$, supp $\sigma$ is noncompact (it is a union of fibers), and the scalar product will be defined by integration over $N$ and not over $T^*N$.

2) Let $(M^{2n}, \omega)$ be a quantizable symplectic manifold which admits compatible Kähler metrics. Then, if $g$ is such a metric, the sheaf $\mathcal{P}$ of germs of holomorphic functions with respect to the corresponding complex structure is a polarization called a **Kähler polarization**. In this case, $K$ is a holomorphic line bundle (e.g., [**7**]), and

$$\mathcal{Q}(M,\mathcal{P}) = \{f \in C^\infty(M) \ / \ X_f = X_f^{1,0} + X_f^{0,1},\ X_f^{1,0} \text{ holomorphic}\},$$

where the upper indices indicate the complex type. Equivalently, if $J$ is the tensor of the complex structure, then $L_{X_f}J = 0$. We say that $X$ is an analytic vector field, and we distinguish in this paper between the terms analytic and holomorphic for vector fields i.e., $X$ is **analytic** and its component $X^{1,0}$ is **holomorphic**. Furthermore, we may forget about halfdensities, and make $\Gamma_{0c}(K)$ into a Hilbert space by the scalar product

$$\langle \sigma_1, \sigma_2 \rangle = \int_M h(\sigma_1, \sigma_2) d(vol\, g) \qquad (\sigma_1, \sigma_2 \in \Gamma_{0c}(K)), \tag{1.9}$$

and the property (1.7) follows again from Stokes' theorem.

## 2. Super-Geometric Prequantization

Now, we proceed to the discussion of super-geometric quantization. We start with a quantizable Poisson manifold $(M,P)$ and a quantization complex line bundle $K$. Let us emphasize that we do not intend to discuss geometric quantization of supermanifolds, as in [**3**]. Neither do we consider any kind of supermanifolds [**1**]. But, we shall use the terminology of superalgebra (e.g., [**4**]).

With $(M,P,K)$, we can associate a natural complex linear superspace

$$\mathcal{S}(K) = \mathcal{S}^+(K) \oplus \mathcal{S}^-(K). \tag{2.1}$$

where

$$\mathcal{S}^+ = \oplus_{i\geq 0} \wedge^{2i}(M,K),\ \mathcal{S}^- = \oplus_{i\geq 0} \wedge^{2i+1}(M,K),$$

and $\wedge^h(M,K)$ are the spaces of $K$-valued forms on $M$, and it is possible to extend the Kostant-Souriau prequantization (1.2), (1.5) to $\mathcal{S}(K)$.

We did this in [**8**] as follows. Since $\wedge^h(M,K) = \Gamma((\wedge^h T^*M)\otimes K)$, one has the well-known **covariant exterior differential**

$$D(\alpha\otimes\sigma) = (d\alpha)\otimes\sigma + (-1)^{deg\,\alpha}\alpha\wedge\nabla\sigma, \tag{2.2}$$

and the **covariant Lie derivative**

$$L^{\nabla}_X(\alpha\otimes\sigma) = (L_X\alpha)\otimes\sigma + \alpha\otimes\nabla_X\sigma, \tag{2.3}$$

where $\alpha\in\wedge^h M,\ \sigma\in\Gamma(K)$, and $X$ is a vector field on $M$. These operators have the same global expressions as $d$ and $L_X$, except for the fact that the action of $X$ on functions is replaced by the action of $\nabla_X$ on sections of $K$. Notice also the formula

$$L^{\nabla}_X = Di(X) + i(X)D \tag{2.4}$$

which follows from (2.2) and (2.3).

