WEIGHTED ENDPOINT ESTIMATES FOR MULTILINEAR LITTLEWOOD-PALEY OPERATORS

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ABSTRACT. In this paper, we prove weighted endpoint estimates for multilinear Littlewood-Paley operators.

1. INTRODUCTION AND RESULTS

Let ψ be a fixed function on R^n which satisfies the following properties:

(1) $\int \psi(x)dx = 0$,
(2) $|\psi(x)| \leq C(1+|x|)^{-(n+1)}$,
(3) $|\psi(x+y) - \psi(x)| \leq C|y|(1+|x|)^{-(n+2)}$ when $2|y| < |x|$;

Let m be a positive integer and A be a function on R^n. The multilinear Littlewood-Paley operator is defined by

$$g_\mu^A(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\mu} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2}, \quad \mu > 1,$$

where

$$F_t^A(f)(x,y) = \int_{R^n} \frac{R_{m+1}(A;x,z)}{|x-z|^m} \psi_t(y-z) f(z)dz,$$

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$$R_{m+1}(A;x,y)=A(x)-\sum_{|\alpha|\le m}\frac{1}{\alpha!}D^\alpha A(y)(x-y)^\alpha,$$

and $\psi_t(x)=t^{-n}\psi(x/t)$ for $t>0$. We denote by $F_t(f)(y)=f*\psi_t(y)$. We also define

$$g_\mu(f)(x)=\left(\int\int_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}|F_t(f)(y)|^2\frac{dydt}{t^{n+1}}\right)^{1/2},$$

which is the Littlewood-Paley operator (see [17]).

Let H be the Hilbert space $H=\left\{h:||h||=\left(\int\int_{R_+^{n+1}}|h(t)|^2dydt/t^{n+1}\right)^{1/2}<\infty\right\}$. Then for each fixed $x\in R^n$, $F_t^A(f)(x,y)$ may be viewed as a mapping from $(0,+\infty)$ to H, and it is clear that

$$\begin{aligned} g_\mu^A(f)(x) &= \left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}F_t^A(f)(x,y)\right\|, \\ g_\mu(f)(x) &= \left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}F_t(f)(y)\right\|. \end{aligned}$$

We also consider the variant of g_μ^A, which is defined by

$$\tilde{g}_\mu^A(f)(x)=\left[\int\int_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}|\tilde{F}_t^A(f)(x,y)|^2\frac{dydt}{t^{n+1}}\right]^{1/2},\quad \mu>1,$$

where

$$\tilde{F}_t^A(f)(x,y)=\int_{R^n}\frac{Q_{m+1}(A;x,z)}{|x-z|^m}\psi_t(y-z)f(z)dz$$

and

$$Q_{m+1}(A;x,z)=R_m(A;x,z)-\sum_{|\alpha|=m}D^\alpha A(x)(x-z)^\alpha.$$

Note that when $m=0$, g_μ^A is just the commutator of Littlewood-Paley operator (see [1], [14], [15]). It is well known that multilinear operators, as an extension of commutators, are of great interest in harmonic analysis and have been widely studied by many authors (see [4] – [8], [12], [13]). In [11], [16], the endpoint boundedness properties of commutators generated by the Calderon-Zygmund operator and BMO functions are obtained. The main purpose of this paper is to study the weighted endpoint boundedness of the multilinear Littlewood-Paley operators. Throughout this paper, $M(f)$ will denote the Hardy-Littlewood maximal function of f, Q will denote a cube of R^n with side parallel to the axes. For a cube Q and any locally integral function f on R^n, we denote that $f(Q)=\int_Q f(x)dx$, $f_Q=|Q|^{-1}\int_Q f(x)dx$ and $f^\#(x)=\sup_{x\in Q}|Q|^{-1}\int_Q|f(y)-f_Q|dy$. Moreover, for a weight functions $w\in A_1$ (see [10]), f is said to belong BMO(w) if $f^\#\in L^\infty(w)$ and define $||f||_{\mathrm{BMO}(w)}=||f^\#||_{L^\infty(w)}$, if $w=1$, we denote that BMO(R^n) = BMO(w). Also, we give the concepts of atomic and weighted H^1 space. A function a is called a $H^1(w)$ atom if there exists a cube Q such that a is supported on Q, $||a||_{L^\infty(w)}\le w(Q)^{-1}$ and $\int a(x)dx=0$. It is well known that, for $w\in A_1$, the weighted Hardy space $H^1(w)$ has the atomic decomposition characterization (see [2]).

