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APOLLONIUS' CONTACT PROBLEM IN n–SPACE IN VIEW OF ENUMERATIVE GEOMETRY

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1. Introduction

Let Σ_i^{n-1}, $i = 1, \dots, n+1$, be (as a quadric) non-degenerated hyperspheres of $\mathbb{R}^n$, $n \geq 2$, in "general position". Apollonius' contact problem asks to the hyperspheres of $\mathbb{R}^n$ which contact each of the hyperspheres Σ_i^{n-1}. There are finitely many ones. The number is 2^{n+1} as one can read, for instance, from [**9**].

In our present paper we want to establish them (and those related to the customary variants of the Apollonius problem) — quasi as a footnote to the theory — with help of Schubert's calculus of the enumerative geometry – in the sense of a solution of Hilbert's 15^{th} problem [**8**]. For this purpose we have to consider the spheres over an algebraically closed field ($\mathbb{C}$) and to parameterize and to compactify their set. We shall do this, in a first step, via the equations by the complex projective space $\mathbf{P}^{n+1}$ of the coefficients.

The set of hyperspheres contacting a hypersphere Σ_i^{n-1} then corresponds to a hypersurface B_i of order 2 in $\mathbf{P}^{n+1}$. The set of hyperspheres contacting all of the given spheres Σ_i^{n-1} corresponds to the intersection of $n+1$ such quadrics. In "general position" written as an intersection product this yields $B^{n+1} := B_1 \dots B_{n+1} = 2^{n+1}$ points (spheres) because of Bézout's theorem. At first, in Chapter 3 we shall prove that this argument is correct.

The set of hyperspheres contacting a hyperplane $T_i^{n-1} \subset \mathbb{C}^n$ also corresponds to a hyperquadric $L_i \subset \mathbf{P}^{n+1}$. The set of hyperspheres which contain a given point corresponds to a hyperplane $P_i \subset \mathbf{P}^{n+1}$. Using Bézout's theorem we would get $B^r L^q P^{n+1-r-q} = 2^{r+q}$. This is correct if $0 \leq q < n+1$ and $0 \leq r \leq n+1-q$. The case $q = n+1$ is that in which we consider the number of hyperspheres contacting $n+1$ hyperplanes. But this number has to be 2^n regarding Schoute's result [**9**]. Therefore, Bézout's theorem can not be applied to this case.

We compactified the set of hyperspheres by "degenerations": hyperplanes and cones — which are at least yet hypersurfaces of $\mathbb{C}^n$ — , but also by the point $\pi \in \mathbf{P}^{n+1}$ which corresponds to the hyperplane at infinity of $\mathbb{C}^n$. We see that $\pi \notin B_i$

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and $\pi \notin P_i$, but $\pi \in L_i$. Of course, Schoute did not count the hyperplane at infinity under the solutions. But π is not an exceptional point of $\mathbf{P}^{n+1}$. If we use Bézout's theorem to count the solutions it may be counted with an — ad hoc — unknown multiplicity.

We ask for an other compactification of the open set of regular spheres so that we know by which formula we have to replace the Bézout formula. Blowing-ups seem to be adequate. We start with a blowing-up in π.

The point π is an exceptional one in the sense that it is invariant (stable) under the group $\mathcal{G}$ of the complex similarities. This is a canonical group in context with spheres. By the way, we use $\mathcal{G}$ to define correctly what we mean by the word "general".

After we shall have done a second blowing-up the image of L_i under it does not contain any $\mathcal{G}$-invariant subset. It makes sense to ask for a correct formula on this compactification. We represent it in the cohomology ring.

It is an comfortable fact that all this remains true if we replace for some i the hyperspheres Σ_i^{n-1} or the hyperspaces T_i^{n-1} which have to be contacted by d-spheres Σ_i^d resp. d-spaces T_i^d of smaller dimension $d < n-1$. In the case of hypersurfaces of order 2 this is not true. There one needs for each d a particular blowing-up [**7**], [**10**].

Projectively spoken $(n-1)$-spheres are $(n-1)$-quadrics which contain the $(n-2)$-quadric $\Omega : x_1^2 + \cdots + x_n^2 = 0$ in the plane at infinity of $\mathbb{C}^n$. The Apollonius problem could be a special case of a problem about "complete quadrics" [**10**], [**2**]. But under the solutions of the problem about quadrics are those which fulfill the contact conditions partially or at all in Ω. They are not solutions of the original problem. One had to determine them and there multiplicities. In the case $n = 2$ it is done in [**5**], [**6**].

An other generalization of the classical Apollonius problem to plane curves one can found in [**4**].