Now, if (1.2) is extended to $\mathcal{S}(K)$ by

$$\hat{f}A = L^{\nabla}_{X_f}A + 2\pi\sqrt{-1}fA \qquad (A\in\mathcal{S}(K)), \tag{2.5}$$

it follows from (1.4) that the commutator condition (1.3) is still valid. Indeed [**8**], using (2.3) we get

$$L^{\nabla}_X L^{\nabla}_Y A - L^{\nabla}_Y L^{\nabla}_X A - L^{\nabla}_{[X,Y]}A = \Omega(X,Y)A, \tag{2.6}$$

where $\Omega$ is the curvature of $\nabla$, and then, (1.4) is obtained by a straightforward computation.

The operators $\hat{f}$ preserve the degree of a form. Thus, if we want to give a role to the structure (2.1), it is natural to define a **super-geometric prequantization** of $M$ on $K$ as a prolongation of (2.5) of the form

$$\tilde{f}A = \hat{f}A + 2\pi\sqrt{-1}l(f)(A) \qquad (A \in \mathcal{S}(K)), \tag{2.7}$$

where $l(f)$ is an odd endomorphism of $\mathcal{S}(K)$, such that the following commutation condition holds

$$\widetilde{\{f, g\}} = {}^{s}[\tilde{f}, \tilde{g}]. \tag{2.8}$$

In the right hand side of (2.8), one has the supercommutator [**4**] of the operators $\tilde{f}$, $\tilde{g}$, and we denoted it by the index $s$. Brackets without this index will denote usual commutators.

**Proposition 2.1.** *The operation* $*$ *defined by*

$$f * \theta = [\hat{f}, \theta] = \hat{f}\theta - \theta\hat{f}, \tag{2.9}$$

*($f \in C^{\infty}(M)$, $\theta \in End\,\mathcal{S}(K)$) is a representation of the Lie algebra ($C^{\infty}(M)$, $\{\ ,\ \}$) on $End\,\mathcal{S}(K)$ which leaves $End_{-}\mathcal{S}(K)$ invariant, and* (2.7) *is a super-geometric prequantization iff $l$ is a* 1*-cocycle with values in $End_{-}\mathcal{S}(K)$, and with respect to the representation* (2.9)*, such that*

$$l^2(f) = 0, \qquad \forall f \in C^{\infty}(M). \tag{2.10}$$

*Proof.* The results are rather straightforward since, in view of (1.3), (2.8) is equivalent to

$$l(\{f, g\}) = [\hat{f}, l(g)] + [l(f), \hat{g}],\ \ l(f)l(g) + l(g)l(f) = 0, \tag{2.11}$$

$\forall f, g \in C^{\infty}(M)$. □

**Corollary 2.2.** *$\forall c \in End_{-}\mathcal{S}(K)$ such that $[\hat{f}, c]^2 = 0$, $\forall f \in C^{\infty}(M)$, the operators*

$$\tilde{f}_c(A) = L^{\nabla}_{X}A + 2\pi\sqrt{-1}fA + 2\pi\sqrt{-1}[\hat{f}, c](A) \tag{2.12}$$

*($A \in \mathcal{S}(K)$) define a super-geometric prequantization.*

*Proof.* $[\hat{f}, c]$ is the coboundary of $c$ in the Lie algebra cohomology mentioned in Proposition 2.1. □

We note some important particular cases in

**Proposition 2.3.** *Let $\theta$ be a complex valued 1-form, and $V$ be a complex vector field on the Poisson manifold $(M, P)$. Then,* (2.7) *is a super-geometric prequantization for each of the following choices of $l$:*

$$l_1(f) = e(L_{X_f}\theta), \quad l_2(f) = i([X_f, V]), \\ l_3(f) = l_1(f) + l_2(f). \tag{2.13}$$

*Proof.* In (2.13), $e$ means "exterior product by", and $i$ means "interior product by". $l_1$ is obtained by using Corollary (2.2) for $c = e(\theta)$, and $l_2$ is obtained for $c = i(V)$. □

**Remark 2.4.** If we take $c = D$, then, using (2.4) and the well known fact that $D^2 = e(\Omega)$, we get

$$l(f) = e(i(X_f)\Omega - 2\pi\sqrt{-1}df),$$

which is 0 in the symplectic case because of (1.4).

