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SOLUTION OF STEFAN PROBLEMS BY FULLY DISCRETE LINEAR SCHEMES

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Abstract. This paper deals with a class of nonlinear parabolic problems with nonlinear boundary conditions. Stefan problems and porous medium equations are included. The enthalpy formulation and the variational technique are used. Convergence of a fully discrete linear approximation scheme is studied.

1. Introduction

In our paper we shall consider the convergence of a fully discrete linear approximation scheme. This scheme can be used in solving nonlinear Stefan-like parabolic problems with nonlinear boundary conditions:

$$\begin{aligned} \partial_t u - \nabla \cdot (k(\beta(u))\nabla\beta(u)) &= f(t,x,\beta(u)) && \text{on } Q := (0,T)\times\Omega, \\ u(0,x) &= u_p(x) && \text{on } \Omega, \\ -\nu\cdot k(\beta(u))\nabla\beta(u) &= g(t,x,\beta(u)) && \text{for } x\in\Gamma,\ t\in(0,T), \end{aligned}$$

where $\Omega \subset \mathbb{R}^d$ is a polygonal convex domain with the boundary Γ, $T < \infty$, ν is the outward normal to Γ, the functions f, g, β, k are Lipschitz continuous, $\beta\colon \mathbb{R} \to \mathbb{R}$ is nondecreasing and $k(s)$ is a positive definite symmetric $d \times d$-matrix for any $s \in \mathbb{R}$.

The use of linear approximation schemes for solving these problems from both the theoretical and numerical point of view has been extensively studied. A linear approximation scheme based on the so-called nonlinear Chernoff formula with constant relaxation parameter μ was studied in [**1**], [**12**], [**14**], [**8**].

Another linear approximation scheme was investigated in [**5**], [**6**], [**7**], [**3**]. There the authors used an approximation scheme of the type

$$\begin{gathered} \mu_i(\theta_i - \beta(u_{i-1})) - \tau\Delta\theta_i = \tau f(t_i, x, \beta(u_{i-1})) \quad \text{in } \ \Omega, \\ -\partial_\nu\theta_i = g(t_i, x, \theta_{i-1}) \quad \text{or} \quad u_i = 0 \quad \text{on } \ \Gamma, \\ |\beta(u_{i-1} + \mu_i(\theta_i - \beta(u_{i-1}))) - \beta(u_{i-1})| \le \ \alpha|\theta_i - \beta(u_{i-1})| + o\Big(\frac{1}{\sqrt{n}}\Big), \\ u_i := u_{i-1} + \mu_i(\theta_i - \beta(u_{i-1})), \quad i = 1, \dots, n \end{gathered}$$

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where θ_i is an approximation of the function $\beta(u)$ at time t_i, $\mu_i \in L_\infty(\Omega)$, $0 < \delta \le \mu_i \le K$ and where K^{-1}, $1 - 2\alpha$ and δ are sufficiently small constants. This scheme has the disadvantage of not being explicit with respect to θ_i and μ_i and therefore some iterative method for determining them must be used. In the papers mentioned above, the convergence and energy error estimates for this semi-discretization scheme were established for both strictly monotone ([**5**]) and nondecreasing ([**7**]) function β. The idea of our linear approximation scheme is similar but we introduce the following two novelties:

• We consider a fully discrete scheme, that means we discretize not only in time, but we also use the finite element method for space discretization.

• We linearize not only the function β but also nonlinearities in the right hand side of the equation and the nonlinearity in the boundary condition.

In Subsection 4, we prove the convergence of an iterative method for finding the functions μ_i, θ_i at each time step t_i. In Subsections 5 and 6, the convergence of our linear approximation scheme is proved.

Some of our proofs are only sketched; the details can be found in [**4**].

2. Basic Notations, Assumptions and Basic Results of the Finite Element Method

We denote

$$(\mathrm{H}_\Omega) \qquad \begin{cases} \Omega \subset \mathbb{R}^d \ (d \ge 1) \text{ is polygonal convex domain with boundary } \Gamma, \\ Q := I \times \Omega, \text{ where } I = (0,T),\ 0 < T < \infty \text{ and } T \text{ is fixed.} \end{cases}$$

Further we shall use function spaces in [**10**]: $L_2(\Omega)$ with norm $||\cdot||$, $L_2(I, L_2(\Omega)) = L_2(Q)$ with norm $||\cdot||_{L_2(I,L_2(\Omega))}$ and $H^1(\Omega) := W^{1,2}(\Omega)$ with norm $||.||_H$. We denote the dual space to $H^1(\Omega)$ by $H^*(\Omega)$. Finally, we shall use function space $L_2(\Gamma)$ with norm $||\cdot||_\Gamma$ and inner product $\langle\cdot,\cdot\rangle_\Gamma$. We use the notation $\langle\cdot,\cdot\rangle$ both for the inner product in $L_2(\Omega)$ and the duality pairing between $H^1(\Omega)$ and $H^*(\Omega)$.

We shall assume that:

$$(\mathrm{H}_\beta) \qquad \begin{cases} \beta\colon \mathbb{R} \to \mathbb{R} \text{ is a nondecreasing Lipschitz continuous function,} \\ \beta(0) = 0,\ 0 \le l_\beta \le \beta'(s) \le L_\beta < \infty \text{ for almost all } s \in \mathbb{R}, \\ \text{there exists } \lim\limits_{|s|\to\infty} \beta'(s) = c_1 > 0, \end{cases}$$

$$(\mathrm{H}_f) \qquad \begin{cases} f\colon I \times \Omega \times \mathbb{R} \to \mathbb{R} \text{ is a Lipschitz continuous function with} \\ \text{Lipschitz constant } L_f, \text{ and there exists a constant } C \text{ such that:} \\ |f(t,x,0)| \le C \text{ for a.a. } t \in I,\ x \in \Omega, \end{cases}$$

$$(\mathrm{H}_g) \qquad \begin{cases} g\colon I\times\Gamma\times\mathbb{R}\to\mathbb{R} \text{ is a Lipschitz continuous function with} \\ \text{Lipschitz constant } L_g, \text{ and there exists a constant } C \text{ such that:} \\ |g(t,x,0)|\le C \text{ for a.a. } t\in I,\ x\in\Gamma, \end{cases}$$

$$(\mathrm{H}_{u_p}) \qquad u_p\in H^1(\Omega),\ \beta(u_p)\in H^1(\Omega),$$

$$(\mathrm{H}_k) \qquad \begin{cases} k\colon\mathbb{R}\to\mathbb{R}^{d\times d} \text{ is Lipschitz continuous and bounded,} \\ k(s) \text{ is a symmetric and (uniformly) positive definite matrix;} \\ \text{that is there exist positive constants } k_0, K_0\in\mathbb{R} \text{ such that} \\ k_0|\xi|^2\le\xi^T k(s)\xi\le K_0|\xi|^2 \text{ for all } \xi\in\mathbb{R}^d \text{ and } s\in\mathbb{R} \end{cases}$$

Let $\{S_h\}_h$ be a family of decompositions

$$S_h=\{S_k\}_{k=1}^{K_h}$$

of Ω into closed d-simplices so that: $\bar\Omega=\cup_{k=1}^{K_h}S_k$ where h is the mesh size: $h=\sup_{k=1,\dots,K_h}\operatorname{diam}S_k$.

We assume that

$$(\mathrm{H}_{S_h}) \qquad \begin{cases} \text{the family } \{S_h\} \text{ has the usual properties of finite elements,} \\ \text{i.e. it is regular and also the assumption of inverse inequality} \\ \text{holds in the sense of Ciarlet ([\textbf{2}, p. 134, p. 142]).} \end{cases}$$

The assumptions above will be simply denoted by (H), that is

$$(\mathrm{H}) \qquad (\mathrm{H}_\Omega),\ (\mathrm{H}_\beta),\ (\mathrm{H}_k),\ (\mathrm{H}_f),\ (\mathrm{H}_g),\ (\mathrm{H}_{S_h}),\ (\mathrm{H}_{u_p})$$

Further we denote:

$$V_h^1:=\{\psi\in C^0(\bar\Omega):\psi/_{S_k} \text{ is linear for all } k=1,\dots,K_h\},$$
$$V_h^0:=\{\psi:\psi/_{S_k} \text{ is constant for all } k=1,\dots,K_h\}.$$

On the boundary Γ, we define the discrete inner product by:

$$\langle\psi,\phi\rangle_{h,\Gamma}:=\sum_{k=1}^{K_h}\int_{S_k\cap\Gamma}\Pi_h(\psi\phi)\,ds \tag{2.1}$$

for any piecewise continuous functions ψ,ϕ in $\bar\Omega$, where Π_h stands for the local linear interpolation operator ([**2**]).