2. The $\mathbf{P}^{n+1}$ as a Variety of $(n-1)$-spheres

The set of all hyperspheres (of a real euklidian space $\mathbb{R}^n$, $n \geq 2$) is embedded by the equations

$$p_0 x^\top x + 2c^\top x + p_{n+1} = 0, \qquad x \in \mathbb{C}^n,\ c \in \mathbb{C}^n, \tag{1}$$

in a complex projective space $\mathbf{P}^{n+1}$ with coordinates

$$p^\top = (p_0, p_1, \ldots, p_n, p_{n+1}) = (p_0, c^\top, p_{n+1}). \tag{2}$$

The set of hyperspheres is invariant under the group $\mathcal{G}$ of the complex similarities

$$g \colon x \mapsto \varrho U x + a, \tag{3}$$

$x \in \mathbb{C}^n$, U an orthogonal matrix over $\mathbb{C}$, $0 \neq \varrho \in \mathbb{C}$, $a \in \mathbb{C}^n$.

With respect to (1) it acts on $\mathbf{P}^{n+1}$ by

$$(g,(p_0,c^\top,p_{n+1}))\mapsto(p_0,\varrho c^\top U^\top-p_0a^\top,\varrho^2p_{n+1}-2\varrho c^\top U^\top a+p_0a^\top a). \tag{4}$$

Definition 2.1. We say that $\mathbf{P}^{n+1}$ together with the action (4) of $\mathcal{G}$ is the variety of $(n-1)$-spheres.

This variety contains degenerated $(n-1)$-spheres and that in the following $\mathcal{G}$-invariant algebraic subsets: the (non-degenerated) n-quadric Q with the ideal $\mathfrak{q}=(c^\top c-p_0p_{n+1})$, the tangent hyperplane H to Q with $\mathfrak{h}=(p_0)$, the $(n-1)$-cone Φ with the ideal $\mathfrak{q}+\mathfrak{h}$, and the contact point π of H and Q with the ideal $(p_0,\dots,p_n)$. The $\mathcal{G}$-orbits in $\mathbf{P}^{n+1}$ are: $\mathbf{P}^{n+1}\setminus(Q\cup H)$ (the non-degenerated complex spheres of $\mathbb{C}^n$), $Q\setminus\Phi$ (the cones containing the non-degenerated hyperquadric Ω with the equation $x^\top x=0$ in the hyperplane at infinity of $\mathbb{C}^n$), $H\setminus\Phi$ (the hyperplanes of $\mathbb{C}^n$, which do not contact Ω), $\Phi\setminus\pi$ (the hyperplanes, which contact Ω), and π (the hyperplane at infinity).

3. Apollonius Conditions

Let $\dot\Sigma^d$, $d=0,\dots,n-1$, be the complex d-sphere with the ideal

$$(x_1^2+\cdots+x_{d+1}^2+1,x_{d+2},\dots,x_n).$$

The minimal subvariety of $\mathbf{P}^{n+1}$ which contains all $(n-1)$-spheres of $\mathbb{C}^n$ contacting $\dot\Sigma^d$ is the quadric $\dot B_d$ with the ideal

$$\dot{\mathfrak{b}}_d=(4(p_1^2+\cdots+p_{d+1}^2)+(p_0-p_{n+1})^2). \tag{5}$$

Further, let $\dot T^d$, $d=1,\dots,n-1$, be the linear d-space with the ideal

$$(x_{d+1},\dots,x_n).$$

The minimal subvariety of $\mathbf{P}^{n+1}$ which contains all $(n-1)$-spheres of $\mathbb{C}^n$ contacting $\dot T^d$ is the quadric $\dot L_d$ with the ideal

$$\dot{\mathfrak{l}}_d=(p_1^2+\cdots+p_d^2-p_0p_{n+1}). \tag{6}$$

Finally, let $\dot P$ the linear space in $\mathbf{P}^{n+1}$ of all $(n-1)$-spheres containing the origin of $\mathbb{C}^n$.

Definition 3.1. Let $g\in\mathcal{G}$. $B=g\dot B_d$ is the contact condition to the d-sphere $\Sigma^d=g\dot\Sigma^d$. $L=g\dot L_d$ is the contact condition to the linear d-space $T^d=g\dot T^d$. $P=g\dot P$ is the condition to contain $x=g0$. These conditions are the Apollonius conditions.