Now, as in Section 1, we can relate super-geometric prequantization with a scalar product. Namely, we consider again the complex line bundle $\mathcal{D}$ of halfdensities over $M$, and use the bundle $K \otimes \mathcal{D}$ instead of $K$. Then, instead of (2.1), we have

$$\tilde{\mathcal{S}}(K) := \mathcal{S}(K \otimes \mathcal{D}) := \tilde{\mathcal{S}}^+(K) \oplus \tilde{\mathcal{S}}^-(K), \tag{2.14}$$

which consists of forms with values in $K \otimes \mathcal{D}$ organized as those in (2.1).

Furthermore, we put on $M$ a Riemannian metric $g$, and define a scalar product of $\wedge^p_c(M, K \otimes \mathcal{D})$ (i.e., forms with a compact support) by

$$\langle \alpha_1 \otimes \sigma_1 \otimes \rho_1, \alpha_2 \otimes \sigma_2 \otimes \rho_2 \rangle = \int_M g(\alpha_1, \alpha_2) h(\sigma_1, \sigma_2)\rho_1\bar{\rho}_2\,, \tag{2.15}$$

where $\alpha_i \in \wedge^p(M)$, $\sigma_i \in \Gamma(K)$, $\rho_i \in \Gamma(\mathcal{D})$ $(i = 1, 2)$. Then, we get

**Proposition 2.5.** *Assume that* (2.7) *is a super-geometric prequantization where the odd cocycle $l$ is Hermitian with respect to $gh$. Then, the extension of* (2.7) *defined by*

$$\tilde{f}(A \otimes \rho) = (\tilde{f}A) \otimes \rho + A \otimes L_{X_f}\rho \tag{2.16}$$

*satisfies the commutator property* (2.8)*, and, if $X_f$ is a Killing vector field for $g$, $\tilde{f}$ is anti-Hermitian with respect to* (2.15).

*Proof.* That $\tilde{f}$ of (2.16) also satisfies (2.8) follows by a straightforward calculation. (Notice that $l(f)$ extends to $\tilde{\mathcal{S}}(K)$ by $l(f)(A \otimes \rho) = (l(f)A) \otimes \rho$.) Furthermore, by the metric $gh$ we mean

$$gh(\alpha_1 \otimes \sigma_1, \alpha_2 \otimes \sigma_2) = g(\alpha_1, \alpha_2)h(\sigma_1, \sigma_2),$$

and $l(f)$ are supposed to be $gh$-Hermitian. The anti-Hermitian character (1.7) of the present situation follows by using Stokes' theorem under the form (e.g., [**6**])

$$\int_M L_{X_f}(g(\alpha_1, \alpha_2)h(\sigma_1, \sigma_2)\rho_1\bar{\rho}_2) = 0.$$

□

## 3. Super-Geometric Quantization

Now, we combine super-geometric prequantization with a polarization and this process is **super-geometric quantization**.

Let $(M, P)$ be a Poisson manifold endowed with the prequantization (2.7), (2.16), and the scalar product (2.15), and let $\mathcal{P}$ be a polarization of $M$. Then, we shall define the linear superspace

$$\mathcal{S}_0(K) = \{A \in \mathcal{S}(K) \;/\; L^{\nabla}_{X_\varphi} A = 0,\ i(X_\varphi)A = 0,\ \forall\varphi \in \mathcal{P}\}. \tag{3.1}$$

Using (2.5), (2.6) and (1.4), we see easily that $\forall A \in \mathcal{S}_0(K)$, $\forall f \in \mathcal{Q}(M, \mathcal{P})$, one has $\hat{f}A \in \mathcal{S}_0(K)$. We recall that (Section 1)

$$\mathcal{Q}(M, \mathcal{P}) = \{f \in C^\infty(M) \;/\; \{\varphi, f\} \in \mathcal{P}, \forall\varphi \in \mathcal{P}\}.$$