For any $\psi, \varphi \in V_h^1$, we have ([**15**]):

$$|\langle \psi, \varphi \rangle_{h,\Gamma}| \leq C_3 ||\psi||_H ||\varphi||_H \tag{2.2}$$

and

$$|\langle \psi, \varphi \rangle_\Gamma - \langle \psi, \varphi \rangle_{h,\Gamma}| \leq C_5 h ||\psi||_H ||\varphi||_H. \tag{2.3}$$

The estimate (2.3) holds also for functions $\varphi \in V_h^1$ and $\psi = b \cdot v$, where $v \in V_h^1$ and $b \in V_h^0$; $||b||_{L_\infty} \leq K$, $K > 0$.
We denote by $a(\cdot,\cdot)$ the inner product in $H^1(\Omega)$:

$$a(w, z) := \langle \nabla w, \nabla z \rangle + \langle w, z \rangle.$$

We now introduce L_2-projection operator $P_h^0 \colon L_2(\Omega) \to V_h^0$, which is defined by

$$\langle P_h^0 z, \psi \rangle = \langle z, \psi \rangle \text{ for any } \psi \in V_h^0,\ z \in L_2(\Omega)$$

and satisfies

$$||z - P_h^0 z||_{H^{-s}(\Omega)} \leq C_8 h^{r+s} ||z||_{H^r(\Omega)}, \quad 0 \leq s, r \leq 1. \tag{2.4}$$

We also introduce the discrete H^1-projection operator $P_h^1 \colon H^1(\Omega) \to V_h^1$, which is defined by

$$a(z - P_h^1 z, \psi) = 0 \text{ for any } \psi \in V_h^1,\ z \in H^1(\Omega)$$

and satisfies ([**15**])

$$||z - P_h^1 z||_{H^s(\Omega)} \leq C_9 h^{2-(r+s)} ||z||_{H^{2-r}(\Omega)}, \quad 0 \leq s, r \leq 1. \tag{2.5}$$

We shall use the following well-known facts (see e.g. [**13**, p. 15].)

$$ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2} \text{ for any } a, b \in \mathbb{R} \text{ and } \varepsilon > 0, \tag{2.6}$$

$$2a(a-b) = a^2 - b^2 + (a-b)^2 \text{ for any } a, b \in \mathbb{R}, \tag{2.7}$$

$$||v||_\Gamma^2 \leq C_{10} \left(\varepsilon ||\nabla v||^2 + \frac{1}{\varepsilon} ||v||^2 \right) \text{ for any } v \in W^{1,2}(\Omega) \text{ and } \varepsilon > 0 \text{ small.} \tag{2.8}$$

We conclude with some notations concerning the time discretization. Let $\tau = \frac{T}{n}$ be the time step and $t_i = i\tau$, $I_i = (t_{i-1}, t_i]$ for $1 \leq i \leq n$. We also set $z^i := z(\cdot, t_i)$ or $\bar{z}^i := \frac{1}{\tau} \int_{I_i} z(\cdot, t)\, dt$ for any continuous or integrable (in time) function z defined in Q, respectively, and $\partial z^i := (z^i - z^{i-1})/\tau$, $1 \leq i \leq n$ for any given family $\{z^i\}_{i=0}^n$. Similarly we denote $\delta u_i = (u_i - u_{i-1})/\tau$ for $i = 1, \ldots, n$ and $u_i, u_{i-1} \in V_h^0$.

For simplicity we shall denote $f(s) := f(x, t, s)$ and $g(s) := g(x, t, s)$. By C we denote a generic constant which is independent of time and space discretization τ and h.

3. Formulations of Problems

The variational formulation of our problem is as follows.
Problem (P): Find $\{u(t,x), \theta(t,x)\}$ such that

$$u(t,x) \in L_2(I; L_2(\Omega)),\ \partial_t u(t,x) \in L_2(I; H^*(\Omega)) \text{ and } u(0,x) = u_p(x) \tag{3.1}$$

$$\theta(t,x) \in L_2(I; H^1(\Omega)) \tag{3.2}$$

$$\theta(t,x) = \beta(u(t,x)) \text{ for a.e.}(t,x) \in Q \tag{3.3}$$

and for all $\varphi \in L_2(I, H^1(\Omega))$, the following equation holds

$$\int_I \big(\langle \partial_t u, \varphi\rangle + \langle k(\theta)\nabla\theta, \nabla\varphi\rangle + \langle g(\theta), \varphi\rangle_\Gamma\big)\, dt = \int_I \langle f(\theta), \varphi\rangle\, dt. \tag{3.4}$$

As we have mentioned above, the following approximation scheme is based on the ideas of [**5**] and [**7**] (see the approximation scheme in Introduction). The basic idea of this linear scheme consists of determining a variable relaxation parameter $\mu_i = \mu_i(x)$ to be used in the algebraic correction $u_i := u_{i-1} + \mu_i(\theta_i - \beta(u_{i-1}))$, $i = 1, \ldots, n$, where θ_i is the approximation of $\beta(u)$ at time t_i. Moreover, the same iterative method for finding functions μ_i and θ_i can be used also for improving the approximation of nonlinearities on the boundary and on the right hand side. In the same way as in the case of the nonlinearity $\beta(u)$ we can use the variable relaxation parameters also for nonlinearities on the boundary (function $g(\beta(u))$ and on the right hand side (function $f(\beta(u))$. We denote these relaxation parameters by ω_i (for the function g) and ρ_i (for the function f). We use a fully discrete scheme including the finite element method.

Therefore, our discrete problem is of the form:
Problem $(P_{h,\tau})$: For any $1 \le i \le n$, find $\{u_i(x), \theta_i(x)\}$ such that $u_i \in V_h^0$ and $\theta_i \in V_h^1$, and the functions μ_i, ρ_i and $\omega_i \in V_h^0$ which have the following properties:

$$u_0 := P_h^0 u_p(x), \quad \theta_0 := P_h^1(\beta(u_p(x))), \tag{3.5}$$

$$\begin{aligned} &\langle \mu_i P_h^0 \theta_i, \psi\rangle + \tau\langle k_{i-1}\nabla\theta_i, \nabla\psi\rangle + \tau\langle g(\theta_{i-1}), \psi\rangle_{h,\Gamma} + \tau\langle \omega_i(\theta_i - \theta_{i-1}), \psi\rangle_{h,\Gamma} \\ &\quad = \langle \mu_i \beta(u_{i-1}), \psi\rangle + \tau\langle f(\beta(u_{i-1})), \psi\rangle + \tau\langle \rho_i(P_h^0\theta_i - \beta(u_{i-1})), \psi\rangle \\ &\qquad + \langle p_i, \psi\rangle \text{ for all } \psi \in V_h^1, \end{aligned} \tag{3.6}$$

$$\begin{aligned} &||\beta(u_{i-1} + \mu_i(P_h^0\theta_i - \beta(u_{i-1}))) - \beta(u_{i-1})||_{L_2,\mu_i} \\ &\quad \le \alpha ||P_h^0\theta_i - \beta(u_{i-1})||_{L_2,\mu_i} + o\Big(\frac{1}{\sqrt{n}}\Big), \quad \text{(convergence condition)} \end{aligned} \tag{3.7}$$

$$u_i := u_{i-1} + \mu_i(P_h^0\theta_i - \beta(u_{i-1})), \quad \text{(algebraic correction)} \tag{3.8}$$

for $i = 1, \dots, n$, where the "error" functional p_i satisfies

$$
\begin{aligned}
&|\langle p_i, \psi\rangle| \le ||p_i||_{H^*(\Omega)} ||\psi||_H \text{ for all } \psi \in H^1(\Omega) \text{ and} \\
&||p_i||_{H^*} = O\Big(\frac{1}{n^\sigma}\Big), \quad \text{where } \sigma > 1.
\end{aligned} \tag{3.9}
$$

Moreover, we have $0 < \delta \le \mu_i \le K$, $|\rho_i| \le K$, $|\omega_i| \le K$ and $k_{i-1} = k(\beta(u_{i-1}))$ and all members of this matrix are functions from V_h^0. Further $\alpha > 0$ and $K^{-1}, 1-2\alpha, \delta$ are sufficiently small positive constants. By $||\cdot||_{L_2,\mu_i}$ we denote the norm in the weighted L_2-space with the weight function μ_i (e.g. [**10**]). This scheme has the disadvantage of not being explicit with respect to θ_i and μ_i. We determine the values of the functions $\theta_i, \mu_i, \rho_i, \omega_i$ by an iterative process that we describe later.

In case the function β is not strictly monotone (e.g. for the Stefan problem), we approximate the function β by a strictly monotone Lipschitz continuous function β_n (with Lipschitz constant L_{β_n}), such that

$$
||\beta_n - \beta||_{L_\infty(R)} = o\Big(\frac{1}{\sqrt{n}}\Big), \ \beta_n \in C^2, \ 0 < \delta_n < \beta_n'(s) \le L_{\beta_n} \le L_\beta \tag{3.10}
$$

for all n.

4. Convergence of an Iterative Method

As we have mentioned above, the approximate scheme (3.5)–(3.8) has the disadvantage of not being explicit with respect to the functions μ_i and θ_i. There are several iterative schemes for solving this problem (see for example [**4**]). The scheme that we establish in this paper is based on the idea of an iterative method in [**11**].