Lemma 3.2. *There exist* $g_1, \dots, g_{r+s+q}$, $r+s+q \le n+1$, *in* $\mathcal{G}$ *so that the intersection*

$$g_1\dot{B}_{d_1} \cap \dots \cap g_r\dot{B}_{d_r} \cap g_{r+1}\dot{P} \cap \dots \cap g_{r+s}\dot{P} \cap g_{r+s+1}\dot{L}_{d_{r+s+1}} \cap \dots \cap g_{r+s+q}\dot{L}_{d_{r+s+q}}$$

is $(n-r-s-q+1)$*-dimensional.*

Proof. Obviously, the only $\mathcal{G}$-invariant subset contained in a Apollonius condition A is the point π. Therefore, for each algebraic subset U of $\mathbf{P}^{n+1}$ with $\dim_{\mathbb{C}} U > 0$ an element $g \in \mathcal{G}$ exist so that $\dim_{\mathbb{C}}(U \cap gA) = \dim_{\mathbb{C}} U - 1$. Namely, if $U \in gA$ for all $g \in \mathcal{G}$ then a $\mathcal{G}$-invariant subset S exists with $U \subseteq S \subseteq A$ [**3**]. An iterative argument completes the proof. □

In the case $r+s+q = n+1$ we get the

Theorem 3.3. *Let* r *and* s *in* $\mathbb{N}$ *with* $r+s>0$. *Then there exist* $g_1, \dots, g_{n+1}$ *in* $\mathcal{G}$ *so that the intersection*

$$g_1\dot{B}_{d_1} \cap \dots \cap g_r\dot{B}_{d_r} \cap g_{r+1}\dot{P} \cap \dots \cap g_{r+s}\dot{P} \cap g_{r+s+1}\dot{L}_{d_{r+s+1}} \cap \dots \cap g_{n+1}\dot{L}_{d_{n+1}}$$

consists of a finite number of points of the $\mathcal{G}$*-orbit* $\mathbf{P}^{n+1} \setminus (Q \cup H)$. *Including multiplicities this number is* 2^{n+1-s}.

Proof. Obviously, neither the conditions $\dot{B}_d$ nor the condition $\dot{P}$ contain the point π and therefore a $\mathcal{G}$-invariant subset at all. Therefore, for each algebraic subset U of $\mathbf{P}^{n+1}$ an element $g \in \mathcal{G}$ exists so that

$$U \not\subseteq g\dot{B}_d \quad \text{resp.} \quad U \not\subseteq g\dot{P}.$$

See [**3**][1].

We start an iteration of intersections with $U = Q \cup H$ and end it with a $g_i\dot{B}_{d_i}$ or a $g_j\dot{P}$ obtaining the empty set. So the points are not in $Q \cup H$. Therefore, we can compute their number in the cohomology ring $H^*\mathbf{P}^{n+1}$. It is generated by the class of hyperplanes represented by any P. Because each B and each L is of order two we have $B \sim 2P$ and $L \sim 2P$ in $H^*\mathbf{P}^{n+1}$. It follows

$$B^r L^q P^s \sim 2^{r+q} P^{n+1} \quad \text{for} \quad r+s+q = n+1 \tag{7}$$

in $H^*\mathbf{P}^{n+1}$. This proves the last part of the Theorem. □

Let Y be a hyperplane of $\mathbf{P}^{n+1}$, $Y \ne H$, $\pi \in Y$. Then $Y \sim H \sim P$ in $H^*\mathbf{P}^{n+1}$.

Each contact condition L contains the $\mathcal{G}$-invariant point π. The hyperplane of infinity of $\mathbb{C}^n$ suffices each L. If we would determine the number of elements in the intersection of $n+1$ such conditions via $H^*\mathbf{P}^{n+1}$ this hyperplane of infinity will be counted with an certain multiplicity. The number 2^{n+1} is not the number of the proper solutions of this Apollonius problem[2]. Therefore, we shall blow up $\mathbf{P}^{n+1}$ in the point π to the variety $\bar{M}^{n+1}$.

[1] $\mathbf{P}^{n+1}$ is $\mathcal{G}$-complete with respect to B_d and P [**3**].

[2] The point p $\mathcal{G}$-properly satisfies L if $p \in L$ but a $g \in \mathcal{G}$ exists so that $gp \notin L$ [**1**].

4. The First Blowing-Up

Definition 4.1. Let $\bar{M}^{n+1}$ be the closure of the graph of the map which attaches to each non-degenerated $(n-1)$-sphere its midpoint $-\frac{1}{p_0}c$.