Furthermore, in order to deal with the odd part of (2.7), we restrict ourselves to $\mathcal{Q}'(M, \mathcal{P}) \subseteq \mathcal{Q}(M, \mathcal{P})$, where we define that $f \in \mathcal{Q}'(M, \mathcal{P})$ if it satisfies the following supplementary conditions

$$[L^{\nabla}_{X_\varphi}, l(f)] = 0, \quad {}^s[i(X_\varphi), l(f)] = 0, \qquad \forall\varphi \in \mathcal{P}. \tag{3.2}$$

Then, we get

**Proposition 3.1.** *$\forall f \in \mathcal{Q}'(M, \mathcal{P})$ and $\forall A \in \mathcal{S}_0(K)$, we have $\tilde{f}A \in \mathcal{S}_0(K)$, for $\tilde{f}$ defined by* (2.7). *In particular, if the* 1*-form $\theta$ and the vector field $V$ of $M$ are such that $L_{X_\varphi}\theta = i(X_\varphi)\theta = 0$, $[X_\varphi, V] = 0$, $\forall\varphi \in \mathcal{P}$, the prequantizations of Proposition* 2.3 *induce quantization formulas on $\mathcal{S}_0(K)$, $\forall f \in \mathcal{Q}(M, \mathcal{P})$.*

*Proof.* The first assertion follows straightforwardly from the definitions. For the second assertion, we check that (3.2) holds for the cocycles $l_1$ and $l_2$ of (2.13), and $\forall B \in \mathcal{S}(K)$:

$$\begin{aligned}
L^{\nabla}_{X_\varphi}((L_{X_f}\theta) \wedge B) - (L_{X_f}\theta) \wedge L^{\nabla}_{X_\varphi} B &= (L_{X_\varphi} L_{X_f}\theta) \wedge B \\
&= (L_{X_f} L_{X_\varphi}\theta + L_{X_{\{\varphi,f\}}}\theta) \wedge B = 0, \\
i(X_\varphi)((L_{X_f}\theta) \wedge B) + (L_{X_f}\theta) \wedge (i(X_\varphi)B) &= (i(X_\varphi)L_{X_f}\theta)B \\
&= (L_{X_f} i(X_\varphi)\theta + i(X_{\{\varphi,f\}})\theta)B = 0, \\
L^{\nabla}_{X_\varphi} i([X_f, V])B - i([X_f, V])L^{\nabla}_{X_\varphi} B &= i([X_\varphi, [X_f, V]])B \\
&= i([X_{\{\varphi,f\}}, V])B + i([X_f, [X_\varphi, V]])B = 0, \\
i(X_\varphi)i([X_f, V])B + i([X_f, V])i(X_\varphi)B &= 0.
\end{aligned}$$

□

Furthermore, if we want a good scalar product, we may try to adapt conveniently formula (2.15), but, since we know from Proposition 2.5 that we shall need to ask $X_f$ to be a Killing vector field for $g$, it is simpler to look at the subspace $\mathcal{S}_{0c}(K)$ of the elements of $\mathcal{S}_0(K)$ which have a compact support, and put

$$\langle \alpha_1 \otimes \sigma_1, \alpha_2 \otimes \sigma_2 \rangle = \int_M g(\alpha_1, \alpha_2) h(\sigma_1, \sigma_2) d(vol\, g) \tag{3.3}$$

(while, of course, $M$ is assumed to be oriented). This scalar product vanishes on forms of different degrees. Hence, it makes $\mathcal{S}_{0c}(K)$ into a pre-Hilbert superspace, which, afterwards, will be completed to a Hilbert superspace. Then, just as for Proposition 2.5, we deduce