We shall prove the following theorems in Section 3.

Theorem 1. *Let $D^\alpha A\in \mathrm{BMO}(R^n)$ for $|\alpha|=m$ and $w\in A_1$. Then g_μ^A is bounded from $L^\infty(w)$ to* BMO(w).

Theorem 2. *Let $D^\alpha A\in \mathrm{BMO}(R^n)$ for $|\alpha|=m$ and $w\in A_1$. Then $\tilde g_\mu^A$ is bounded from $H^1(w)$ to $L^1(w)$.*

Theorem 3. *Let $D^\alpha A\in \mathrm{BMO}(R^n)$ for $|\alpha|=m$ and $w\in A_1$. Then g_μ^A is bounded from $H^1(w)$ to weak $L^1(w)$.*

Theorem 4. *Let $D^\alpha A \in \mathrm{BMO}(R^n)$ for $|\alpha| = m$ and $w \in A_1$.*

(i) *If for any $H^1(w)$-atom a supported on certain cube Q and $u \in 3Q \setminus 2Q$, there is*

$$\int_{(4Q)^c} \left\| \left(\frac{t}{t+|x-y|}\right)^{n\mu/2} \sum_{|\alpha|=m} \frac{1}{\alpha!} \frac{(x-u)^\alpha}{|x-u|^m} \int_Q \psi_t(y-z) D^\alpha A(z) a(z) dz \right\| w(x) dx \leq C,$$

then g_μ^A is bounded from $H^1(w)$ to $L^1(w)$;

(ii) *If for any cube Q and $u \in 3Q \setminus 2Q$, there is*

$$\frac{1}{w(Q)} \int_Q \left\| \left(\frac{t}{t+|x-y|}\right)^{n\mu/2} \sum_{|\alpha|=m} \frac{1}{\alpha!} (D^\alpha A(x) - (D^\alpha A)_Q) \cdot \int_{(4Q)^c} \frac{(u-z)^\alpha}{|u-z|^m} \psi_t(y-z) f(z) dz \right\| w(x) dx$$
$$\leq C ||f||_{L^\infty(w)},$$

then $\tilde{g}_\mu^A$ is bounded from $L^\infty(w)$ to $\mathrm{BMO}(w)$.

Remark. In general, g_μ^A is not bounded from $H^1(w)$ to $L^1(w)$.

2. Some Lemmas

We begin with two preliminary lemmas.

Lemma 1. *(see [7].) Let A be a function on R^n and $D^\alpha A \in L^q(R^n)$ for $|\alpha| = m$ and some $q > n$. Then*

$$|R_m(A : x, y)| \le C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}(x,y)$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

Lemma 2. *Let $w \in A_1$, $1 < p < \infty$ $1 < r \le \infty$, $1/q = 1/p + 1/r$ and $D^\alpha A \in \mathrm{BMO}(R^n)$ for $|\alpha| = m$. Then g_μ^A is bounded from $L^p(w)$ to $L^q(w)$, that is*

$$||g_\mu^A(f)||_{L^q(w)} \le C \sum_{|\alpha|=m} ||D^\alpha A||_{\mathrm{BMO}} ||f||_{L^p(w)}.$$

Proof. By Minkowski inequality and the condition of ψ, we have

$$\begin{aligned}
& g_\mu^A(f)(x) \\
\le \; & \int_{R^n} \frac{|f(z)||R_{m+1}(A; x, z)|}{|x-z|^m} \left(\int_{R_+^{n+1}} |\psi_t(y-z)|^2 \left(\frac{t}{t+|x-y|} \right)^{n\mu} \frac{dydt}{t^{1+n}} \right)^{1/2} dz \\
\le \; & C \int_{R^n} \frac{|f(z)||R_{m+1}(A; x, z)|}{|x-z|^m} \left(\int_0^\infty \int_{R^n} \frac{t^{-2n}}{(1+|y-z|/t)^{2n+2}} \cdot \left(\frac{t}{t+|x-y|} \right)^{n\mu} \frac{dydt}{t^{1+n}} \right)^{1/2} dz \\
\le \; & C \int_{R^n} \frac{|f(z)||R_{m+1}(A; x, z)|}{|x-z|^m} \left[\int_0^\infty \left(t^{-n} \int_{R^n} \left(\frac{t}{t+|x-y|} \right)^{n\mu} \cdot \frac{dy}{(t+|y-z|)^{2n+2}} \right) tdt \right]^{1/2} dz,
\end{aligned}$$