Lemma 4.2. *The projection $\bar{\chi}\colon \bar{M}^{n+1} \to \mathbf{P}^{n+1}$ is a blowing-up with the center π.*

Proof. Let $\mathbf{Y}^n$ a projective n-space with coordinates $y = (y_0, y_1, \dots, y_n)^\top = (y_0, z^\top)^\top$. We embed $\bar{M}^{n+1}$ in $\mathbf{P}^{n+1} \times \mathbf{Y}^n$. Then $\bar{M}^{n+1}$ is described by the ideal

$$\bar{\mathfrak{m}} = (y_i p_j - y_j p_i)_{(i,j)\in\{0,1,\dots,n\}^2} \tag{8}$$

in the doubly graded ring $\mathbb{C}[p; y]$. □

We denote the inverse image $\bar{\chi}^{-1}D$ of an algebraic set D in $\mathbf{P}^{n+1}$ by $\bar{D}$. Then $\bar{\mathfrak{q}} = \mathfrak{q} + \bar{\mathfrak{m}} + (z^\top c - y_0 p_{n+1})$ is the ideal of $\bar{Q}$, $\bar{\mathfrak{h}} = \bar{\mathfrak{m}} + (p_0, y_0)$ is the ideal of $\bar{H}$, and $\bar{\dot{\mathfrak{l}}}_d = \dot{\mathfrak{l}}_d + \bar{\mathfrak{m}} + (y_1 p_1 + \cdots + y_d p_d - y_0 p_{n+1})$ is the ideal of $\bar{\dot{L}}_d$.

We denote the complete inverse image of π by $\bar{E}$. The ideal of it is $\bar{\mathfrak{e}} = (p_0, p_1, \dots, p_n)$. If one wants to interprete a point (π, y) of $\bar{E}$ as a (degenerated) sphere then it is the hyperplane at infinity of $\mathbb{C}^n$ with any point of the projective closure of $\mathbb{C}^n$ as midpoint.

The action of the group $\mathcal{G}$ is lifted to $\mathbf{Y}^n$ and $\bar{M}^{n+1}$. $\mathcal{G}$ acts on $\mathbf{Y}^n$ resp. $\bar{E}$ because of (8) and (4) by

$$(g, (y_0, z)) \mapsto (y_0, \varrho U z - y_0 a). \tag{9}$$

Lemma 4.3. *The closures of $\mathcal{G}$-orbits of $\bar{M}^{n+1}$ are $\bar{E}$, $\bar{Q}$, $\bar{H}$, $\bar{\Phi}$, $\bar{\Xi} = \bar{H} \cap \bar{E} = \bar{Q} \cap \bar{E}$ and $\bar{\Psi} = \bar{\Phi} \cap \bar{E} = \bar{\Phi} \cap \bar{\Xi}$.*

Proof. The subsets $\bar{E}$, $\bar{Q}$, $\bar{H}$, and $\bar{\Phi}$ are $\mathcal{G}$-invariant because they are inverse images of $\mathcal{G}$-invariant subsets. The rest of the closures of $\mathcal{G}$-orbits on $\bar{M}^{n+1}$ must be in $\bar{E}$. We see by (9) that $\bar{E} \setminus \bar{\Xi}$ is a $\mathcal{G}$-orbit.

For the points of $\bar{\Xi}$ holds $y_0 = 0$. (That means that the midpoint lies in the hyperplane at infinity of $\mathbb{C}^n$.) $\mathcal{G}$ acts on $\bar{\Xi}$ by

$$(g, z) \mapsto \varrho U z. \tag{10}$$

The subset $\bar{\Psi}$ is described by $z^\top z = 0$. It is an exercise to show that the orbits under the action

$$O(\mathbb{C}, n) \times \mathbb{C}^n \setminus \{0\} \to \mathbb{C}^n \setminus \{0\} \tag{11}$$

of the orthogonal group correspond to the values of $z^\top z$. Therefore $\bar{\Xi} \setminus \bar{\Psi}$ and $\bar{\Psi}$ are the $\mathcal{G}$-orbits of $\bar{\Xi}$. So we know the closures of the orbits on $\bar{M}^{n+1}$. □

Remark 4.4. Each of it contains at least one point of $\bar{\Xi}$.

This can be seen in the following manner. Because $\pi \in \Phi \subset H$ we have $\bar{\Xi} \cap \bar{\Phi} \cap \bar{E} = \bar{\Phi} \cap \bar{E} \neq \emptyset$ and therefore $\bar{\Xi} \cap \bar{\Phi} \neq \emptyset$.

The $\mathcal{G}$-invariant subset $\bar{\Xi}$ is contained in all contact conditions $\bar{L}$. The intersection of $n+1$ such conditions is not zero-dimensional. Therefore, in the next chapter we shall blow up $\bar{M}^{n+1}$ along $\bar{\Xi}$ to a variety $\tilde{M}^{n+1}$.