**Proposition 3.2.** *Assume that the cocycle $l$ is Hermitian with respect to the metric $gh$, and put*

$$\mathcal{Q}''(M, \mathcal{P}) = \{ f \in \mathcal{Q}'(M, \mathcal{P}) \ / \ L_{X_f} g = 0 \}. \tag{3.4}$$

*Then, the operator $\tilde{f}$ of* (2.7)*, associated with any $f \in \mathcal{Q}''(M, \mathcal{P})$ is anti-Hermitian with respect to the metric* (3.3)*.*

*Proof.* Make explicit the Lie derivative in the Stokes' formula

$$\int_M L_{X_f}(g(\alpha_1, \alpha_2) h(\sigma_1, \sigma_2) d(vol\, g)) = 0.$$

□

**Corollary 3.3.** *Assume that, $\forall \varphi \in \mathcal{P}$, $L_{X_\varphi} g = 0$. Assume that there exists a 1-form $\theta$ on $M$ such that $\forall \varphi \in \mathcal{P}$ one has $L_{X_\varphi} \theta = 0$, $i(X_\varphi)\theta = 0$, and define $V = \sharp_g \theta$. Then, the cocycle $l_3$ of* (2.13) *is $g$-selfadjoint, and $\forall f \in \mathcal{Q}(M, \mathcal{P})$ such that $X_f$ is a Killing vector field for $g$ the superquantization $\tilde{f}$ of* (2.7) *with $l = l_3$ is defined on $\mathcal{S}_0(K)$, and it is anti-Hermitian with respect to* (3.3)*.*

*Proof.* By the definition of $V$, we have $g(V, Z) = \theta(Z)$ for any vector field $Z$ of $M$, and the hypotheses of Proposition 3.1 are satisfied. Furthermore, we also see that $\sharp_g(L_{X_f}\theta) = L_{X_f} V = [X_f, V]$. Hence, the $g$-adjoint of $e(L_{X_f}\theta)$ is $i([X_f, V])$, and the result follows. □

## 4. Kähler and Lagrangian Polarizations

Now, we shall apply the general Propositions of Section 3 to the two basic examples mentioned in Section 1 i.e., where $M$ is a symplectic manifold and $\mathcal{P}$ is either a Kähler or a real Lagrangian polarization of $M$.

In the case of a Kähler polarization we get

**Proposition 4.1.** *Let $(M,\omega)$ be a quantizable symplectic manifold, and $\mathcal{P}$ a Kähler polarization of $M$, with the corresponding complex structure $J$ and metric $g$. Then $K$ is a holomorphic line bundle, $\mathcal{S}_0(K)$ is the linear superspace of the $K$-valued holomorphic forms of $(M,J)$, and, $\forall f\in\mathcal{Q}(M,\mathcal{P})$, the Hamiltonian vector field $X_f$ is Killing. Furthermore, if $\theta$ is a holomorphic* 1*-form on $M$,* (2.7) *with $l(f)=e(L_{X_f}\theta)$ is a super-geometric quantization on $\mathcal{S}_0(K)$, $\forall f\in\mathcal{Q}(M,\mathcal{P})$. Moreover, if*

$$\mathcal{Q}_0(M,\mathcal{P}) := \{f\in\mathcal{Q}(M,\mathcal{P})\ /\ \sharp_g L_{X_f}\bar{\theta}\ \textit{is holomorphic}\},$$

*then* (2.7) *with*

$$l(f)=e(L_{X_f}\theta)+i([X_f,\sharp_g\bar{\theta}]) \tag{4.1}$$

*is an anti-Hermitian super-geometric quantization of $\mathcal{Q}_0(M,\mathcal{P})$ on $\mathcal{S}_0(K)$ seen as a Hilbert superspace with the scalar product* (3.3).

*Proof.* We already recalled in Section 1 that $K$ is holomorphic and that, $\forall f\in\mathcal{Q}(M,\mathcal{P})$, $X_f$ is analytic ($L_{X_f}J=0$). Since, of course, $L_{X_f}\omega=0$, we also have $L_{X_f}g=0$. The assertion about the super-geometric quantization with the odd cocycle $e(L_{X_f}\theta)$ follows from Proposition 3.1.