Put:

$$
\gamma_K(s) = \begin{cases} K & \text{if } s > K, \\ s & \text{if } -K \le s \le K, \\ -K & \text{if } s < -K. \end{cases} \tag{4.1}
$$

Let $1 \le i \le n$ be fixed. Then we define our iterative method as follows:

$$
\begin{aligned}
&\langle(\mu_{i,k-1} - \tau\rho_{i,k-1})(P_h^0\theta_{i,k} - \beta(u_{i-1})), \psi\rangle + \tau\langle k_{i-1}\nabla\theta_{i,k}, \nabla\psi\rangle \\
&\qquad + \tau\langle\omega_{i,k-1}(\theta_{i,k} - \theta_{i-1}), \psi\rangle_{h,\Gamma} \\
&\qquad = \tau\langle f(\beta(u_{i-1})), \psi\rangle - \tau\langle g(\theta_{i-1}), \psi\rangle_{h,\Gamma} \text{ for all } \psi \in V_h^1. \\
&\mu_{i,0} := \gamma_K\left(\frac{1}{\beta_n'(u_{i-1})}\right), \\
&\rho_{i,0} := \gamma_K(P_h^0(g'(\theta_{i-1}))),
\end{aligned} \tag{4.2}
$$

$$
\begin{aligned}
&\omega_{i,0} := \gamma_K(P_h^0(f'(\theta_{i-1}))).\\
&\bar{\mu}_{i,k} := \gamma_K\left(\frac{\beta_n^{-1}(\beta_n(u_{i-1}) + \alpha(P_h^0\theta_{i,k} - \beta(u_{i-1}))) - u_{i-1}}{P_h^0\theta_{i,k} - \beta(u_{i-1})}\right)\\
&\mu_{i,k} := \bar{\mu}_{i,k}, \quad \text{for } 1 \le k \le k_0,\ k_0 \in N \text{ be fixed}\\
&\mu_{i,k} := \min\{\bar{\mu}_{i,k}, \mu_{i,k-1}\}, \ \text{for } k > k_0.\\
&\omega_{i,k} := \gamma_K\left(\frac{g(P_h^0\theta_{i,k}) - g(P_h^0\theta_{i-1})}{P_h^0(\theta_{i,k} - \theta_{i-1})}\right)\\
&\rho_{i,k} := \gamma_K\left(\frac{f(P_h^0\theta_{i,k}) - f(\beta(u_{i-1}))}{P_h^0\theta_{i,k} - \beta(u_{i-1})}\right) \quad \text{for } k = 1, \ldots
\end{aligned}
\tag{4.3}
$$

If $P_h^0\theta_{i,k} = \beta(u_{i-1})$ then $\mu_{i,k} := \mu_{i,k-1}$ and $\rho_{i,k} := \rho_{i,k-1}$. Similarly, if $P_h^0(\theta_{i,k} - \theta_{i-1}) = 0$ then $\omega_{i,k} := \omega_{i,k-1}$.

The solvability of (4.2)–(4.3) follows from the theory of monotone operators (see [**4**]).

The sequences $\{\mu_{i,k}\}$, $\{\rho_{i,k}\}$, $\{\omega_{i,k}\}$, $\{\theta_{i,k}\}$ of iterations help us to find functions μ_i, ρ_i, ω_i, θ_i as it is formulated in the following theorem:

Theorem 1. *Assume* (H). *Let $1 \le i \le n$ be fixed and $\tau \le \tau_0$, $h \le h_0$ for sufficiently small τ_0, h_0. If $\{\mu_{i,k}\}$, $\{\rho_{i,k}\}$, $\{\omega_{i,k}\}$ and $\{\theta_{i,k}\}$ are sequences from* (4.2)–(4.3) *then there exists index $l \in N$ such that the functions $\mu_i := \mu_{i,l}$, $\rho_i := \rho_{i,l}$, $\omega_i := \omega_{i,l}$ from V_h^0 and $\theta_i := \theta_{i,l} \in V_h^1$ satisfy the equation* (3.6) *and the inequality* (3.7).

Proof.

First we denote $v_k = \mu_{i,k} - \tau\rho_{i,k}$. It is obvious that this function is positive for sufficiently small τ_0, hence $v_k \ge \gamma$ for some $\gamma > 0$.

The monotonicity of the sequence $\{\mu_{i,k}\}$ for $k \ge k_0$ implies $\mu_{i,k} \to \mu$ pointwise in Ω. Since the functions $\mu_{i,k}$ are bounded, we have $\mu_{i,k} \to \mu$ in $L_p(\Omega)$ for all $p > 1$.

We can easily prove that the sequence $\{\theta_{i,k}\}$ is bounded in $H^1(\Omega)$:

In (4.2) we choose as a test function $\psi = \theta_{i,k}$ and after some rearrangement we obtain:

$$
\begin{aligned}
&\langle v_{k-1}(P_h^0\theta_{i,k} - \beta(u_{i-1})), \theta_{i,k}\rangle + \tau k_0||\nabla\theta_{i,k}||^2 + \tau\langle\omega_{i,k-1}(\theta_{i,k} - \theta_{i-1}), \theta_{i,k}\rangle_{h,\Gamma}\\
&\qquad \le \tau\langle f(\beta(u_{i-1})), \theta_{i,k}\rangle - \tau\langle g(\theta_{i-1}), \theta_{i,k}\rangle_{h,\Gamma}.
\end{aligned}
$$

Using the properties of the functions v_k and $\omega_{i,k}$, and due also to (2.2), (2.3), (2.4) and (2.6), we have:

$$
\begin{aligned}
\gamma||\theta_{i,k}||^2 + \tau k_0||\nabla\theta_{i,k}||^2 \le\ & \frac{\gamma}{2}||\theta_{i,k}||^2 + C(h^2+\tau^2)||\theta_{i,k}||_H^2 + C||\beta(u_{i-1})||^2\\
&+ \tau\frac{k_0}{4}||\nabla\theta_{i,k}||^2 + C\tau||\theta_{i,k}||^2 + C||\theta_{i-1}||^2\\
&+ C||f(\beta(u_{i-1}))||^2 + C||g(\theta_{i-1})||_H^2.
\end{aligned}
$$

Now using the properties of the functions $f, g, \theta_{i-1}, \beta(u_{i-1})$, we obtain for sufficiently small $h < h_0$:

$$||\theta_{i,k}||_H \leq C.$$

From the properties of the sequence $\{\mu_{i,k}\}$ it follows that there exists an index l such that $||\mu_{i,l-1} - \mu_{i,l}||_{L_p(\Omega)} \leq C\tau^\sigma$ for $\sigma > 1$. We set $\theta_i := \theta_{i,l}$, $\mu_i := \mu_{i,l}$, $\rho_i := \rho_{i,l-1}$ and $\omega_i := \omega_{i,l-1}$. We must only verify, that for these functions, the conditions (3.6) and (3.7) hold. It is clear that $\mu_i = \mu_{i,l} \leq \bar{\mu}_{i,l}$ and that is why (3.7) holds.

For $\theta_i := \theta_{i,l}$ the equation (4.2) holds. We have:

$$\begin{aligned}
&\langle \mu_i(P_h^0\theta_i - \beta(u_{i-1})), \psi\rangle + \tau\langle k_{i-1}\nabla\theta_i, \nabla\psi\rangle \\
&\qquad + \tau\langle \omega_i(\theta_i - \theta_{i-1}), \psi\rangle_{h,\Gamma} - \tau\langle \rho_i(P_h^0\theta_i - \beta(u_{i-1})), \psi\rangle \\
&\quad = \langle (\mu_{i,l} - \mu_{i,l-1})(P_h^0\theta_i - \beta(u_{i-1}), \psi\rangle + \tau\langle f(\beta(u_{i-1})), \psi\rangle - \tau\langle g(\theta_{i-1}), \psi\rangle_{h,\Gamma}.
\end{aligned}$$

We now set $\langle p_i, \psi\rangle = \langle (\mu_{i,l} - \mu_{i,l-1})(P_h^0\theta_i - \beta(u_{i-1}), \psi\rangle$. From the properties of $\mu_{i,k}$, $P_h^0\theta_i$ and $\beta(u_{i-1})$ using generalized Hölder's inequality we obtain that the functional p_i satisfies (3.9). This completes the proof. □

5. Stability of the Discrete Scheme

The problem (3.6)–(3.8) can be rewritten as follows

$$\begin{aligned}
(5.1)\qquad &\langle \delta u_i, \psi\rangle + \langle k_{i-1}\nabla\theta_i, \nabla\psi\rangle + \langle g(\theta_{i-1}), \psi\rangle_{h,\Gamma} \\
&\qquad + \langle \omega_i(\theta_i - \theta_{i-1}), \psi\rangle_{h,\Gamma} \\
&\quad = \langle f(\beta(u_{i-1})), \psi\rangle + \langle \rho_i(P_h^0\theta_i - \beta(u_{i-1})), \psi\rangle + \frac{1}{\tau}\langle p_i, \psi\rangle \\
&\qquad \text{for all } \psi \in V_h^1 \text{ and } ||p_i||_{H^*(\Omega)} = O(\tau^\sigma).
\end{aligned}$$