Nevertheless, we shall compute the cohomology ring $H^*\bar{M}^{n+1}$ for we need it in the next chapter.

Lemma 4.5. *The cohomology ring can be generated by two elements represented by $\bar{P}$ and $\bar{E}$ with the relations*

$$\bar{P}\bar{E} \sim 0 \tag{12}$$

and

$$\bar{E}^{n+1} \sim (-1)^n \bar{P}^{n+1}. \tag{13}$$

Furthermore, it holds

$$\bar{Y} \sim \bar{H} \sim \bar{P} - \bar{E} \quad \text{and} \quad \bar{L} \sim 2\bar{P} - \bar{E}. \tag{14}$$

Proof. We have

$$\bar{\chi}^* P = \bar{P}, \quad \bar{\chi}^* H = \bar{H} + \bar{E}, \quad \bar{\chi}^* Y = \bar{Y} + \bar{E} \text{ and } \bar{\chi}^* L = \bar{L} + \bar{E} \tag{15}$$

because $\pi \notin P$ resp. because π lies simply on Y, H and L.

Therefore, (14) is true. (12) follows also from the fact that $\pi \notin P$.

If $\pi\colon \bar{M}^{n+1} \to \mathbf{Y}^n$ is the projection and Y' is a linear divisor in $\mathbf{Y}^n$ then $\pi^* Y' \sim \bar{Y}$. Because of ${Y'}^{n+1} \sim 0$ in $H^*\mathbf{Y}^n$ we get

$$\bar{Y}^{n+1} \sim 0 \tag{16}$$

in $H^*\bar{M}^{n+1}$. From this with help of (12) we get (13).

Thus the relations (12) and (13) reduce the $(n+1)$-th graduation to a one-dimensional module. Using general arguments this suffices to complete the proof of the Lemma 4.5. To be quite sure one can check the other graduations. Because of the exact cohomology sequences it has to be $\dim H^i\bar{M}^{n+1} = 2$ if $0 < i < n+1$. □

5. The Second Blowing-Up

Definition 5.1. Let $\tilde{\chi}\colon \tilde{M}^{n+1} \to \bar{M}^{n+1}$ the blowing-up of $\bar{M}^{n+1}$ along $\bar{\Xi}$.

We describe $\bar{\Xi}$ by the forms

$$s_{0,n+1} = y_0 p_{n+1}, \quad s_{ij} = y_i p_j,\ (i,j) \in \{0,\dots,n\}^2. \tag{17}$$

Then we obtain $\tilde{M}^{n+1}$ in a product space $\mathbf{P}^{n+1} \times \mathbf{Y}^n \times \mathbf{S}^{(n+1)^2}$, where $\mathbf{S}^{(n+1)^2}$ is a projective space with coordinates $s = (s_{0,n+1},\ s_{ij}), (i,j) \in \{0,\dots,n\}^2$. Let $\tilde{\mathfrak{m}}$ be its ideal in the multiply graded ring $\mathbb{C}[p,y,s]$ and let S the symmetric matrix $zc^\top = (s_{ij})_{(i,j)\in\{1,\dots,n\}^2}$.

We denote the inverse image $\tilde{\chi}^{-1}\overline{D}$ of an algebraic set $\bar{D}$ in $\bar{M}^{n+1}$ by $\tilde{D}$. Then $\tilde{\mathfrak{q}} = \bar{\mathfrak{q}} + \tilde{\mathfrak{m}} + (\operatorname{trace} S - s_{0,n+1})$ is the ideal of $\tilde{Q}$, $\tilde{\mathfrak{h}} = \tilde{\mathfrak{m}} + (p_0, y_0, s_{0j})_{j\in\{0,\dots,n+1\}}$ is the ideal of $\tilde{H}$, $\tilde{\mathfrak{e}} = \tilde{\mathfrak{m}} + (p_k, s_{ij})_{k\in\{0,\dots,n\},(i,j)\in\{0,\dots,n\}^2}$ is the ideal of $\tilde{E}$, and $\tilde{\bar{\mathfrak{l}}}_d = \bar{\mathfrak{l}}_d + \tilde{\mathfrak{m}} + (\sum_{j=1}^d s_{jj} - s_{0,n+1})$ is the ideal of $\tilde{\bar{L}}_d$.

We denote the complete inverse image of $\bar{\Xi}$ by $\tilde{X}$ and its ideal by $\tilde{\mathfrak{x}}$. The coordinates $s_{00}, s_{01}, \dots, s_{0n}$ vanish on $\tilde{X}$.