noting that

$$t^{-n}\int_{R^n}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\frac{dy}{(t+|y-z|)^{2n+2}}\le CM\left(\frac{1}{(t+|\cdot-z|)^{2n+2}}\right)(x)\le C\frac{1}{(t+|x-z|)^{2n+2}}$$

and

$$\int_0^\infty\frac{tdt}{(t+|x-z|)^{2n+2}}=C|x-z|^{-2n},$$

we obtain

$$\begin{aligned} g_\mu^A(f)(x) &\le C\int_{R^n}\frac{|f(z)|}{|x-z|^m}|R_{m+1}(A;x,z)|\left(\int_0^\infty\frac{tdt}{(t+|x-z|)^{2n+2}}\right)^{1/2}dz \\ &= C\int_{R^n}\frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^{m+n}}dz, \end{aligned}$$

thus, the lemma follows from [8] [9]. □

3. Proof of Theorems

Proof of Theorem 1. It is only to prove that there exists a constant C_Q such that

$$\frac{1}{w(Q)}\int_Q |g_\mu^A(f)(x) - C_Q|w(x)dx \le C||f||_{L^\infty(w)}$$

holds for any cube Q. Fix a cube $Q = Q(x_0, l)$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A)_{\tilde{Q}} x^\alpha$, then $R_m(A; x, y) = R_m(\tilde{A}; x, y)$ and $D^\alpha \tilde{A} = D^\alpha A - (D^\alpha A)_{\tilde{Q}}$ for $|\alpha| = m$. We write $F_t^A(f) = F_t^A(f_1) + F_t^A(f_2)$ for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n\setminus\tilde{Q}}$, then

$$\begin{aligned}
& \frac{1}{w(Q)}\int_Q |g_\mu^A(f)(x) - g_\mu^A(f_2)(x_0)|w(x)dx \\
= & \frac{1}{w(Q)}\int_Q \left| ||(\frac{t}{t+|x-y|})^{n\mu/2} F_t^A(f)(x,y)|| - ||(\frac{t}{t+|x_0-y|})^{n\mu/2} F_t^A(f_2)(x_0,y)|| \right| w(x)dx \\
\le & \frac{1}{w(Q)}\int_Q g_\mu^A(f_1)(x)w(x)dx + \frac{1}{w(Q)}\int_Q \left|\left| \left(\frac{t}{t+|x-y|}\right)^{n\mu/2} F_t^A(f_2)(x,y) \right.\right. \\
& \left.\left. - \left(\frac{t}{t+|x_0-y|}\right)^{n\mu/2} F_t^A(f_2)(x_0,y) \right|\right| w(x)dx \\
:= & I(x) + II(x).
\end{aligned}$$

Now, let us estimate I and II. First, by the L^∞ boundedness of g_μ^A (Lemma 2), we gain

$$I \le ||g_\mu^A(f_1)||_{L^\infty(w)} \le C||f||_{L^\infty(w)}.$$

To estimate II, we write

$$
\begin{aligned}
& \left(\frac{t}{t+|x-y|}\right)^{n\mu/2} F_t^A(f_2)(x,y) - \left(\frac{t}{t+|x_0-y|}\right)^{n\mu/2} F_t^A(f_2)(x_0,y) \\
= \quad & \left(\frac{t}{t+|x-y|}\right)^{n\mu/2} \int \left[\frac{1}{|x-z|^m} - \frac{1}{|x_0-z|^m}\right] \psi_t(y-z) R_m(\tilde{A};x,z) f_2(z)dz \\
& + \left(\frac{t}{t+|x-y|}\right)^{n\mu/2} \int \frac{\psi_t(y-z)f_2(z)}{|x_0-z|^m}[R_m(\tilde{A};x,z) - R_m(\tilde{A};x_0,z)]dz \\
& + \int \left[\left(\frac{t}{t+|x-y|}\right)^{n\mu/2} - \left(\frac{t}{t+|x_0-y|}\right)^{n\mu/2}\right] \frac{\psi_t(y-z)R_m(\tilde{A};x_0,z)f_2(z)}{|x_0-z|^m} dz \\
& - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int \left[(\frac{t}{t+|x-y|})^{n\mu/2} \frac{(x-z)^\alpha}{|x-z|^m} \right. \\
& \qquad \left. - \left(\frac{t}{t+|x_0-y|}\right)^{n\mu/2} \frac{(x_0-z)^\alpha}{|x_0-z|^m}\right] \psi_t(y-z) D^\alpha \tilde{A}(z) f_2(z) dz \\
:= \quad & II_1^t(x) + II_2^t(x) + II_3^t(x) + II_4^t(x),
\end{aligned}
$$