Finally, we claim that, $\forall f\in\mathcal{Q}_0(M,\mathcal{P})$, the conditions (3.2) are also satisfied for the cocycle $l(f)=i([X_f,\sharp_g\bar{\theta}])$. Indeed, the second condition (3.2) is well known, and, as shown during the proof of Proposition 3.1, the first condition (3.2) is satisfied if

$$[X_\varphi,[X_f,\sharp_g\bar{\theta}]]=0,\qquad \forall\varphi\in\mathcal{P}. \tag{4.2}$$

By taking $\varphi$ equal to the local complex coordinates $z^i$ of $(M,J)$, we see that the antiholomorphic tangent bundle $T_{0,1}M$ of $(M,J)$ has local bases of the form $\{X_{\varphi_i}\}$, for some $\varphi_i\in\mathcal{P}$. Hence, (4.2) means that $[X_f,\sharp_g\bar{\theta}]$ preserves $T_{0,1}M$. On the other hand, since $X_f$ is Killing, we have

$$[X_f,\sharp_g\bar{\theta}]=\sharp_g(L_{X_f}\bar{\theta}), \tag{4.3}$$

and this is a vector field of the complex type $(1,0)$. Accordingly, (4.2) holds iff $[X_f,\sharp_g\bar{\theta}]$ is a holomorphic vector field, as claimed.

Now, if we use again Proposition 3.1, and the argument of Corollary 3.3, namely, that the adjoint of $e(\theta)$ is $i(\bar{\theta})$, we obtain the last assertion of Proposition 4.1. □

We shall also add a few more results about the space $\mathcal{Q}(M,\mathcal{P})$ of a Kähler polarization.

**Proposition 4.2.** *i) For a Kähler polarization $\mathcal{P}$, $f \in \mathcal{Q}(M,\mathcal{P})$ iff*

$$\nabla_{\bar{i}}\left(\frac{\partial f}{\partial \bar{z}^j}\right) = 0, \tag{4.4}$$

*where $(z^i)$ are complex coordinates and $\nabla$ is the Riemannian connection of the Kähler manifold $(M,g,J)$.*

*ii) If the Kähler manifold $M$ is compact, $f \in \mathcal{Q}(M,\mathcal{P})$ iff*

$$\Delta df - 2\sharp_r^{-1}\sharp_g df = 0, \tag{4.5}$$

*where $\Delta$ is the Laplace operator and $r$ is the Ricci tensor of $g$.*

*Proof.* i) The condition (4.4) follows immediately from the local coordinate expression of a Hamiltonian vector field $X_f$.

ii) In (4.5) the definition of $\sharp_r^{-1}$ is similar to that of $\sharp_g^{-1}$, but $\sharp_r$ may not exist. It is well known that, if $M$ is compact, $X_f$ is analytic iff

$$\Delta(\sharp_g^{-1}X_f) - 2\sharp_r^{-1}X_f = 0$$

(e.g., see Proposition 2.140 in [**2**]). But, it follows easily that $\sharp_g^{-1}X_f = -df \circ J$, and, using the known properties of $\Delta$ in the Kähler case, the previous relation becomes

$$(\Delta df) \circ J + 2\sharp_r^{-1}J\sharp_g df = 0.$$

If this equality is composed by $J$, and if we remember that $r$ is compatible with $J$, (4.5) follows. □

**Remark 4.3.** 1) If M is a compact connected Kähler-Einstein manifold, (4.5) becomes

$$\Delta f - 2\kappa f = \text{const.}, \tag{4.6}$$

where $\kappa$ is the (constant) scalar curvature of $g$.

2) If the Ricci curvature of the compact Kähler manifold $M$ is negative definite, $\mathcal{Q}(M,\mathcal{P}) = \boldsymbol{R}$. Indeed, in this case $M$ has no non zero analytic vector fields (e.g., Proposition 2.138 in [**2**]).

