COMPARISON THEOREMS FOR PSEUDOCONJUGATE POINTS OF HALF-LINEAR ORDINARY DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

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Abstract. This paper generalizes well known comparison theorems for linear differential equations of the second order to half-linear differential equations of second order. We are concerned with pseudoconjugate and deconjugate points of solutions of these equations.

1. Introduction

In this paper we are concerned with the behavior of solutions of nonlinear ordinary differential equations of the form

$$l_\alpha[y] \equiv [r(x)|y'|^{\alpha-1}y']' + p(x)|y|^{\alpha-1}y = 0, \quad x \geq x_0 > 0,$$

where $\alpha > 0$ is a constant and r and p are continuous functions defined on an interval $I \subset [x_0, \infty)$ with $r(x) > 0$ for $x \in I$, which, with notation

$$u^{\alpha*} = |u|^\alpha \operatorname{sgn} u = (|u|^{\alpha-1}u), \quad u \in \mathbb{R},$$

can be written shortly as

$$l_\alpha[y] \equiv [r(x)(y')^{\alpha*}]' + p(x)y^{\alpha*} = 0. \tag{1}$$

The equations of the form (1) are sometimes called *half-linear* because if y is a solution of the equation $l_\alpha[y] = 0$ and c is any real constant, then the function cy is also the solution of the equation (1).

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The domain $D_{l_\alpha}(I)$ of the operator l_α is defined to be the set of all continuous functions y defined on I such that y and $r(x)(y')^{\alpha *}$ are continuously differentiable on I.

Let $y(x)$ be a solution of Eq. (1) satisfying the condition $y(a) = 0$ for some $a \in I$. A value $x = b$ from I is called a *conjugate* (resp. *pseudoconjugate* point to $x = a$ if $b > a$ and $y(b) = 0$ (resp. $y'(b) = 0$) (see [7]).
If $y(x)$ is a solution of (1) satisfying $y'(a) = 0$ for some $a \in I$, then a value $x = b \in I$ is called a *focal* (resp. *deconjugate*) point to $x = a$ if $b > a$ and $y(b) = 0$ (resp. $y'(b) = 0$) (see [7]).

Along with the equation (1) consider also another half-linear equation

$$L_\alpha[z] \equiv [R(x)(z')^{\alpha *}]' + P(x)z^{\alpha *} = 0, \quad x \geq x_0, \tag{2}$$

where R and P are continuous on I with $R(x) > 0$ for $x \in I$. The domain $D_{L_\alpha}(I)$ of the half-linear operator L_α is defined similarly as $D_{l_\alpha}(I)$.

In the case $\alpha = 1$, i.e. if equations (1) and (2) are linear, the following comparison results concerning pseudoconjugate points (Theorem A) and deconjugate points (Theorem B) are known (see [4]).

Theorem A *If $z(x)$ is a solution of*

$$(R(x)z')' + P(x)z = 0 \tag{3}$$

for which $z(a) = z'(c) = 0$ with $z'(x) \neq 0$ on $[a, c)$ and if

$$\int_a^c [(R - r)(z')^2 + (p - P)z^2]dx \geq 0, \tag{4}$$

then any nontrivial solution $y(x)$ of

$$(r(x)y')' + p(x)y = 0 \tag{5}$$

with $y(a) = 0$ has the property that $y'(\xi) = 0$ for some point $x = \xi \in (a, c]$, with $\xi = c$ only if $y(x) = kz(x)$, where k is a constant.

Theorem B *Let $r(x)$, $R(x)$, $p(x)$ and $P(x)$ be positive and continuous on the interval $[a, b]$. If the derivative $z'(x)$ of a solution $z(x)$ of the equation (3) has consecutive zeros at $x = c_1$ and $x = c_2$ $(a \le c_1 < c_2 \le b)$, and if*

$$R(x) \ge r(x), \quad p(x) \ge P(x)$$

holds on $[a, b]$, then the derivative $y'(x)$ of any nontrivial solution $y(x)$ of the equation (5) with the property $y'(c_1) = 0$ will have a zero on the interval $(c_1, c_2]$.

The purpose of this paper is to generalize Theorems A and B to the case of half-linear equations, i.e., nonlinear differential equations of the form (1) and (2). The proofs are based on a half-linear version of the well known Picone's identity (see [3]) and the reciprocity principle (see [1]) which connects the pair of equations (1) and (2) with another pair of the half-linear equations

$$\left((p(x))^{-1/\alpha}(y_1')^{(1/\alpha)*}\right)' + (r(x))^{-1/\alpha}(y_1)^{(1/\alpha)*} = 0$$

and

$$\left((P(x))^{-1/\alpha}(z_1')^{(1/\alpha)*}\right)' + (R(x))^{-1/\alpha}(z_1)^{(1/\alpha)*} = 0.$$

2. Comparison theorem for pseudoconjugate points

In what follows we employ the following result from [3].