Note that $|x-z| \sim |x_0-z|$ for $x \in \tilde{Q}$ and $z \in R^n \setminus \tilde{Q}$, similar to the proof of Lemma 2 and by Lemma 1, we have

$$\begin{aligned}
&\frac{1}{w(Q)}\int_Q ||II_1^t(x)||w(x)dx \\
&\leq \frac{C}{w(Q)}\int_Q \left(\int_{R^n\setminus\tilde{Q}} \frac{|x-x_0||f(z)|}{|x-z|^{n+m+1}}|R_m(\tilde{A};x,z)|dz\right) w(x)dx \\
&\leq \frac{C}{w(Q)}\int_Q \left(\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \frac{|x-x_0||f(z)|}{|x-z|^{n+m+1}}|R_m(\tilde{A};x,z)|dz\right) w(x)dx \\
&\leq C\sum_{k=0}^{\infty}\frac{kl(2^k l)^m}{(2^k l)^{n+m+1}}\sum_{|\alpha|=m}||D^\alpha A||_{\mathrm{BMO}}(\int_{2^{k+1}\tilde{Q}}|f(z)|dz) \\
&\leq C\sum_{|\alpha|=m}||D^\alpha A||_{\mathrm{BMO}}||f||_{L^\infty(w)}\sum_{k=0}^{\infty}k2^{-k} \leq C\sum_{|\alpha|=m}||D^\alpha A||_{\mathrm{BMO}}||f||_{L^\infty(w)};
\end{aligned}$$

For $II_2^t(x)$, by the formula (see [7]):

$$R_m(\tilde{A};x,z) - R_m(\tilde{A};x_0,z) = R_m(\tilde{A};x,x_0) + \sum_{0<|\beta|<m}\frac{1}{\beta!}R_{m-|\beta|}(D^\beta\tilde{A};x_0,z)(x-x_0)^\beta$$

and Lemma 1, we get

$$|R_m(\tilde{A};x,z) \quad - \quad R_m(\tilde{A};x_0,z)| \leq C\sum_{|\alpha|=m}||D^\alpha A||_{\mathrm{BMO}}(|x-x_0|^m + \sum_{0<|\beta|<m}|x_0-z|^{m-|\beta|}|x-x_0|^{|\beta|}),$$

thus, for $x \in Q$,

$$
\begin{aligned}
&||II_2^t(x)|| \\
&\leq \quad C\int_{R^n}\frac{|f_2(z)|}{|x-z|^{m+n}}|R_m(\tilde{A};x,z)-R_m(\tilde{A};x_0,z)|dz \\
&\leq \quad C\sum_{|\alpha|=m}|D^\alpha A||_{\mathrm{BMO}}\int_{R^n}\frac{|x-x_0|^m+\sum_{0<|\beta|<m}|x_0-z|^{m-|\beta|}|x-x_0|^{|\beta|}}{|x_0-z|^{m+n}}|f_2(z)|dz \\
&\leq \quad C\sum_{|\alpha|=m}||D^\alpha A||_{\mathrm{BMO}}\sum_{k=0}^{\infty}\frac{kl^m}{(2^k l)^{m+n}}\int_{2^{k+1}\tilde{Q}}|f(z)|dz \\
&\leq \quad C\sum_{|\alpha|=m}||D^\alpha A||_{\mathrm{BMO}}||f||_{L^\infty(w)}\sum_{k=1}^{\infty}k2^{-km} \\
&\leq \quad C\sum_{|\alpha|=m}||D^\alpha A||_{\mathrm{BMO}}||f||_{L^\infty(w)};
\end{aligned}
$$