Lemma 1. *Assume* (H). *Then for $\tau \leq \tau_0$ and $h \leq h_0$ such that the equation*

$$(5.2)\qquad \tau = C_* h^\xi, \ \text{for } 0 < \xi \leq 2$$

is fulfilled and τ_0, h_0 are sufficiently small, there exists a constant C, independent of the discretization parameters such that

$$(5.3)\qquad \max_{1\leq i\leq n} ||\beta(u_i)|| + \sum_{i=1}^n ||u_i - u_{i-1}||^2 + \sum_{i=1}^n \tau||\nabla\theta_i||^2 \leq C.$$

Proof. The function $\tau\theta_i$ is an admissible test function in (5.1) because $\theta_i \in V_h^1$. So let $\tau\theta_i$ be a test function and sum (5.1) over i from 1 to $m \le n$. We have:

$$
\begin{aligned}
(5.4) \qquad & \sum_{i=1}^{m} \langle u_i - u_{i-1}, \theta_i\rangle + \sum_{i=1}^{m} \tau \langle k_{i-1} \nabla\theta_i, \nabla\theta_i\rangle + \sum_{i=1}^{m} \tau \langle g(\theta_{i-1}), \theta_i\rangle_{h,\Gamma} \\
& + \sum_{i=1}^{m} \tau \langle \omega_i(\theta_i - \theta_{i-1}), \theta_i\rangle_{h,\Gamma} =: I + II + III + IV \\
& = \sum_{i=1}^{m} \tau \langle f(\beta(u_{i-1})), \theta_i\rangle + \sum_{i=1}^{m} \tau \langle \rho_i(P_h^0\theta_i - \beta(u_{i-1})), \theta_i\rangle + \sum_{i=1}^{m} \langle p_i, \theta_i\rangle \\
& =: V + VI + VII.
\end{aligned}
$$

We shall estimate each resulting term. Some of these terms are similar as in [**5**] and [**7**] so we comment them only briefly.

First note that from (3.8) it holds:

$$P_h^0\theta_i = \frac{u_i - u_{i-1}}{\mu_i} + \beta(u_{i-1}).$$

We use (see e.g. [**12**]) the convex function

$$\Phi_\lambda(s) = \int_0^s \lambda(z)\,dz \quad \text{for all } s \in \mathbb{R},$$

where the function $\lambda\colon \mathbb{R} \to \mathbb{R}$ is absolutely continuous, $\lambda(0) = 0$ and $0 \le \lambda' \le \Lambda < \infty$. The function Φ_λ has the following properties:

$$\frac{1}{2\Lambda}\lambda^2(s) \le \Phi_\lambda(s) \le \frac{\Lambda}{2}s^2 \quad \text{for } s \in \mathbb{R}.$$

The difference $u_i - u_{i-1} \in V_h^0$, hence the definition of P_h^0 and the monotonicity of β imply

$$
\begin{aligned}
\sum_{i=1}^{m} \langle u_i - u_{i-1}, \theta_i\rangle &= \sum_{i=1}^{m} \langle u_i - u_{i-1}, P_h^0\theta_i\rangle \\
&\ge \sum_{i=1}^{m} \left\langle u_i - u_{i-1}, \frac{u_i - u_{i-1}}{\mu_i} \right\rangle + \sum_{i=1}^{m} \int_\Omega \left(\Phi_\beta(u_i) - \Phi_\beta(u_{i-1})\right) dX \\
&\quad - \sum_{i=1}^{m} \langle u_i - u_{i-1}, \beta(u_i) - \beta(u_{i-1})\rangle.
\end{aligned}
$$

Further in the same way as in [**5**] we arrive at

$$(5.5) \qquad I \ge (1 - 2\alpha) \sum_{i=1}^{m} ||u_i - u_{i-1}||^2_{L_2, \frac{1}{\mu_i}} + \frac{1}{2L_\beta}||\beta(u_m)||^2 - C.$$

We estimate the second term in (5.4) easily using (H_k):

$$\tau\sum_{i=1}^{m}\langle k_{i-1}\nabla\theta_i,\nabla\theta_i\rangle \geq \tau k_0\sum_{i=1}^{m}||\nabla\theta_i||^2. \tag{5.6}$$

In the third term in (5.4) we use (2.3), (H_g), (H_β), (2.6), (2.8):

$$\begin{aligned}|III| &\leq \tau\sum_{i=1}^{m}\Big|\langle g(\theta_{i-1}),\theta_i\rangle_\Gamma\Big| + \tau\sum_{i=1}^{m}\Big|\langle g(\theta_{i-1}),\theta_i\rangle_\Gamma - \langle g(\theta_{i-1}),\theta_i\rangle_{h,\Gamma}\Big|\\ &\leq \tau C\sum_{i=1}^{m}||\theta_i||^2 + \tau\frac{k_0}{8}\sum_{i=1}^{m}||\nabla\theta_i||^2 + \tau hC\sum_{i=1}^{m}(||\theta_i||^2 + ||\nabla\theta_i||^2) + C.\end{aligned}$$

Now for sufficiently small h_0 we get

$$\big|III\big| \leq \tau C\sum_{i=1}^{m}||\theta_i||^2 + \tau\frac{k_0}{4}\sum_{i=1}^{m}||\nabla\theta_i||^2 + C. \tag{5.7}$$

For the fourth term we have:

$$\big|IV\big| \leq \tau C\sum_{i=1}^{m}||\theta_i||^2 + \tau\frac{k_0}{8}\sum_{i=1}^{m}||\nabla\theta_i||^2 + \tau hC\sum_{i=1}^{m}||\theta_i||_H^2 + C,$$

where we used (2.3), (2.6), (2.8), (2.7).

Now again for sufficiently small h_0 we have:

$$\big|IV\big| \leq \tau C\sum_{i=1}^{m}||\theta_i||^2 + \tau\frac{3k_0}{16}\sum_{i=1}^{m}||\nabla\theta_i||^2 + C, \tag{5.8}$$

In the term V, we use (H_f), the definition of the operator P_h^0 ($u_i \in V_h^0$), (H_β) and the relation for θ_i as in the first term:

$$\begin{aligned}\big|V\big| &\leq \tau L_f\sum_{i=1}^{m}\Big|\Big\langle\big|\beta(u_{i-1})\big|,\Big|\frac{u_i-u_{i-1}}{\mu_i}+\beta(u_{i-1})\Big|\Big\rangle\Big|\\ &\quad + \tau\frac{\delta}{4}\sum_{i=1}^{m}||\frac{u_i-u_{i-1}}{\mu_i}+\beta(u_{i-1})||^2 + \frac{C}{8\delta}||f(0)||^2\\ &\leq \tau C\sum_{i=1}^{m}||\beta(u_i)||^2 + \tau\sum_{i=1}^{m}||u_i-u_{i-1}||^2_{L_2,\frac{1}{\mu_i}} + C\,.\end{aligned} \tag{5.9}$$

For the term VI, we use the property of P_h^0 and (3.9), (3.8), (2.6):

$$\big|VI\big| \leq \tau C\sum_{i=1}^{m}||u_i-u_{i-1}||^2_{L_2,\frac{1}{\mu_i}} + \tau C\sum_{i=1}^{m}||\beta(u_i)||^2 + C. \tag{5.10}$$

From the property of the functional p_i in the last term we easily get:

$$(5.11)\quad \left|VII\right| \le \sum_{i=1}^{m} |\langle p_i, \theta_i\rangle| \le \frac{4}{\tau k_0}\sum_{i=1}^{m} ||p_i||^2_{H^*(\Omega)} + \tau\frac{k_0}{16}\sum_{i=1}^{m} ||\theta_i||^2_H$$
$$\le C + \tau\frac{k_0}{16}\sum_{i=1}^{m}\left(||\nabla\theta_i||^2 + ||\theta_i||^2\right).$$

Now for sufficiently small τ_0 and from (5.5)–(5.11) we have:

$$\frac{7(1-2\alpha)}{8}\sum_{i=1}^{m} ||u_i - u_{i-1}||^2_{L_2,\frac{1}{\mu_i}} + \frac{1}{2L_\beta}||\beta(u_m)||^2 + \frac{k_0}{2}\sum_{i=1}^{m}\tau||\nabla\theta_i||^2$$
$$\le C + C_b\tau\sum_{i=1}^{m}||\beta(u_i)||^2 + \tau C_t\sum_{i=1}^{m}||\theta_i||^2,$$

where C_b and C_t are constants independent of time and space discretization.