Lemma 1. *Let y, z, $r(y')^{\alpha*}$ and $R(z')^{\alpha*}$ be continuously differentiable functions on an interval I and let $y(x) \neq 0$ in I. Then*

$$\begin{aligned}
&\frac{d}{dx}\Big\{\frac{z}{y^{\alpha*}}\Big[y^{\alpha*}R(z')^{\alpha*} - z^{\alpha*}r(y')^{\alpha*}\Big]\Big\} \\
&\quad = (R-r)|z'|^{\alpha+1} + r\Big[|z'|^{\alpha+1} + \alpha\Big|\frac{z}{y}y'\Big|^{\alpha+1} - (\alpha+1)z'\Big(\frac{z}{y}y'\Big)^{\alpha*}\Big] \\
&\qquad + z(R(z')^{\alpha*})' - \frac{|z|^{\alpha+1}}{y^{\alpha*}}(r(y')^{\alpha*})'.
\end{aligned} \tag{6}$$

Our first result is comparison theorem for pseudoconjugate points which generalizes Theorem A from the introduction.

Theorem 1. *If $z(x)$ is a solution of Eq. (2) for which $z(a) = z'(c) = 0$ with $z'(x) \neq 0$ on $[a, c)$ and if*

$$V_\alpha[z] \equiv \int_a^c [(R-r)|z'|^{\alpha+1} + (p-P)|z|^{\alpha+1}]dx \geq 0, \tag{7}$$

then any nontrivial solution $y(x)$ of (1) with $y(a) = 0$ has the property that $y'(\xi) = 0$ for some point $x = \xi \in (a, c]$, with $\xi = c$ only if $y(x) = kz(x)$, where k is a constant.

Proof. We can suppose that $y(x) \neq 0$ on the whole interval $(a, c]$ because otherwise the proof of the theorem would be trivial.

If, in the Picone's identity (6), we use that y and z are solutions of the equations (1) and (2), respectively, then we obtain

$$\begin{aligned}
\frac{d}{dx}\Big\{\frac{z}{y^{\alpha*}}\Big[y^{\alpha*}R(z')^{\alpha*} - z^{\alpha*}r(y')^{\alpha*}\Big]\Big\} = {} & (R-r)|z'|^{\alpha+1} + (p-P)|z|^{\alpha+1} \\
& + r\Big[|z'|^{\alpha+1} + \alpha\Big|\frac{z}{y}y'\Big|^{\alpha+1} - (\alpha+1)z'\Big(\frac{z}{y}y'\Big)^{\alpha*}\Big].
\end{aligned} \tag{8}$$

Integrating (8) on $[u, v]$ and passing to the limit as $u \to a^+$ and $v \to c^-$, we have

$$
(9) \qquad \begin{aligned}
\lim_{\substack{v\to c^-\\ u\to a^+}} \left[\frac{z}{y^{\alpha*}}\Big(y^{\alpha*}R(z')^{\alpha*} - z^{\alpha*}r(y')^{\alpha*}\Big)\right]_u^v &= \int_a^c [(R-r)|z'|^{\alpha+1} + (p-P)|z|^{\alpha+1}]dx \\
&+ \lim_{u\to a^+}\int_u^c r\left[|z'|^{\alpha+1} + \alpha\Big|\frac{z}{y}y'\Big|^{\alpha+1} - (\alpha+1)z'\Big(\frac{z}{y}y'\Big)^{\alpha*}\right]dx.
\end{aligned}
$$

If $y(a) = 0$ (resp.$z'(c) = 0$), then due to the fact that zeros of nontrivial solutions of second order half-linear equations are simple (see [6]) $y'(a)$ (resp. $z(c)$) must be a nonzero finite value. Since, obviously, $\lim_{u\to a^+} z(u)r(u)(y'(u))^{\alpha*} = 0$ and also

$$
\lim_{u\to a^+}\frac{(z(u))^{\alpha*}}{(y(u))^{\alpha*}} = \lim_{u\to a^+}\left(\frac{z(u)}{y(u)}\right)^{\alpha*} = \left(\lim_{u\to a^+}\frac{z(u)}{y(u)}\right)^{\alpha*} = \left(\lim_{u\to a^+}\frac{z'(u)}{y'(u)}\right)^{\alpha*} < \infty
$$

by l'Hospital rule , we have

$$
\lim_{u\to a^+}\frac{z(u)}{(y(u))^{\alpha*}}\left[(y(u))^{\alpha*}R(u)(z'(u))^{\alpha*} - (z(u))^{\alpha*}r(u)(y'(u))^{\alpha*}\right] = 0.
$$