For $II_3^t(x)$, by the inequality: $a^{1/2}-b^{1/2}\leq(a-b)^{1/2}$ for $a\geq b>0$, we obtain, similar to the estimate of Lemma 2 and II_1,

$$
\begin{aligned}
&||II_3^t(x)|| \\
&\le \quad C\int_{R^n}\left(\int_{R^{n+1}_+}\left[\frac{t^{n\mu/2}|x-x_0|^{1/2}|\psi_t(y-z)||f_2(z)|}{(t+|x-y|)^{(n\mu+1)/2}|x_0-z|^m}|R_m(\tilde{A};x_0,z)|\right]^2\frac{dydt}{t^{n+1}}\right)^{1/2}dz \\
&\le \quad C\int_{R^n}\frac{|f_2(z)||x-x_0|^{1/2}|R_m(\tilde{A};x_0,z)|}{|x_0-z|^m} \\
&\qquad\qquad \cdot\left(\int\int_{R^{n+1}_+}\left(\frac{t}{t+|x-y|}\right)^{n\mu+1}\frac{t^{-n}dydt}{(t+|y-z|)^{2n+2}}\right)^{1/2}dz \\
&\le \quad C\int_{R^n}\frac{|f_2(z)||x-x_0|^{1/2}|R_m(\tilde{A};x_0,z)|}{|x_0-z|^m}\left(\int_0^\infty\frac{dt}{(t+|x-z|)^{2n+2}}\right)^{1/2}dz \\
&\le \quad C\int_{R^n}\frac{|f_2(z)||x-x_0|^{1/2}|R_m(\tilde{A};x_0,z)|}{|x_0-z|^{m+n+1/2}}dz \\
&\le \quad C\sum_{k=0}^\infty\frac{kl^{1/2}(2^kl)^m}{(2^kl)^{n+m+1/2}}\sum_{|\alpha|=m}||D^\alpha A||_{\mathrm{BMO}}\Big(\int_{2^{k+1}\tilde{Q}}|f(z)|dz\Big) \\
&\le \quad C\sum_{|\alpha|=m}||D^\alpha A||_{\mathrm{BMO}}||f||_{L^\infty(w)}\sum_{k=0}^\infty k2^{-k/2} \\
&\le \quad C\sum_{|\alpha|=m}||D^\alpha A||_{\mathrm{BMO}}||f||_{L^\infty(w)};
\end{aligned}
$$

For $II_4^t(x)$, similar to the estimates of $II_1^t(x)$ and $II_3^t(x)$, we have

$$\begin{aligned} ||II_4^t(x)|| &\leq C\int_{R^n\setminus\tilde{Q}}\left(\frac{|x-x_0|}{|x-z|^{n+1}}+\frac{|x-x_0|^{1/2}}{|x-z|^{n+1/2}}\right)\sum_{|\alpha|=m}|D^\alpha\tilde{A}(z)||f(z)|dz \\ &\leq C\sum_{|\alpha|=m}||D^\alpha A||_{\mathrm{BMO}}||f||_{L^\infty(w)}\sum_{k=0}^{\infty}k(2^{-k}+2^{-k/2}) \\ &\leq C\sum_{|\alpha|=m}||D^\alpha A||_{\mathrm{BMO}}||f||_{L^\infty(w)}. \end{aligned}$$

Combining these estimates, we complete the proof of Theorem 1. $\square$

Proof of Theorem 2. It suffices to show that there exists a constant $C>0$ such that for every $H^1(w)$-atom a (that is that a satisfies: $\operatorname{supp} a\subset Q=Q(x_0,r)$, $||a||_{L^\infty(w)}\leq w(Q)^{-1}$ and $\int a(y)dy=0$ (see [8])), we have

$$||\tilde{g}_\mu^A(a)||_{L^1(w)}\leq C.$$

We write

$$\int_{R^n}\tilde{g}_\mu^A(a)(x)w(x)dx=\left[\int_{|x-x_0|\leq 2r}+\int_{|x-x_0|>2r}\right]\tilde{g}_\mu^A(a)(x)w(x)dx:=J+JJ.$$