In order to exemplify the case of a Lagrangian polarization, we consider the basic situation of a cotangent bundle $M = T^*N$ with the symplectic form

$$\omega = -d\theta + p^*F, \tag{4.7}$$

where $\theta$ is the Liouville form, $p\ T^*N \to N$ is the natural projection, and $F$ is an exact 2-form $F = d\lambda$ of $N$ (the **electromagnetic term**). Thus, if $q^i$ are local coordinates on $N$, and $p_i$ are covector coordinates, we have (with the Einstein summation convention)

$$\theta = p_i dq^i, \ \lambda = \lambda_i(q)dq^i. \tag{4.8}$$

Then, $K$ may be taken trivial, the $K$-valued forms are just complex valued forms, the connection $\nabla$ can be defined by the global, flat connection form $2\pi\sqrt{-1}(\theta - \lambda)$, and the prequantization formula (2.5) becomes

$$\hat{f}A = L_{X_f}A + 2\pi\sqrt{-1}(\theta(X_f) - \lambda(X_f) + f)A \qquad (A \in \wedge M \otimes \boldsymbol{C}). \tag{4.9}$$

Furthermore, the polarization $\mathcal{P}$ is defined as the sheaf of germs of lifts to $T^*N$ of functions on $N$ (i.e., functions of the $(q^i)$ alone), and

$$\mathcal{Q}(M, \mathcal{P}) = \{f \in C^\infty(M) \ / \ f = \mu(Y) + \varphi\}, \tag{4.10}$$

where $Y$ is a tangent vector field of $N$, $\mu(Y)$ is its **momentum** $\mu(Y) = p_i Y^i$ $(Y = Y^i(\partial/\partial q^i))$, and $\varphi \in \mathcal{P}$ (e.g., [**10**]).

We shall use the notions of complete and vertical lift as defined, for instance, in [**Y**]. Then, it is easy to obtain

$$X_\varphi = \text{vertical lift of } d\varphi = \frac{\partial \varphi}{\partial q^i}\frac{\partial}{\partial p_i}, \qquad \forall \varphi \in \mathcal{P}, \tag{4.11}$$

and, for a vector field $Y$ of $N$

$$\begin{aligned} X_{\mu(Y)} &= -\text{complete lift of } Y - \text{vertical lift of } i(Y)F \\ &= -Y^i \frac{\partial}{\partial q^i} + \text{vertical part} \end{aligned} \tag{4.12}$$

(**vertical** means tangent to the fibers of $T^*M$).

From (4.11) we see easily that $\mathcal{S}_0(K)$ can be identified with the linear superspace of the complex valued differential forms of the base manifold $N$.

An odd cocycle $l$ is provided by the Liouville form $\theta$ and, as we know, it is $l(f) = e(L_{X_f}\theta)$ $(f \in C^\infty(T^*N))$. In particular, using (4.11) and (4.12), we get for $f = \mu(Y) + \varphi \in \mathcal{Q}(M, \mathcal{P})$

$$l(\mu(Y) + \varphi) = e(-i(Y)F + d\varphi), \tag{4.13}$$

which is a 1-form on $N$. Hence, this cocycle $l$ defines a super-geometric quantization of $\mathcal{Q}(M, \mathcal{P})$ on $\mathcal{S}_0(K)$. Moreover, we can prove

**Proposition 4.4.** *With the notation above, and with respect to a fixed Riemannian metric $g$ on the base manifold $N$, the formula*

$$\tilde{f}A = -L_Y A + 2\pi\sqrt{-1}(\varphi + \lambda(Y))A + 2\pi\sqrt{-1}(-i(Y)F + d\varphi)\wedge A + 2\pi\sqrt{-1}i(-i(Y)F + d\varphi)A \quad (A \in \wedge^* N \otimes \boldsymbol{C}) \tag{4.14}$$