We rearrange the last term of this inequality using (2.4) and the property of $P_h^0\theta_i$ in this way:

$$\tau C_t\sum_{i=1}^{m}||\theta_i||^2 \le \tau\frac{4C_t}{\delta}\sum_{i=1}^{m}||u_i - u_{i-1}||^2_{L_2,\frac{1}{\mu_i}} + \tau 4C_t\sum_{i=1}^{m}||\beta(u_i)||^2$$
$$+ \tau h^2 2C_t C_8^2\sum_{i=1}^{m}||\theta_i||^2_H.$$

Using this estimate for sufficiently small τ_0 we have:

$$\frac{3(1-2\alpha)}{4}\sum_{i=1}^{m} ||u_i - u_{i-1}||^2_{L_2,\frac{1}{\mu_i}} + \frac{1}{2L_\beta}||\beta(u_m)||^2 + \frac{k_0}{2}\sum_{i=1}^{m}\tau||\nabla\theta_i||^2$$
$$\le C + (C_b + 4C_t)\tau\sum_{i=1}^{m}||\beta(u_i)||^2 + \tau h^2 C\sum_{i=1}^{m}||\theta_i||^2_H.$$

We add to the both sides of this inequality the term $\frac{k_0}{4}\tau||\theta_m||^2$ and we estimate this term on the right hand side as above:

$$\frac{k_0}{4}\tau||\theta_m||^2 \le \tau\frac{k_0}{\delta}||u_m - u_{m-1}||^2_{L_2,\frac{1}{\mu_m}} + \tau k_0||\beta(u_{m-1})||^2 + \tau h^2 k_0 C_8^2||\theta_m||^2_H.$$

Now let the ratio between τ and h be chosen in the following way:

$$\tau = C_* h^\xi,\ \text{where } 0 < \xi \le 2.$$

We conclude:

$$\frac{(1-2\alpha)}{2K}\sum_{i=1}^{m}||u_i-u_{i-1}||^2+\frac{1}{2L_\beta}||\beta(u_m)||^2+\tau\frac{k_0}{4}||\theta_m||_H^2+\frac{k_0}{4}\sum_{i=1}^{m}\tau||\nabla\theta_i||^2$$
$$\leq C+C\tau\sum_{i=1}^{m}||\beta(u_i)||^2+\tau^2Ch_0^{2-\xi}\sum_{i=1}^{m}||\theta_i||_H^2.$$

And now for sufficiently small τ_0 the assertion (5.3) follows as a consequence of the discrete Gronwall inequality. □

Consequence. *Let the assumptions of Lemma* 1 *hold. Then there exists constant C, independent of discretization parameters such that:*

$$\max_{i=1,\ldots,n}||P_h^0\theta_i||+\max_{i=1,\ldots,n}||u_i||\leq C, \tag{5.12}$$

$$\sum_{i=1}^{n}\tau||\theta_i||^2\leq C, \tag{5.13}$$

$$\sum_{i=1}^{n}\tau||\theta_i||_H^2\leq C. \tag{5.14}$$

Proof. From the results of Lemma 1 we have:

$$||P_h^0\theta_i||\leq C$$

uniformly for $i=1,\ldots,n$. Now (5.12) follows immediately from Lemma 1 and (H_β). Combining now these results and the results of Lemma 1 we have:

$$\sum_{i=1}^{n}\tau||\theta_i||^2\leq 2\sum_{i=1}^{n}\tau||\theta_i-P_h^0\theta_i||^2+2\sum_{i=1}^{n}\tau||P_h^0\theta_i||^2$$
$$\leq 2\tau\sum_{i=1}^{n}C_8^2h^2||\theta_i||_H^2+2TC\leq 2C+2C\tau h^2\sum_{i=1}^{n}(||\theta_i||^2+||\nabla\theta_i||^2)$$
$$\leq C+Ch^2\sum_{i=1}^{n}\tau||\nabla\theta_i||^2+2C\tau h^2\sum_{i=1}^{n}||\theta_i||^2.$$

Using now also (5.3) we have

$$\tau(1-2Ch^2)\sum_{i=1}^{n}||\theta_i||^2\leq C$$

and for $h\leq h_0$ sufficiently small we get (5.13) and (5.14) too. □

6. Convergence of the Method

We denote $\gamma := (\tau, h)$ the pair of discretization parameters, where τ and h have the same meaning as before. Then Rothe's function is defined as follows:

$$\theta^{(\gamma)}(t,x) := \theta_{i-1}(x) + \frac{t - t_{i-1}}{\tau}(\theta_i - \theta_{i-1}) \tag{6.1}$$

for $t \in \langle t_{i-1}, t_i \rangle$, $\theta_i \in V_h^1$ a $i = 1, \dots, n$.

We define the step function as:

$$\begin{aligned} \bar{\theta}^{(\gamma)}(t,x) &= \theta_i(x) \quad \text{for } t \in (t_{i-1}, t_i\rangle,\ i = 1, \dots, n, \\ \bar{\theta}^{(\gamma)}(0,x) &= \beta(u_0(x)). \end{aligned} \tag{6.2}$$

We define also the function

$$\begin{aligned} P_h^0 \bar{\theta}^{(\gamma)}(t,x) &= P_h^0 \theta_i \quad \text{for } t \in (t_{i-1}, t_i\rangle,\ i = 1, \dots, n, \\ P_h^0 \bar{\theta}^{(\gamma)}(0,x) &= \beta(u_0(x)). \end{aligned} \tag{6.3}$$

Analogously, we can define also Rothe's and the step function $u^{(\gamma)}(t,x)$ and $\bar{u}^{(\gamma)}(t,x)$, respectively.

We denote the time derivative of Rothe's function $u^{(\gamma)}$ by $u_t^{(\gamma)}$. For $t \in (t_{i-1}, t_i)$ we have:

$$u_t^{(\gamma)} = \frac{u_i - u_{i-1}}{\tau}.$$

We denote the dual space of the space V_h^1 by $V_h^{1,*}$.

We can prolongate all functions defined on Q by zero outside Q.

Lemma 2. *Under the assumption* (H) *and* (5.2) *there exists a constant* $C > 0$*, independent of mesh parameters, such that*

$$||u_t^{(\gamma)}||_{L_2(I, V_h^{1,*})} \le C \tag{6.4}$$

$$||u^{(\gamma)} - \bar{u}^{(\gamma)}||_{L_2(I, L_2(\Omega))} + ||\bar{u}^{(\gamma)}(. + \tau) - \bar{u}^{(\gamma)}(.)||_{L_2(I, L_2(\Omega))} \le \frac{C}{\sqrt{n}} \tag{6.5}$$

$$||\theta^{(\gamma)} - \bar{\theta}^{(\gamma)}||_{L_2(I, L_2(\Omega))} + ||\bar{\theta}^{(\gamma)} - \beta(\bar{u}^{(\gamma)})||_{L_2(I, L_2(\Omega))} \le \frac{C}{\sqrt{n}} \tag{6.6}$$

Proof. The proof of (6.4) is very simple using the duality argument in (5.1) and Lemma 1.

We obtain the estimate (6.5) also very easily using the results of Lemma 1. The second part of (6.6) can be estimated as follows:

$$\begin{aligned} ||\bar{\theta}^{(\gamma)} - \beta(\bar{u}^{(\gamma)})||_{L_2(I, L_2(\Omega))} &= \left(\sum_{i=1}^{n} \int_{I_i} ||\theta_i - \beta(u_i)||^2 \, dt \right)^{\frac{1}{2}} \\ &\le C\sqrt{\tau} \left(\sum_{i=1}^{n} ||P_h^0 \theta_i - \beta(u_{i-1})||^2 + \sum_{i=1}^{n} ||\beta(u_{i-1}) - \beta(u_i)||^2 + \sum_{i=1}^{n} ||\theta_i - P_h^0 \theta_i||^2 \right)^{\frac{1}{2}} \\ &\le C\sqrt{\tau} + Ch, \end{aligned}$$

where (H_β), (2.4), (5.3) and (5.14) were used. The final result can be obtained immediately due to (5.2).

Using this result we obtain

$$||\theta^{(\gamma)} - \bar{\theta}^{(\gamma)}||_{L_2(I,L_2(\Omega))} \le C\sqrt{\tau}\left(\sum_{i=1}^{n} ||P_h^0\theta_i - \theta_{i-1}||^2 + \sum_{i=1}^{n} ||\theta_i - P_h^0\theta_i||^2\right)^{\frac{1}{2}}$$
$$\le C\sqrt{\tau}\left(\sum_{i=1}^{n} ||u_i - u_{i-1}||^2 + \sum_{i=1}^{n} ||\beta(u_{i-1}) - \theta_{i-1}||^2 + C_8 h^2 \sum_{i=1}^{n} ||\theta_i||_H^2\right)^{\frac{1}{2}}$$
$$\le C\sqrt{\tau}, \qquad \square$$

Lemma 3. *Under the same assumptions of Lemma* 2 *we have the following estimate:*

(6.7) $$\int_0^{T-z} ||\bar{\theta}^{(\gamma)}(t+z) - \bar{\theta}^{(\gamma)}(t)||^2\, dt \le C(z + n^{-\frac{1}{2}})$$

uniformly for n *and* $0 < z \le z_0$.