For J, by the following equality

$$Q_{m+1}(A;x,y)=R_{m+1}(A;x,y)-\sum_{|\alpha|=m}\frac{1}{\alpha!}(x-y)^\alpha(D^\alpha A(x)-D^\alpha A(y)),$$

we have, similar to the proof of Lemma 2,

$$\tilde{g}_\mu^A(a)(x) \le g_\mu^A(a)(x) + C\sum_{|\alpha|=m}\int \frac{|D^\alpha A(x) - D^\alpha A(y)|}{|x-y|^n}|a(y)|dy,$$

thus, $\tilde{g}_\mu^A$ is L^∞-bounded by Lemma 2 and [3]. We see that

$$J \le C||\tilde{g}_\mu^A(a)||_{L^\infty(w)}w(2Q) \le C||a||_{L^\infty(w)}w(Q) \le C.$$

To obtain the estimate of JJ, we denote that $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A)_{2B}x^\alpha$. Then $Q_m(A;x,y) = Q_m(\tilde{A};x,y)$. We write, by the vanishing moment of a and $Q_{m+1}(A;x,y) = R_m(A;x,y) - \sum_{|\alpha|=m}\frac{1}{\alpha!}(x-y)^\alpha D^\alpha A(x)$, for $x \in (2Q)^c$,

$$\begin{aligned}
&\tilde{F}_t^A(a)(x,y)\\
&= \int \frac{\psi_t(y-z)R_m(\tilde{A};x,z)}{|x-z|^m}a(z)dz - \sum_{|\alpha|=m}\frac{1}{\alpha!}\int \frac{\psi_t(y-z)D^\alpha\tilde{A}(z)(x-z)^\alpha}{|x-z|^m}a(z)dz\\
&= \int \left[\frac{\psi_t(y-z)R_m(\tilde{A};x,z)}{|x-z|^m} - \frac{\psi_t(y-x_0)R_m(\tilde{A};x,x_0)}{|x-x_0|^m}\right]a(z)dz\\
&\quad - \sum_{|\alpha|=m}\frac{1}{\alpha!}\int \left[\frac{\psi_t(y-z)(x-z)^\alpha}{|x-z|^m} - \frac{\psi_t(y-x_0)(x-x_0)^\alpha}{|x-x_0|^m}\right]D^\alpha\tilde{A}(x)a(z)dz,
\end{aligned}$$

thus, similar to the proof of II in Theorem 1, we obtain

$$||\tilde{F}_t^A(a)(x,y)|| \le C\frac{|Q|^{1+1/n}}{w(Q)}(\sum_{|\alpha|=m}||D^\alpha A||_{\mathrm{BMO}}|x-x_0|^{-n-1} + |x-x_0|^{-n-1}|D^\alpha\tilde{A}(x)|),$$

note that if $w \in A_1$, then $\frac{w(Q_2)}{|Q_2|}\frac{|Q_1|}{w(Q_1)} \le C$ for all cubes Q_1, Q_2 with $Q_1 \subset Q_2$. Thus, by Holder' inequality and the reverse of Holder' inequality for $w \in A_1$, taking $p > 1$ and $1/p + 1/p' = 1$, we obtain

$$\begin{aligned} JJ &\le C \sum_{|\alpha|=m} ||D^\alpha A||_{\mathrm{BMO}} \sum_{k=1}^{\infty} 2^{-k} \left(\frac{|Q|}{w(Q)} \frac{w(2^{k+1}Q)}{|2^{k+1}Q|} \right) \\ &\quad + C \sum_{|\alpha|=m} \sum_{k=1}^{\infty} 2^{-k} \frac{|Q|}{w(Q)} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |D^\alpha \tilde{A}(x)|^p dx \right)^{1/p} \cdot \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} w(x)^{p'} dx \right)^{1/p'} \\ &\le C \sum_{|\alpha|=m} ||D^\alpha A||_{\mathrm{BMO}} \sum_{k=1}^{\infty} k 2^{-k} \left(\frac{w(2^{k+1}Q)}{|2^{k+1}Q|} \frac{|Q|}{w(Q)} \right) \le C, \end{aligned}$$

which together with the estimate for J yields the desired result. This finishes the proof of Theorem 2. □