*defines a super-geometric quantization of the observables $f = \mu(Y)+\varphi \in \mathcal{Q}(M,\mathcal{P})$, such that $Y$ is a $g$-Killing vector field of $N$, on the linear superspace $\wedge^*_c N \otimes \boldsymbol{C}$ ($c$ means "with compact support") with the odd-even grading. This quantization is by anti-Hermitian operators with respect to the scalar product defined by $g$ on the forms of $N$.*

*Proof.* In the right hand side of (4.14), the first two terms are $\hat{f}A$ (as one can see by using (4.9), (4.11), (4.12)), and the third term is the odd cocycle (4.13). Moreover, the operator of the fourth term is the $g$-adjoint of the operator of the third term. Therefore, we must only check that this fourth term behaves like a superquantization 1-cocycle i.e., it satisfies the conditions (2.11), $\forall f = \mu(Y) + \varphi$, $g = \mu(Z) + \psi$, where $Y, Z$ are $g$-Killing vector fields of $N$, $\varphi, \psi \in C^\infty(N)$. The second condition (2.11) is obvious, and, for the first, we compute the corresponding expressions for $l(f) = i(-i(Y)F + d\varphi)$, and in the following cases.

a) $f = \varphi$, $g = \psi$. Then, with (4.11), $\{f,g\} = 0$, and $l(\{f,g\}) = 0$. Furthermore, $[\hat{\varphi}, l(\psi)] = 0$, $[l(\varphi), \hat{\psi}] = 0$.

b) $f = \varphi$, $g = \mu(Z)$. Then, $\{f,g\} = X_f g = Zf$, and $l(\{f,g\}) = i(dZ\varphi)$. Furthermore, we obtain

$$[\hat{\varphi}, l(g)] + [l(\varphi), \hat{g}] = i([Z, \sharp_g d\varphi]) = i(\sharp_g dZ\varphi) = i(dZ\varphi).$$

We used that $\forall \alpha \in \wedge^1(M)$, $i(\alpha) := i(\sharp_g \alpha)$, and that $Z$ is Killing i.e., $L_Z \sharp_g = 0$.

c) $f = \mu(Y)$, $g = \mu(Z)$. Then

$$\{f,g\} = X_{\mu(Y)}\mu(Z) \overset{(4.12)}{=} -\mu([Y,Z]) - F(Y,Z), \tag{4.15}$$

and, since $dF = 0$,

$$\begin{aligned} l(\{f,g\}) &= i(i([Y,Z])F - d(F(Y,Z))) \\ &= -i(i(Z)L_Y F - L_Y i(Z)F + d(F(Y,Z))) \\ &= -i(i(Z)di(Y)F - i(Y)di(Z)F + 2d(F(Y,Z))). \end{aligned} \tag{4.16}$$

Furthermore, using again the general relations that exist among $L_X$, $i(X)$, $d$ for any vector field $X$, we get

$$\begin{aligned} [\hat{f}, l(g)] + [l(f), \hat{g}] &= i([Z, \sharp_g i(Y)F] - [Y, \sharp_g i(Z)F]) \\ &= i(\sharp_g L_Z i(Y)F - \sharp_g L_Y i(Z)F) \end{aligned} \tag{4.17}$$

(because $Y$, $Z$ are Killing vector fields), and the final result will be the same as in (4.16). □

**Remark 4.5.** If $\lambda$ is used instead of $\theta$, the same results as in Proposition 4.4, can be proven in the same way for

$$\tilde{f}A = -L_Y A + 2\pi\sqrt{-1}(\varphi + \lambda(Y))A - \\ - 2\pi\sqrt{-1}(L_Y\lambda) \wedge A - 2\pi\sqrt{-1}i([Y, \sharp_g\lambda])A. \tag{4.18}$$

In (4.18), the notation and the hypotheses are the same as in Proposition 4.4.

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