The action of the group $\mathcal{G}$ is lifted to $\tilde{M}^{n+1}$. Because of (17), (4) and (9) the action on $\tilde{X}$ is given by

$$(g, (s_{0,n+1}, S)) \mapsto (s_{0,n+1}, USU^\top). \tag{18}$$

Lemma 5.2. *No contact condition $\tilde{L}$ contains any $\mathcal{G}$-invariant subset of $\tilde{M}^{n+1}$.*[3]

Proof. We shall determine the minimal $\mathcal{G}$-invariant subsets and shall show that they are not in $\tilde{\bar{L}}_d$ for all d.

Because of Remark 4.4 the closure of each $\mathcal{G}$-orbit of $\tilde{M}^{n+1}$ contains a point of the complete inverse image $\tilde{X}$ of $\bar{\Xi}$. Therefore all minimal $\mathcal{G}$-invariant subsets of $\tilde{M}^{n+1}$ are in $\tilde{X}$.

The subsets $\tilde{V}(\lambda,\mu)$, $\lambda \in \mathbb{C}$, $\mu \in \mathbb{C}$, $(\lambda,\mu) \neq (0,0)$, with the ideals

$$\tilde{\mathfrak{v}}(\lambda,\mu) = \tilde{\mathfrak{x}} + (\lambda s_{0,n+1} - \mu \operatorname{trace} S) \tag{19}$$

are $\mathcal{G}$-invariant because of (18). $\tilde{V}(\lambda,\mu)$ are varieties for $\lambda \neq 0$. By the way, $\tilde{V}(1,1) = \tilde{Q} \cap \tilde{X}$ and $\tilde{V}(1,0) = \tilde{H} \cap \tilde{X}$. $\tilde{V}(0,1)$ is the union of the $\mathcal{G}$-invariant varieties $\tilde{E} \cap \tilde{X}$ and $\tilde{W}$ where $\tilde{W}$ is the bundle over $\bar{\Psi}$ with the ideal

$$\tilde{\mathfrak{x}} + (z^\top z,\ \operatorname{trace} S). \tag{20}$$

The intersection of any two sets $\tilde{V}(\lambda,\mu)$ is the $\mathcal{G}$-invariant $(n-2)$-dimensional set $\tilde{\Psi}_0$ with the ideal

$$\tilde{\mathfrak{x}} + (z^\top z, s_{0,n+1},\ \operatorname{trace} S). \tag{21}$$

Each point of $\tilde{X} \setminus \tilde{\Psi}_0$ lies in one and only one $\tilde{V}(\lambda,\mu)$.

[3] $\tilde{M}^{n+1}$ is $\mathcal{G}$-complete with respect to $\tilde{L}$ [**3**].

The varieties $\tilde{V}(\lambda,\mu)$ for $\lambda \neq 0$ and $\tilde{E} \cap \tilde{X}$ are sections in the bundle $\tilde{X}$. That means that to each point of $\bar{\tilde{\Xi}}$ and to each point $\lambda : \mu$ of $\mathbf{P}^1$ exact one point exists in $\tilde{V}(\lambda,\mu)$. Because of the action (11) $\tilde{\Psi}_0$ and $\tilde{V}(\lambda,\mu) \setminus \tilde{\Psi}_0$ for $\lambda \neq 0$ are $\mathcal{G}$-orbits.

Just so $\tilde{\Psi}_\infty = \tilde{E} \cap \tilde{W}$ and $\tilde{E} \setminus \tilde{\Psi}_\infty$ are $\mathcal{G}$-orbits.

Because of the action (11), too, each matrix $S \neq O$ with $\operatorname{trace} S = 0$ can be transformed by $O(\mathbb{C}, n)$ in any matrix $S' \neq O$ with $\operatorname{trace} S' = 0$. Therefore, $\tilde{W} \setminus (\tilde{\Psi}_\infty \cup \tilde{\Psi}_0)$ is a $\mathcal{G}$-orbit, too.

Thus the minimal $\mathcal{G}$-invariant subvarieties of $\tilde{M}^{n+1}$ are $\tilde{\Psi}_\infty$ with the ideal

$$\tilde{\mathfrak{x}} + (z^\top z, p_k, s_{ij})_{k\in\{0,\ldots,n\},(i,j)\in\{0,\ldots,n\}^2}, \tag{22}$$

and $\tilde{\Psi}_0$ with the ideal

$$\tilde{\mathfrak{x}} + (z^\top z, s_{0,n+1}, \operatorname{trace} S). \tag{23}$$

Both do not lie in $\dot{\tilde{L}}_d$ for all d. □

Lemma 5.3. *There exist $g_1, \ldots, g_{n+1}$ in $\mathcal{G}$ so that the intersection*

$$\bigcap_{i=1}^{n+1} g_i \dot{\tilde{L}}_{d_i}$$

is a zero-dimensional set in the $\mathcal{G}$-orbit $\tilde{M}^{n+1} \setminus (\tilde{Q} \cup \tilde{H} \cup \tilde{E} \cup \tilde{X})$.