Proof of Theorem3. By the equality

$$R_{m+1}(A; x, y) = Q_{m+1}(A; x, y) + \sum_{|\alpha|=m} \frac{1}{\alpha!}(x-y)^\alpha (D^\alpha A(x) - D^\alpha A(y))$$

and similar to the proof of Lemma 2, we get

$$g_\mu^A(f)(x) \le \tilde{g}_\mu^A(f)(x) + C \sum_{|\alpha|=m} \int \frac{|D^\alpha A(x) - D^\alpha A(y)|}{|x-y|^n} |f(y)| dy,$$

by Theorem 1 and 2 with [3], we obtain

$$
\begin{aligned}
& w(\{x \in R^n : g_\mu^A(f)(x) > \lambda\}) \\
\le\ & w(\{x \in R^n : \tilde{g}_\mu^A(f)(x) > \lambda/2\}) + w(\{x \in R^n : \sum_{|\alpha|=m} \int \frac{|D^\alpha A(x) - D^\alpha A(y)|}{|x-y|^n} |f(y)|dy > C\lambda\}) \\
\le\ & C||f||_{H^1(w)}/\lambda.
\end{aligned}
$$

This completes the proof of Theorem 3. □

Proof of Theorem 4. **(i)** It suffices to show that there exists a constant $C > 0$ such that for every $H^1(w)$-atom a with supp$a \subset Q = Q(x_0, d)$, there is

$$||g_\mu^A(a)||_{L^1(w)} \le C.$$

Let $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A)_{\tilde{Q}} x^\alpha$. We write, by the vanishing moment of a and for $u \in 3Q \setminus 2Q$,

$$
\begin{aligned}
& F_t^A(a)(x,y) \\
=\ & \chi_{4Q}(x) F_t^A(a)(x,y) \\
& + \chi_{(4Q)^c}(x) \int_{R^n} \left[\frac{R_m(\tilde{A}; x, z)\psi_t(y-z)}{|x-z|^m} - \frac{R_m(\tilde{A}; x, u)\psi_t(y-u)}{|x-u|^m} \right] a(z)dz \\
& - \chi_{(4Q)^c}(x) \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \left[\frac{(x-z)^\alpha}{|x-z|^m} - \frac{(x-u)^\alpha}{|x-u|^m} \right] \psi_t(y-z) D^\alpha A(z) a(z)dz \\
& - \chi_{(4Q)^c}(x) \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \frac{(x-u)^\alpha}{|x-u|^m} \psi_t(y-z) D^\alpha A(z) a(z)dz,
\end{aligned}
$$

then

$$
\begin{aligned}
g_\mu^A(a)(x) &= \left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2} F_t^A(a)(x,y)\right\| \\
&\leq \chi_{4Q}(x)\left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2} F_t^A(a)(x,y)\right\| \\
&\quad + \chi_{(4Q)^c}(x)\left\|(\frac{t}{t+|x-y|})^{n\mu/2}\int_{R^n}\left[\frac{R_m(\tilde{A};x,z)\psi_t(y-z)}{|x-z|^m}-\frac{R_m(\tilde{A};x,u)\psi_t(y-u)}{|x-u|^m}\right]a(z)dz\right\| \\
&\quad + \chi_{(4Q)^c}(x)\left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{R^n}\left[\frac{(x-z)^\alpha}{|x-z|^m}-\frac{(x-u)^\alpha}{|x-u|^m}\right]\cdot\ \psi_t(y-z)D^\alpha A(z)a(z)dz\right\| \\
&\quad + \chi_{(4Q)^c}(x)\left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{R^n}\frac{(x-u)^\alpha}{|x-u|^m}\psi_t(y-z)a(z)dz\right\| \\
&= L_1(x)+L_2(x)+L_3(x,u)+L_4(x,u).
\end{aligned}
$$

By the $L^\infty(w)$-boundedness of g_μ^A, we get

$$
\int_{R^n} L_1(x)w(x)dx \;=\; \int_{4Q} g_\mu^A(a)(x)w(x)dx \leq ||g_\mu^A(a)||_{L^\infty(w)}w(4Q)\leq C||a||_{L^\infty(w)}w(Q)\leq C;
$$