Thus, (9) is reduced to

$$
-(y'(c))^{\alpha*}\frac{|z(c)|^{\alpha+1}r(c)}{(y(c))^{\alpha*}} = V_\alpha[z] + \int_a^c r\left[|z'|^{\alpha+1} + \alpha\Big|\frac{z}{y}y'\Big|^{\alpha+1} - (\alpha+1)z'\Big(\frac{z}{y}y'\Big)^{\alpha*}\right]dx.
$$

Since $|z'|^{\alpha+1} + \alpha\Big|\frac{z}{y}y'\Big|^{\alpha+1} - (\alpha+1)z'\Big(\frac{z}{y}y'\Big)^{\alpha*}$ is nonnegative (see [3]), then

$$
(10) \qquad -(y'(c))^{\alpha*}\frac{|z(c)|^{\alpha+1}r(c)}{(y(c))^{\alpha*}} \geq 0
$$

holds.

We may suppose without loss of generality that $y'(a)$ and $z'(a)$ are positive, i.e., $y(x) > 0$ and $z(x) > 0$ on $(a, c]$.
If $y'(x)$ does not have a zero on $a < x \leq c$, i.e., $y'(c) > 0$, then we obtain contradiction with (2). Thus $y'(c) \leq 0$ and so there exists a value $x = \xi$ on $(a, c]$ with the property $y'(\xi) = 0$. The case $y'(c) = 0$ occurs when $|z'|^{\alpha+1} + \alpha\left|\frac{z}{y}y'\right|^{\alpha+1} - (\alpha+1)z'\left(\frac{z}{y}y'\right)^{\alpha*} = 0$, i.e., if $z' = \frac{z}{y}y'$ which is equivalent with the fact that $y(x) = kz(x)$ where k is a constant. The proof is complete. $\square$

Remark. Theorem 1 says that the solution $y(x)$ will have a maximum resp. minimum on $(a, c]$ not later than $z(x)$.

Corollary. If $R(x) \geq r(x), \quad p(x) \geq P(x)$ on [a,c], then the assertion of Theorem 1 is valid.

3. Comparison theorem for deconjugate points

In this section we generalize comparison theorem for deconjugate points for half-linear equations.

Theorem 2. *Let $r(x)$, $R(x)$, $p(x)$ and $P(x)$ be positive and continuous on the interval $[a, b]$. If the derivative $z'(x)$ of a solution $z(x)$ of the equation (2) has consecutive zeros at $x = c_1$ and $x = c_2$ $(a \leq c_1 < c_2 \leq b)$, and if*

$$R(x) \geq r(x), \quad p(x) \geq P(x) \tag{11}$$

holds on$[a, b]$, then the derivative $y'(x)$ of any nontrivial solution $y(x)$ of the equation (1) with the property $y'(c_1) = 0$ will have a zero on the interval $(c_1, c_2]$.

In the proof we will use the following theorem (see [3]):

Sturm-Picone comparison theorem *Let A, a, B and b be continuous functions on an interval $[\alpha, \beta]$ with $A(x) > 0$ and $a(x) > 0$ on $[\alpha, \beta]$ and $\gamma > 0$ be a constant. If $u(x)$ is a solution of the half-linear differential* *equation*

$$[A(x)(u')^{\gamma *}]' + B(x)u^{\gamma *} = 0$$

for which $u(\alpha) = u(\beta) = 0$ *and if* $A(x) \geq a(x)$, $b(x) \geq B(x)$ *on* $[\alpha, \beta]$, *then any nontrivial solution* $v(x)$ *of*

$$[a(x)(v')^{\gamma *}]' + b(x)v^{\gamma *} = 0$$

with $v(\alpha) = 0$ *has the property that* $v(c) = 0$ *for some point* $x = c \in (\alpha, \beta]$, *with* $c = \beta$ *only if* $v(x) = ku(x)$, *where* k *is a constant.*