Proof. Because of Lemma 5.2 we can use the same arguments as in the proofs of Lemma 3.2 and Theorem 3.3. □

Theorem 5.4. *There exist $g_1, \ldots, g_{n+1}$ in $\mathcal{G}$ so that the number of the non-degenerated $(n-1)$-spheres*[4] *satisfying the contact conditions $g_i \dot{L}_{d_i}$, $i = 1, \ldots, n+1$, is 2^n.*

Proof. The projection $\bar{\chi}\tilde{\chi}$ isomorphically maps the $\mathcal{G}$-orbit $\tilde{M}^{n+1} \setminus (\tilde{Q} \cup \tilde{H} \cup \tilde{E} \cup \tilde{X})$ onto the $\mathcal{G}$-orbit $\mathbf{P}^{n+1} \setminus (Q \cup H)$. Therefore, because of Lemma 5.3 we can compute the number in question using the cohomology of $\tilde{M}^{n+1}$.

Lemma 5.5. *The cohomology ring $H^*\tilde{M}^{n+1}$ can be generated by three elements represented by $\tilde{P}$, $\tilde{E}$ and $\tilde{X}$ with the relations*

$$\tilde{P}\tilde{E} \sim 0, \tag{24}$$

$$\tilde{P}\tilde{X} \sim 0, \tag{25}$$

$$\tilde{E}^2 + 2\tilde{E}\tilde{X} \sim 0, \tag{26}$$

$$(\tilde{E} + \tilde{X})^n \tilde{X} \sim 0, \tag{27}$$

$$\tilde{P}^{n+1} \sim (-1)^n (\tilde{E} + \tilde{X})^{n+1}. \tag{28}$$

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[4]These spheres are the proper solutions of this Apollonius problem.

Proof. We have

$$\tilde{\chi}^*\bar{P} = \tilde{P} \text{ and } \tilde{\chi}^*\bar{Y} = \tilde{Y} \tag{29}$$

because $\bar{\Xi}$ is neither in $\bar{P}$ nor in $\bar{Y}$, and

$$\tilde{\chi}^*\bar{E} = \tilde{E} + \tilde{X} \text{ and } \tilde{\chi}^*\bar{H} = \tilde{H} + \tilde{X} \tag{30}$$

because $\bar{\Xi}$ simply lies on $\bar{E}$ and on $\bar{H}$.

With (14) and (12) it follows

$$\tilde{Y} \sim \tilde{P} - \tilde{E} - \tilde{X}, \quad \tilde{H} \sim \tilde{P} - \tilde{E} - 2\tilde{X} \tag{31}$$

and

$$\tilde{P}(\tilde{E} + \tilde{X}) \sim 0. \tag{32}$$

(13) leads to (28).

We get (25) because $\bar{P}_{|\bar{\Xi}} \sim 0$ in $H^*\bar{\Xi}$ (or because $\tilde{P} \cap \tilde{X} = \emptyset$), and from this and (32) we have (24).

Moreover,

$$\tilde{E}\tilde{H} \sim 0 \tag{33}$$

because $\bar{E}$ and $\bar{H}$ transversally intersect in $\bar{\Xi}$.

From (33), (31) and (24) it follows (26).

It is

$$\tilde{Y}^n\tilde{X} \sim 0 \tag{34}$$

because $\bar{Y}^n_{|\bar{\Xi}} \sim 0$ in $H^*\bar{\Xi}$. Therefore, with (31), (25) and (24) we get (27).

In the $(n+1)$-th graduation, at first, we have

$$\tilde{P}^i\tilde{E}^{n-i+1} \sim 0, \; \tilde{P}^i\tilde{X}^{n-i+1} \sim 0 \text{ and } (-2)^i\tilde{E}^{n-i+1}\tilde{X}^i \sim \tilde{E}^{n+1} \tag{35}$$

for $i = 1, \ldots, n$ because of (31), (24) and (26).