Proof. Using the results obtained above, the proof is very similar to that in [**5**] so we can omit it here. □

Lemma 4. *Let the assumptions of Lemma* 1 *hold. Then the sequence* $\{\theta^{(\gamma)}\}$ *defined in* (6.1) *is relatively compact in* $L_2(Q)$.

Proof. For the proof we easily verify Kolmogoroff's compactness argument in $L_2(Q_T)$ (see the Riesz theorem in [**10**, p. 88], for the proof see also [**4**, pp. 45–48]).□

The time derivative can be prolongated as in [**7**]:

Definition. For arbitrary $\varphi \in L_2(I, H^1(\Omega))$ and for arbitrary $I^{'} \subset I$, we define the function $F_\gamma \in L_2(I, H^*(\Omega))$ by

(6.8) $$\int_{I'} \langle F_\gamma(t), \varphi(t)\rangle\, dt := \int_{I'} \langle u_t^{(\gamma)}(t), P_h^1\varphi(t)\rangle\, dt$$

Lemma 5. *The sequence* F_γ *defined in* (6.8) *is bounded in* $L_2(I, H^*(\Omega))$.

Proof. The assertion is a direct consequence of Lemma 2 and (6.8). □

We denote now by $\to$ and $\rightharpoonup$ the strong and the weak convergence, respectively.

Lemma 6. *Assume* (H). *Then there exists a function* $u \in L_2(I, L_2(\Omega))$ *such that* $u_t \in L_2(I, H^*(\Omega))$ *and* $\beta(u) \in L_2(I, H^1(\Omega))$, *and a subsequence (which we denote again by* $\{\gamma\}$*) such that*

$$u^{(\gamma)} \rightharpoonup u \ v \ L_2(I, L_2(\Omega)) \tag{6.9}$$

$$F_\gamma \rightharpoonup u_t \ v \ L_2(I, H^*(\Omega)) \tag{6.10}$$

$$\theta^{(\gamma)} \to \theta = \beta(u) \ v \ L_2(I, L_2(\Omega)) \tag{6.11}$$

$$\bar\theta^{(\gamma)} \to \beta(u) \ v \ L_2(I, L_2(\Omega)) \tag{6.12}$$

$$\beta(\bar u^{(\gamma)}) \to \beta(u) \ v \ L_2(I, L_2(\Omega)) \tag{6.13}$$

$$\bar\theta^{(\gamma)} \rightharpoonup \beta(u) \ v \ L_2(I, H^1(\Omega)). \tag{6.14}$$

Proof. Lemma 4 directly implies the existence of a function $\theta \in L_2(I, L_2(\Omega))$ and a subsequence $\{\gamma_1\}$ such that $\theta^{(\gamma_1)} \to \theta$ in $L_2(I, L_2(\Omega))$; moreover, $\bar\theta^{(\gamma_1)} \to \theta$ and $\beta(\bar u^{(\gamma)}) \to \theta$ in $L_2(I, L_2(\Omega))$ due to (6.6). From (5.12), (6.4) and the reflexivity of $L_2(I, L_2(\Omega))$ it follows that there exists $u \in L_2(I, L_2(\Omega))$ and a subsequence $\{\gamma_2\}$ such that $u^{(\gamma_2)} \rightharpoonup u$, $\bar u^{(\gamma_2)} \rightharpoonup u$ in $L_2(I, L_2(\Omega))$ (where we choose $\{\gamma_2\}$ as a subsequence of $\{\gamma_1\}$). From the boundedness of $\bar\theta^{(\gamma)}$ in $L_2(I, H^1(\Omega))$ (see Lemma 1 and its Consequence) and the reflexivity of $L_2(I, H^1(\Omega))$ we obtain weak convergence of some subsequence $\{\gamma_3\}$ (chosen as a subsequence of $\{\gamma_2\}$) in this space.

By standard arguments we can obtain the fact that θ is identical with $\beta(u)$ (see [**7**], [**4**]). Lemma 5 implies the existence of $F \in L_2(I, H^*(\Omega))$ and a subsequence $\{\gamma\}$ (chosen from $\{\gamma_3\}$), such that $F_{(\gamma)} \rightharpoonup F$ v $L_2(I, H^*(\Omega))$.

We conclude the proof by proving $\frac{du}{dt} = F$ in $L_2(I, H^*(\Omega))$, that means $\frac{du(t)}{dt} = F(t)$ for a.a. $t \in I$.

From the fact that $F_\gamma \rightharpoonup F$ v $L_2(I, H^*(\Omega))$ we have for any $\varphi \in L_2(I, H^1(\Omega))$

$$\int_I \langle u^{(\gamma)} - u_0, \varphi\rangle\, dt = \int_I \left\langle \int_0^t u_s^{(\gamma)}(s)\, ds, P_h^1\varphi(t)\right\rangle dt + \int_I \langle u^{(\gamma)} - u_0, \varphi - P_h^1\varphi\rangle\, dt.$$

From the standard properties of Bochner's integral, the definition of F_γ, the boundedness of $u^{(\gamma)}$ in $C(I, L_2(\Omega))$ and (2.5) it follows that passing to the limit as $\gamma \to 0$ (i.e. $h \to 0$), we obtain

$$\int_I \langle u(t) - u_p, \varphi(t)\rangle\, dt \le \int_I \left\langle \int_0^t F(s)\, ds, \varphi(t)\right\rangle dt.$$

On the other hand,

$$\begin{aligned}\int_I \int_0^t \langle F_\gamma(s), \varphi(t)\rangle\, ds\, dt &= \int_I \int_0^t \langle u_s^{(\gamma)}(s), P_h^1\varphi(t)\rangle\, ds\, dt \\ &= \int_I \langle u^{(\gamma)}(t) - u_0, \varphi(t)\rangle\, dt + \int_I \langle u^{(\gamma)}(t) - u_0, P_h^1\varphi(t) - \varphi(t)\rangle\, dt.\end{aligned}$$

Again, for $\gamma \to 0$ we obtain:

$$\int_I \left\langle \int_0^t F(s)\,ds, \varphi(t) \right\rangle\,dt \le \int_I \langle u(t) - u_p, \varphi(t)\rangle\ dt.$$

Combining these two results and basic properties of abstract functions we obtain

$$u(t) = u_0 + \int_0^t F(s)\,ds,$$

that means

$$\frac{du(t)}{dt} = F(t)$$

(see [**9**]). □

For proving strong the convergence of some subsequence $\bar{\theta}^{(\gamma)}$ to the function $\beta(u)$ in $L_2(I, H^1(\Omega))$ we need the following lemma (for its proof see [**6**] or [**4**]):

Lemma A. *Let u and a subsequence $\{\bar{u}^{(\gamma)}\}$ be the same as in Lemma 6. Then for almost all $t \in I$ it holds:*

(i) $\displaystyle\lim_{k\to\infty} \int_0^t \langle u_s^{(\gamma_k)}(s), \bar{\theta}^{(\gamma_k)}(s)\rangle\,ds \ge \int_\Omega \Phi_\beta(u(t))\,dx - \int_\Omega \Phi_\beta(u_p)\,dx,$

($\{\gamma_k\}$ is some subsequence of $\{\gamma\}$);

(ii) $\displaystyle\int_0^t \langle u_s, \beta(u(s))\rangle\,ds = \int_\Omega \Phi_\beta(u(t))\,dx - \int_\Omega \Phi_\beta(u_p)\,dx.$

Lemma 7. *Let the assumptions of Lemma 1 hold and let the function u and the subsequence $\{\bar{\theta}^{(\gamma)}\}$ be as in Lemma 6. Then it holds:*

$$\bar{\theta}^{(\gamma)} \to \beta(u)\ \text{in}\ L_2(I, H^1(\Omega)). \tag{6.15}$$

Proof. In the notation above, the functions

$$P_h^1\bar{\beta}^i = \frac{1}{\tau}\int_{I_i} P_h^1\beta(u(t))\,dt, \quad i = 1, \dots, n,$$

belong to V_h^1. Now we choose the function $\psi = \theta_i - P_h^1\bar{\beta}^i$ as a test function in (5.1). After multiplying by τ and rearranging we have

$$\begin{aligned}
&\int_{I_i} \langle \delta u_i, \theta_i - P_h^1\beta(u(t))\rangle\,dt + \int_{I_i} \langle k_{i-1}\nabla\theta_i, \nabla(\theta_i - P_h^1\beta(u(t)))\rangle \\
&\qquad + \int_{I_i} \langle g(\theta_i), \theta_i - P_h^1\beta(u(t))\rangle_{h,\Gamma}\,dt + \int_{I_i} \langle \omega_i(\theta_i - \theta_{i-1}), \theta_i - P_h^1\beta(u(t))\rangle_{h,\Gamma}\,dt \\
&= \int_{I_i} \langle f(\beta(u_{i-1})), \theta_i - P_h^1\beta(u(t))\rangle\,dt + \int_{I_i} \langle \rho_i(P_h^0\theta_i - \beta(u_{i-1})), \theta_i - P_h^1\beta(u(t))\rangle\,dt \\
&\qquad + \frac{1}{\tau}\int_{I_i} \langle p_i, \theta_i - P_h^1\beta(u(t))\rangle\,dt.
\end{aligned}$$