Proof. To prove Theorem 2 substitute $z_1(x) = R(x)(z')^{\alpha *}$ and $y_1(x) = r(x)(y')^{\alpha *}$ in (2) and (1), respectively. It follows that z_1 satisfies the differential equation

$$\left((P(x))^{-1/\alpha}(z_1')^{(1/\alpha)*}\right)' + (R(x))^{-1/\alpha}(z_1)^{(1/\alpha)*} = 0 \tag{12}$$

with $z_1(c_1) = z_1(c_2) = 0$, and y_1 satisfies the differential equation

$$\left((p(x))^{-1/\alpha}(y_1')^{(1/\alpha)*}\right)' + (r(x))^{-1/\alpha}(y_1)^{(1/\alpha)*} = 0 \tag{13}$$

with $y_1(c_1) = 0$. We note that from conditions (11) the inequalities

$$\left(\frac{1}{P(x)}\right)^{\frac{1}{\alpha}} \geq \left(\frac{1}{p(x)}\right)^{\frac{1}{\alpha}}, \quad \left(\frac{1}{r(x)}\right)^{\frac{1}{\alpha}} \geq \left(\frac{1}{R(x)}\right)^{\frac{1}{\alpha}} \tag{14}$$

follows.
An application of the Sturm-Picone comparison theorem to equations (12) and (13) completes the proof. $\square$

4. Generalized sine function

Let $S(x)$ be the solution of the equation

$$((z')^{\alpha *})' + \alpha z^{\alpha *} = 0 \tag{15}$$

determined by the initial conditions $z(0) = 0$ and $z'(0) = 1$. The function $S(x)$ has the properties

$$|S(x)|^{\alpha+1} + |S'(x)|^{\alpha+1} = 1 \qquad \text{and} \qquad (x + \pi_\alpha) = -S(x)$$

for all $x \in (-\infty, \infty)$, where π_α is given by

$$\pi_\alpha = \frac{\dfrac{2\pi}{\alpha+1}}{\sin \dfrac{\pi}{\alpha+1}}$$

and further $S(\frac{\pi_\alpha}{2}) = 1$ and $S'(\frac{\pi_\alpha}{2}) = 0$ (see [2]). The function $S(x)$ is called *generalized sine function* (see [2]). This function will be used in the proof of the next theorem, in which we determine an upper bound for pseudo-conjugate poins of $x = 0$.

Theorem 3. *If there exists a constant $k > 0$ such that*

$$\int_0^{\pi_\alpha/2k} [p(x) - \alpha k^{\alpha+1}] |S(kx)|^{\alpha+1} dx = 0$$

then the derivative of a nontrivial solution $y(x)$ of the differential equation

$$((y')^{\alpha *})' + p(x) y^{\alpha *} = 0$$

with properties $y(0) = 0$, $y'(0) > 0$ will have a zero in the interval $(0, \frac{\pi_\alpha}{2k}]$. The zero will be on the open interval except when $p(x) \equiv \alpha k^{\alpha+1}$.

Proof. Observe that if $y(x)$ has a zero on $(0, \frac{\pi_\alpha}{2k}]$, the theorem is immediate.

Let $y(x)$ have no zero on $(0, \frac{\pi_\alpha}{2k}]$. We consider the following half-linear differential equation

$$((z')^{\alpha *})' + \alpha k^{\alpha+1} z^{\alpha *} = 0 \tag{16}$$

with the initial conditions $z(0) = 0$ and $z'(0) = 1$. As easily seen, a solution of the differential equation (16) is $S(kx)$. To prove the theorem, consider Picone's identity (8)

$$\begin{aligned} \Big\{ \frac{z}{y^{\alpha *}} \Big[y^{\alpha *}(z')^{\alpha *} - z^{\alpha *}(y')^{\alpha *} \Big] \Big\}_0^{\pi_\alpha/2k} &= \int_0^{\pi_\alpha/2k} [(p - \alpha k^{\alpha+1})|z|^{\alpha+1}] dx \, + \\ &+ \int_0^{\pi_\alpha/2k} \Big[|z'|^{\alpha+1} + \alpha \Big| \frac{z}{y} y' \Big|^{\alpha+1} - (\alpha+1) z' \Big(\frac{z}{y} y' \Big)^{\alpha *} \Big] dx, \end{aligned} \tag{17}$$

where $z(x) = S(kx)$. The right-hand side of (17) is positive, which implies that

$$-\frac{(y'(\dfrac{\pi_\alpha}{2k}))^{\alpha *}}{(y(\dfrac{\pi_\alpha}{2k}))^{\alpha *}} > 0$$

hence $y'(\frac{\pi_\alpha}{2k}) < 0$, i.e., there exists a zero of $y'(x)$ on the interval $(0, \frac{\pi_\alpha}{2k}]$. If y is a constant multiple of z then the second integral in (17) must be zero and this implies that $p(x) \equiv \alpha k^{\alpha+1}$. □

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