With (27) the third relation in (35) leads to

$$2^n\tilde{X}^{n+1} \sim \left(\sum_{i=1}^{n}(-1)^{i+1}2^{n-i}\binom{n}{i-1}\right)\tilde{E}^{n+1}. \tag{36}$$

The coefficient of $\tilde{E}^{n+1}$ in (36) is 0 if n even and it is 1 if n is odd. Thus

$$\tilde{X}^{n+1} \sim 0 \text{ if } n \text{ even and } \tilde{E}^{n+1} \sim 2^n\tilde{X}^{n+1} \text{ if } n \text{ odd.} \tag{37}$$

The relation (28) under (35) translates to

$$(-2)^n \tilde{P}^{n+1} \sim \left(\sum_{j=1}^{n+1} (-1)^{j+1} 2^{n+1-j} \binom{n+1}{j-1} \right) \tilde{E}^{n+1} + 2^n \tilde{X}^{n+1}. \tag{38}$$

The coefficient of $\tilde{E}^{n+1}$ in (38) is 1 if n is even and it is 0 if n is odd. So we have

$$\tilde{X}^{n+1} \sim -\tilde{P}^{n+1} \quad \text{if } n \text{ odd} \tag{39}$$

and with (37)

$$\tilde{E}^{n+1} \sim (-2)^n \tilde{P}^{n+1} \tag{40}$$

for all n.

Thus the relations (24), (25), (26), (27) and (28) reduce the $(n+1)$-th graduation to a one-dimensional module. Using general arguments this suffices to complete the proof of the Lemma 5.5. To be quite sure one can check the other graduations. $\bar{\Xi}$ is isomorphic to a projective $(n-1)$-space and, therefore, it has to be $\dim H^i \tilde{X} = 2$ if $0 < i < n$ and $\dim H^i \tilde{M}^{n+1} = 3$ if $0 < i < n+1$. □

Proof of Theorem 5.4 *continued.* We want to compute $\tilde{L}^{n+1}$ in $H^* \tilde{M}^{n+1}$. It is

$$\tilde{\chi}^* \bar{L} = \tilde{L} + \tilde{X}$$

because $\bar{\Xi}$ simply lies on $\bar{L}$.

Via (14), (29) and (30) we get

$$\tilde{L} \sim 2\tilde{P} - \tilde{E} - 2\tilde{X}.$$

This together with (25), (24) and (35) leads to

$$\tilde{L}^{n+1} \sim 2^{n+1} \tilde{P}^{n+1} + (-1)^{n+1} \left(\sum_{i=0}^{n} (-1)^i \binom{n+1}{i} \right) \tilde{E}^{n+1} + (-2)^{n+1} \tilde{X}^{n+1}.$$

The sum in parenthesis is $(-1)^n$. Therefore, by (37), (40) and (39) we get the result

$$\tilde{L}^{n+1} \sim 2^n \tilde{P}^{n+1}.$$

□

The blowing-up onto $\tilde{M}^{n+1}$ can be interpreted — as in the case of complete quadrics — in terms of the sets of the tangent d-spaces to the $(n-1)$-spheres and that for each dimension d. To describe complete quadrics one needs a separate blowing-up for each d which can be given by the coefficients of the equation in the Grassmann coordinates. In the case of spheres this coefficients are quadratic in p_k for all d and can be represented linear in s_{ij}.

Any point of the fiber over a point (π, y) of $\bar{\Xi}$ can be interpreted to be the set of d-spaces which contact an $(n-2)$-quadric lying in the plane at infinity. The one-dimensional fiber corresponds to the pencil of such $(n-2)$-quadrics which is generated by Ω and by the doubly counted polar at y to Ω. The sections of the fiber by $\tilde{E}$, $\tilde{H}$ resp. $\tilde{Q}$ correspond in the pencil to the quadric Ω, to the polar at y to Ω resp. to the tangent cone to Ω with the apex y. If $(\pi, y) \in \bar{\Psi}$ so $y \in \Omega$.

Example. Let $n = 3$ and let $d_i = 2$, $i = 1, \ldots, 4$. If the four planes T_i^2 are in general position (that means under enough action of $\mathcal{G}$) then they are the support planes of a tetrahedron. In general, the 8 solutions are non-degenerated 2-spheres:

– the inscribed sphere which contacts all faces of the tetrahedron inside,
– the 4 spheres each of which contacts one face outside and the other 3 planes inside and
– the 3 spheres each of which contacts two planes outside and the two others inside whereby for each pairing the position of the planes decides which of the pairs can be contact inside an which can be contact outside.

If the tetrahedron is not general some ore all of the last 3 solutions can be degenerate into the plane at infinity. In the case of a regular tetrahedron all those degenrate. The midpoints y of these degenerations lie in the plane at infinity. They are given by the lines which join the midpoints of opposite edges of the tetrahedron. The polars at the y to $\Omega : x_1^2 + x_2^2 + x_3^2 = 0$ are the diagonals in the quadrilateral formed by the traces of the faces of the tetrahedron in the plane at infinity.

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