For simplicity, we denote

$$\begin{gathered}
k_\gamma(t) = k_{i-1} \text{ pre } t \in (t_{i-1}, t_i\rangle \\
\bar\theta_\tau^{(\gamma)}(t) = \theta_{i-1} \text{ pre } t \in (t_{i-1}, t_i\rangle \\
f_\gamma(\beta(\bar u_\tau^{(\gamma)})) = f(\beta(u_{i-1})) = f(t_i, \beta(u_{i-1})) \text{ pre } t \in (t_{i-1}, t_i\rangle \\
g_\gamma(\bar\theta_\tau^{(\gamma)}) = g(\theta_{i-1}) = g(t_i, \theta_{i-1}) \text{ pre } t \in (t_{i-1}, t_i\rangle \\
\bar p^{(\gamma)}(t) = p_i \text{ pre } t \in (t_{i-1}, t_i\rangle.
\end{gathered}$$

After summing over $i = 1, \dots, n$ we arrive at

$$\begin{aligned}
(6.16)\qquad & \int_I \langle u_t^{(\gamma)}(t), \bar\theta^{(\gamma)}(t) - P_h^1\beta(u(t))\rangle\, dt \\
&\quad + \int_I \langle k_\gamma(t)\nabla\bar\theta^{(\gamma)}(t), \nabla(\bar\theta^{(\gamma)}(t) - P_h^1\beta(u(t)))\rangle\, dt \\
&\quad + \int_I \langle g_\gamma(\bar\theta_\tau^{(\gamma)}(t), \bar\theta^{(\gamma)}(t) - P_h^1\beta(u(t))\rangle_{h,\Gamma}\, dt \\
&\quad + \int_I \langle \bar\omega^{(\gamma)}(t)(\bar\theta^{(\gamma)}(t) - \bar\theta_\tau^{(\gamma)}(t)), \bar\theta^{(\gamma)}(t) - P_h^1\beta(u(t))\rangle_{h,\Gamma}\, dt \\
&= \int_I \langle f_\gamma(\beta(\bar u_\tau^{(\gamma)}(t))), \bar\theta^{(\gamma)}(t) - P_h^1\beta(u(t))\rangle\, dt \\
&\quad + \int_I \langle \bar\rho^{(\gamma)}(t)(P_h^0\bar\theta^{(\gamma)}(t) - \beta(\bar u_\tau^{(\gamma)}(t))), \bar\theta^{(\gamma)}(t) - P_h^1\beta(u(t))\rangle\, dt \\
&\quad + \frac{1}{\tau}\int_I \langle \bar p^{(\gamma)}(t), \bar\theta^{(\gamma)}(t) - P_h^1\beta(u(t))\rangle\, dt.
\end{aligned}$$

We will estimate each term of the equation (6.16).

In the first term we can use the definition of F_γ, the results of Lemma A, (6.10) and the fact $\beta(u) \in L_2(I, H^1(\Omega))$ and passing to the limit as $\gamma \to 0$ we get

$$\lim_{\gamma\to 0}\int_I \langle u_t^{(\gamma)}(t), \bar\theta^{(\gamma)}(t) - P_h^1\beta(u(t))\rangle\, dt \ge 0.$$

We rearrange the second term of (6.16) as follows

$$\begin{aligned}
&\int_I \langle k_\gamma(t)\nabla\bar\theta^{(\gamma)}(t), \nabla(\bar\theta^{(\gamma)}(t) - \beta(u(t)))\rangle\, dt \\
&\qquad + \int_I \langle k_\gamma(t)\nabla\bar\theta^{(\gamma)}(t), \nabla(\beta(u(t)) - P_h^1\beta(u(t)))\rangle\, dt \\
&\quad \ge k_0||\nabla(\bar\theta^{(\gamma)} - \beta(u))||^2_{L_2(I,L_2(\Omega))} + \int_I \langle k_\gamma(t)\nabla\beta(u(t)), \nabla(\bar\theta^{(\gamma)}(t) - \beta(u(t)))\rangle\, dt \\
&\qquad + + \int_I \langle k_\gamma(t)\nabla\bar\theta^{(\gamma)}(t), \nabla(\beta(u) - P_h^1\beta(u(t)))\rangle\, dt
\end{aligned}$$

Now for $\gamma \to 0$, the second term of this inequality converges to 0 because of the properties of the matrix k, the results of Lemma 6 and the Lebesgue dominating convergence theorem. The third term can be estimated as follows

$$\begin{aligned}\int_I \langle k_\gamma(t)\nabla\bar{\theta}^{(\gamma)}(t), \nabla(\beta(u) - P_h^1\beta(u(t)))\rangle\, dt \\ \leq CK_0||\bar{\theta}^{(\gamma)}||_{L_2(I,H^1(\Omega))}||\beta(u) - P_h^1\beta(u)||_{L_2(I,H^1(\Omega))} \\ \leq C||\beta(u) - P_h^1\beta(u)||_{L_2(I,H^1(\Omega))},\end{aligned}$$

where we used again the properties of k_γ and (5.14). The convergence for this term now follows from the interpolation property for the finite element method ([**2**]).

The convergence of the third term in (6.16) is based on the convergence of

$$\left|\int_I \langle g_\gamma(\bar{\theta}_\tau^{(\gamma)}(t)), \bar{\theta}^{(\gamma)}(t) - P_h^1\beta(u(t))\rangle_\Gamma\, dt\right|$$

which follows similarly as in [**7**] so we can omit its proof here.

The next term in (6.16) can be estimated similarly:

$$\begin{aligned}\left|\int_I \langle \bar{\omega}^{(\gamma)}(t)(\bar{\theta}^{(\gamma)}(t) - \bar{\theta}_\tau^{(\gamma)}(t)), \bar{\theta}^{(\gamma)}(t) - P_h^1\beta(u(t))\rangle_\Gamma\, dt\right| \\ \leq C\varepsilon||\nabla\bar{\theta}^{(\gamma)} - \beta(u)||^2_{L_2(I,L_2(\Omega))} + \frac{C}{\varepsilon}||\bar{\theta}^{(\gamma)} - \beta(u)||^2_{L_2(I,L_2(\Omega))} \\ + Ch||\bar{\theta}^{(\gamma)}||_{L_2(I,H^1(\Omega))}||\bar{\theta}^{(\gamma)} - P_h^1\beta(u)||_{L_2(I,H^1(\Omega))}\end{aligned}$$

For sufficiently small values ε and h and again due to the strong convergence of $\{\bar{\theta}^{(\gamma)}\}$ to $\beta(u)$ in $L_2(I, L_2(\Omega))$ and the boundedness of $\bar{\omega}^{(\gamma)}$ and $\{\bar{\theta}^{(\gamma)}\}$ in $L_2(I, H^1(\Omega))$ we obtain the convergence to zero for this term.

The convergence of the first term on the right hand side of (6.16)

$$\int_I \langle f_\gamma(\beta(\bar{u}_\tau^{(\gamma)}(t))), \bar{\theta}^{(\gamma)}(t) - P_h^1\beta(u(t))\rangle\, dt$$

can be omitted again because it is similar as in [**7**].

The next term can be estimated as follows

$$\begin{aligned}\int_I \langle \bar{\rho}^{(\gamma)}(t)(P_h^0\bar{\theta}^{(\gamma)}(t) - \beta(\bar{u}_\tau^{(\gamma)}(t))), \bar{\theta}^{(\gamma)}(t) - P_h^1\beta(u(t))\rangle\, dt \\ \leq K(||P_h^0\bar{\theta}^{(\gamma)}||_{L_2(I,L_2(\Omega))} + ||\beta(\bar{u}^{(\gamma)})||_{L_2(I,L_2(\Omega))}) \\ \times (||\bar{\theta}^{(\gamma)} - \beta(u)||_{L_2(I,L_2(\Omega))} + ||\beta(u) - P_h^1\beta(u)||_{L_2(I,L_2(\Omega))}).\end{aligned}$$

Due to (5.12) and (5.3), we have the boundedness of $\{P_h^0\bar{\theta}^{(\gamma)}\}$ and $\{\beta(\bar{u}^{(\gamma)})\}$ and conclusion is analogous as above.

The estimation of the last term is straightforward with respect to the properties of p_i for $i = 1, \dots, n$:

$$\frac{1}{\tau}\int_I \langle \bar{p}^{(\gamma)}(t), \bar{\theta}^{(\gamma)}(t) - P_h^1\beta(u(t))\rangle\, dt \le \frac{1}{\tau}\int_I ||\bar{p}^{(\gamma)}(t)||_{H^*}||\bar{\theta}^{(\gamma)}(t) - P_h^1\beta(u(t))||_H\, dt$$
$$\le C\tau^{\sigma-1}||\bar{\theta}^{(\gamma)} - P_h^1\beta(u)||^2_{L_2(I,H^1(\Omega))},$$

which converges again to zero.