Similar to the proof of Theorem 1, we obtain

$$
\int_{R^n} L_2(x)w(x)dx \leq C
$$

and

$$\int_{R^n} L_3(x,u)w(x)dx \le C.$$

Thus, using the condition of $L_4(x,u)$, we obtain

$$\int_{R^n} g_\mu^A(a)(x)w(x)dx \le C.$$

(ii) For any cube $Q = Q(x_0, d)$, let $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A)_{\tilde{Q}} x^\alpha$. We write, for $f = f\chi_{4Q} + f\chi_{(4Q)^c} = f_1 + f_2$ and $u \in 3Q \setminus 2Q$,

$$\begin{aligned}
&\tilde{F}_t^A(f)(x,y) \\
&= \tilde{F}_t^A(f_1)(x,y) + \int_{R^n} \frac{R_m(\tilde{A};x,z)}{|x-z|^m}\psi_t(y-z)f_2(z)dz \\
&\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A(x) - (D^\alpha A)_Q) \int_{R^n} \left[\frac{(x-z)^\alpha}{|x-z|^m} - \frac{(u-z)^\alpha}{|u-z|^m}\right]\psi_t(y-z)f_2(z)dz \\
&\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A(x) - (D^\alpha A)_Q) \int_{R^n} \frac{(u-z)^\alpha}{|u-z|^m}\psi_t(y-z)f_2(z)dz,
\end{aligned}$$

then

$$
\begin{aligned}
&\left|\tilde{g}_\mu^A(f)(x) - g_\mu\left(\frac{R_m(\tilde{A};x_0,\cdot)}{|x_0-\cdot|^m}f_2\right)(x_0)\right| \\
&= \left|\left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}\tilde{F}_t^A(f)(x,y)\right\| - \left\|\left(\frac{t}{t+|x_0-y|}\right)^{n\mu/2}F_t\frac{R_m(\tilde{A};x_0,\cdot)}{|x_0-\cdot|^m}f_2)(x_0)\right\|\right| \\
&\le \left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}\tilde{F}_t^A(f)(x,y) - \left(\frac{t}{t+|x_0-y|}\right)^{n\mu/2}F_t\left(\frac{R_m(\tilde{A};x_0,\cdot)}{|x_0-\cdot|^m}f_2\right)(x_0)\right\| \\
&\le \left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}\tilde{F}_t^A(f_1)(x,y)\right\| \\
&\quad + \left\|\int_{R^n}\left[\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}\frac{R_m(\tilde{A};x,z)}{|x-z|^m}\psi_t(y-z)\right.\right. \\
&\qquad \left.\left. - \left(\frac{t}{t+|x_0-y|}\right)^{n\mu/2}\frac{R_m(\tilde{A};x_0,z)}{|x_0-z|^m}\psi_t(x_0-z)\right]f_2(z)dz\right\| \\
&\quad + \left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}\cdot\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A(x)-(D^\alpha A)_Q)\int_{R^n}\left[\frac{(y-z)^\alpha}{|y-z|^m}-\frac{(u-z)^\alpha}{|u-z|^m}\right]\psi_t(y-z)f_2(z)dz\right\| \\
&\quad + \left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}\cdot\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A(x)-(D^\alpha A)_Q)\int_{R^n}\frac{(u-z)^\alpha}{|u-z|^m}\psi_t(y-z)f_2(z)dz\right\| \\
&= M_1(x)+M_2(x)+M_3(x,u)+M_4(x,u).
\end{aligned}
$$

By the $L^\infty(w)$-boundedness of $\tilde{g}_\mu^A$, we get

$$\frac{1}{w(Q)}\int_Q M_1(x)w(x)dx \le ||\tilde{g}_\mu^A(f_1)||_{L^\infty(w)} \le C||f||_{L^\infty(w)};$$

Similar to the proof of Theorem 1, we obtain

$$\frac{1}{w(Q)}\int_Q M_2(x)w(x)dx \le C||f||_{L^\infty(w)}$$

and

$$\frac{1}{w(Q)}\int_Q M_3(x,u)w(x)dx \le C||f||_{L^\infty(w)}.$$

Thus, using the condition of $M_4(x,u)$, we obtain

$$\frac{1}{w(Q)}\int_Q \left|\tilde{g}_\mu^A(f)(x) - g_\mu\left(\frac{R_m(\tilde{A};x_0,\cdot)}{|x_0-\cdot|^m}f_2\right)(x_0)\right| w(x)dx \le C||f||_{L^\infty(w)}.$$

This completes the proof of Theorem 4. □

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