Now we can conclude:

$$||\nabla(\bar{\theta}^{(\gamma)} - \beta(u))||^2_{L_2(I,L_2(\Omega))} \to 0 \quad \text{for } \gamma \to 0.$$

which together with (6.12) gives the assertion of Lemma. □

Consequence. *Let the subsequence* $\{\bar{\theta}^{(\gamma)}\}$ *be as in Lemma* 6*. Then it holds:*

$$\bar{\theta}^{(\gamma)} \to \beta(u) \text{ in } L_2(I, L_2(\Gamma)).$$

Proof. The proof follows in a straightforward way from the imbedding theorem ([**10**]). □

Lemma 8. *Assume* (H) *and let the subsequence* $\{\bar{\theta}^{(\gamma)}\}$ *from* (6.1) *have the properties as above. Let the subsequence* $\{\bar{\theta}_\tau^{(\gamma)}\}$ *be the same as in the proof of Lemma* 7*. Then it holds:*

$$||\bar{\theta}_\tau^{(\gamma)} - \bar{\theta}^{(\gamma)}||_{L_2(I,L_2(\Gamma))} \to 0 \text{ for } \gamma \to 0. \tag{6.17}$$

Proof. We use (2.7) for $\varepsilon = \tau^{\frac{1}{3}}$ and the boundedness of $\bar{\theta}^{(\gamma)}$, $\bar{\theta}_\tau^{(\gamma)}$ (see Consequence of Lemma 1). We have

$$\int_I ||\bar{\theta}_\tau^{(\gamma)}(t) - \bar{\theta}^{(\gamma)}(t)||^2_\Gamma\, dt$$
$$\le C_{10}\tau^{\frac{1}{3}}\int_I ||\nabla(\bar{\theta}_\tau^{(\gamma)}(t) - \bar{\theta}^{(\gamma)}(t))||^2\, dt + C_{10}\frac{1}{\tau^{\frac{1}{3}}}\int_I ||\bar{\theta}_\tau^{(\gamma)}(t) - \bar{\theta}^{(\gamma)}(t)||^2\, dt$$
$$\le C\tau^{\frac{1}{3}} + C_{10}\frac{1}{\tau^{\frac{1}{3}}}\left(\int_0^{T-\tau} ||\bar{\theta}^{(\gamma)}(t+\tau) - \bar{\theta}^{(\gamma)}(t)||^2\, dt + \int_{-\tau}^0 ||\bar{\theta}^{(\gamma)}(t+\tau)||^2\, dt\right)$$
$$\le C\tau^{\frac{1}{6}} \to 0,$$

where similarly as in Lemma 4 we put $\bar{\theta}^{(\gamma)} := 0$ on $\langle -\tau, 0\rangle$ and then we used (6.7) from Lemma 3, (2.5) and Lemma 1. □

Now the main result is:

Theorem 2. *Assume* (H). *Let the ratio between space and time discretization step is as in* (5.2). *Then there exists a pair of functions* $\{u(t,x), \theta(t,x)\}$ *which fulfil the relations* (3.1)–(3.3) *and the variational formulation* (3.4). *Moreover, there exist subsequences* $\{\theta^{\bar\gamma}\}$, $\{u^{\bar\gamma}\}$ *of sequences* $\{\theta^{\gamma}\}$, $\{u^{\gamma}\}$ *defined in* (6.1) *such that*

$$u^{\bar\gamma} \rightharpoonup u \quad \text{and } \theta^{\bar\gamma} \to \theta = \beta(u) \quad \text{in } L_2(I, L_2(\Omega)),$$
$$\bar\theta^{\bar\gamma} \to \beta(u) \quad \text{in } L_2(I, H^1(\Omega))$$

for $\bar\gamma \to 0$.

Proof. Due to Lemma 6 we have the existence of a pair of functions $u(t,x)$, $\theta(t,x) = \beta(u(t,x))$ and subsequences converging to them. Now we must prove that this pair of functions satisfies the variational identity (3.4).

We define
$$P_h^1 \bar v^i = \frac{1}{\tau}\int_{I_i} P_h^1 v(t)\,dt, \quad i = 1,\dots,n,$$
for arbitrary $v \in L_2(I, H^1(\Omega))$. Then $P_h^1\bar v^i$ belongs to V_h^1. Now we use the same notation as in Lemma 7 and we choose as a test function $P_h^1\bar v^i$ for $i = 1,\dots,n$. In an analogous way as in Lemma 7, we can get (we denote subsequences $\{\bar\gamma\}$ again by $\{\gamma\}$):

$$\begin{aligned}
(6.18)\qquad & \int_I \langle u_t^{(\gamma)}(t), P_h^1 v(t)\rangle\,dt + \int_I \langle k_\gamma(t)\nabla\bar\theta^{(\gamma)}(t), \nabla(P_h^1 v(t))\rangle\,dt \\
&\quad + \int_I \langle g_\gamma(\bar\theta_\tau^{(\gamma)}(t), P_h^1 v(t)\rangle_{h,\Gamma}\,dt \\
&\quad + \int_I \langle \bar\omega^{(\gamma)}(t)(\bar\theta^{(\gamma)}(t) - \bar\theta_\tau^{(\gamma)}(t)), P_h^1 v(t)\rangle_{h,\Gamma}\,dt \\
&= \int_I \langle f_\gamma(\beta(\bar u_\tau^{(\gamma)}(t))), P_h^1 v(t)\rangle\,dt \\
&\quad + \int_I \langle \bar\rho^{(\gamma)}(t)(P_h^0\bar\theta^{(\gamma)}(t) - \beta(\bar u_\tau^{(\gamma)}(t))), P_h^1 v(t)\rangle\,dt \\
&\quad + \frac{1}{\tau}\int_I \langle \bar p^{(\gamma)}(t), P_h^1 v(t)\rangle\,dt.
\end{aligned}$$

Now we pass to the limit as $\gamma \to 0$:

For the first term we use (6.8), Lemma 6 (6.10):

$$\int_I \langle u_t^{(\gamma)}(t), P_h^1 v(t)\rangle\,dt = \int_I \langle F_\gamma(t), v(t)\rangle\,dt \to \int_I \langle u_t(t), v(t)\rangle\,dt.$$

In the second term we use the fact $\bar\theta_\tau^{(\gamma)} \to \beta(u)$ in $L_2(I, L_2(\Omega))$ (from Lemma 8), the properties of k, relations (6.13) and (6.14), the boundedness $\{\bar\theta^{(\gamma)}\}$ in

$L_2(I, H^1(\Omega))$ and $P_h^1 v \to v$ v $L_2(I, H^1(\Omega))$. We have:

$$\int_I \langle k_\gamma(t)\nabla\bar{\theta}^{(\gamma)}(t), \nabla(P_h^1 v(t))\rangle\, dt \to \int_I \langle k(t)\nabla\beta(u(t)), \nabla v(t)\rangle\, dt.$$

In the third term we use (2.3) and the boundedness of $\bar{\theta}^{(\gamma)}$ in $L_2(I, L_2(\Gamma))$ (see Consequence of Lemma 1), (H_g) and Lemma 8:

$$\int_I \langle g_\gamma(\bar{\theta}_\tau^{(\gamma)}(t), P_h^1 v(t)\rangle_{h,\Gamma}\, dt \to \int_I \langle g(t, \beta(u(t)), v(t)\rangle_\Gamma\, dt$$

for $\gamma \to 0$.

If we use (2.3) and the Consequence of Lemma 1, Lemma 8 and the boundedness of $\bar{\omega}^{(\gamma)}$ for the fourth term, we obtain:

$$\int_I \langle \bar{\omega}^{(\gamma)}(t)(\bar{\theta}^{(\gamma)}(t) - \bar{\theta}_\tau^{(\gamma)}(t)), P_h^1 v(t)\rangle_{h,\Gamma}\, dt \to 0.$$

Similarly, in the fifth term, we use (H_f), (H_β), Lemma 1 and (6.13):

$$\int_I \langle f_\gamma(\beta(\bar{u}_\tau^{(\gamma)}(t))), P_h^1 v(t)\rangle\, dt \to \int_I \langle f(t, \beta(u(t))), v(t)\, dt.$$

The next term converges to zero due to the boundedness of $\rho^{(\gamma)}$, Lemma 6, (6.13), the inequality (2.4) and the boundedness of $\bar{\theta}^{(\gamma)}$ in $L_2(I, H^1(\Omega))$.

The last term also converges to zero due to the properties of the functionals p_i for $i = 1, \dots, n$ So the pair of functions $u(t,x)$, $\theta(t,x) = \beta(u(t,x))$ satisfies the problem (P). This completes the proof. □

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