# MULTIVARIATE BIRKHOFF-LAGRANGE INTERPOLATION SCHEMES AND CARTESIAN SETS OF NODES

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ABSTRACT. In this paper we study the relevance of cartesian shapes to the solvability of Birkhoff-Lagrange interpolation schemes.

## 1. INTRODUCTION

The bivariate[1] Birkhoff-Lagrange interpolation problem depends on a finite set $Z \subset \mathbb{R}^2$ (of "nodes"), and a lower set $S \subset \mathbb{N}^2$ defining the interpolation space

$$\mathcal{P}_S = \{P \in \mathbb{R}[x, y] : P = \sum_{(i,j)\in S} a_{i,j} x^i y^j\}.$$

Recall [3] that $S$ is lower if

$$(i, j) \in S \Longrightarrow R(i, j) \subset S,$$

where $R(i, j)$ is the rectangle

$$R(i, j) = \{(i', j') \in \mathbb{N}^2 : 0 \leq i' \leq i, 0 \leq j' \leq j\}.$$

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[1]Our results hold in arbitrary dimension, and the only reason for restricting to the bi-variate case is notational simplicity.

Given such a scheme $(Z, S)$, the interpolation problem consists of finding the polynomials $P \in \mathcal{P}_S$ satisfying the equations

$$P(z) = c(z), \qquad \forall\ z \in Z, \tag{1.1}$$

where $c(z)$ are given arbitrary constants. One says that $(Z, S)$ is *solvable* if, for any choice of the constants $c(z)$, (1.1) has at least one solution $P \in \mathcal{P}_S$. If the solution is unique, one says that $(Z, S)$ is *regular*.

The Birkhoff-Lagrange schemes are a particular case of uniform Birkhoff schemes [2], and the present work should be understood in the general context of finding the influence that the shape of $Z$ has on the regularity of the schemes. In this paper we discuss particular shapes (cartesian shapes, cf. Section 2), and we explain the influence that they have on the regularity/solvability of Birkhoff-Lagrange schemes. While referring to the next section for the most general results, we mention here (as the main result) the following:

**Theorem 1.1.** *Given a set of nodes $Z$, there exists at least one lower set $S$ with the property that the Birkhoff-Lagrange scheme $(Z, S)$ is regular. Moreover, $S$ is unique if and only if $Z$ is cartesian.*

## 2. Cartesian sets of nodes

In this section we introduce the notion of cartesian sets of nodes.

**Definition 2.1.** We say that a set $Z$ of nodes is cartesian if there exists a lower set $S$ such that $Z$ can be written as

$$Z = \{(x_i, y_j) : (i, j) \in S\},$$

where the $x_i$'s are distinct real numbers, and similarly the $y_j$'s. We also say that $Z$ is $S$-cartesian.

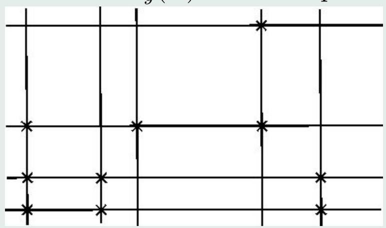
**Remark 2.1.** This notion is useful for understanding "special shapes" for the set of nodes (and not only). For instance, in the case of the rectangles $S = R(p, q)$, one recovers the notion of *rectangular sets of nodes*, i.e. sets which are at the intersection of $(p + 1)$ distinct vertical lines with $(q + 1)$ distinct horizontal lines.

In general, any set of nodes $Z$ induces two lower sets $S_x(Z)$, and $S_y(Z)$, which reflect the shape of $Z$. To describe $S_y(Z)$, one covers $Z$ by lines $l_0, \ldots, l_k$ parallel to the $OY$ axis, and one defines the numbers $n_i$ so that on each line $l_i$ there are exactly $n_i + 1$ points of $Z$. We index the lines so that $n_0 \geq n_1 \geq \ldots \geq n_k$, and we define

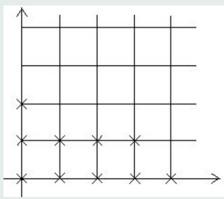
$$S_y(Z) = \{(i, j) : 0 \leq i \leq k, 0 \leq j \leq n_i\}. \tag{2.1}$$

The lower set $S_x(Z)$ is defined similarly, by interchanging the role of $x$ and $y$.

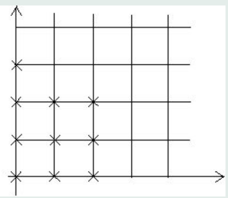
**Remark 2.2.** One should think of $S_y(Z)$ as obtained from $Z$ by moving it downwards (on vertical lines), to the left (on horizontal lines), and reordering the vertical and horizontal lines until one obtains a lower shape. Note also that, in general, $S_x(Z)$ is different from $S_y(Z)$. An examples is shown in the figure.



$\times$: the points of $Z$



$S_y(Z)$



$S_x(Z)$

With these, we have:

**Lemma 2.1.** *A set of nodes $Z$ is cartesian if and only if $S_x(Z) = S_y(Z)$.*

*Proof.* If $Z$ is $S$-cartesian for some lower set $S$, we see that the operations involved in the definition of $S_y(Z)$ produces the lower set $S$, hence $S_y(Z) = S$ and similarly $S_x(Z) = S$. Hence $S_x(Z) = S_y(Z)$. To prove the converse we use induction on the number of elements $n$ of $Z$. First of all, since each of the lines $l_i$ does contain at least one point, it follows that $(i, 0) \in S_y(Z)$ for all $0 \leq i \leq k$. Since $S_x(Z) = S_y(Z)$, it then follows that there is a line $l$, parallel to the $OX$ axis, which intersects each of the lines $l_i$ in a point situated in $Z$. We then see that $S_y(Z \setminus l) = S_x(Z \setminus l)$, and, by the induction hypothesis, $Z \setminus l$ must be cartesian. This clearly implies that $Z$ must be cartesian too. $\square$

**Example 2.1.** Although this example is one of the simplest, it is already very suggestive for the relation to the Lagrange problem, and can be seen as an illustration of the role of $S_x(Z)$ and $S_y(Z)$ for the proof of Theorem 1.1 (next section).

Assume that $Z$ is made of the points $(1, 0)$ and $(0, 1)$ and a third (distinct) one, $(a, b)$, with $a, b \in \mathbb{R}$. The computation of $S_x(Z)$ and $S_y(Z)$ depends on whether $a$ and $b$ belong to $\{0, 1\}$ or not. More precisely, from the definition we deduce that $S_y(Z) = \{(0, 0), (1, 0), (2, 0)\}$ if $a \notin \{0, 1\}$ and $S_y(Z) = \{(0, 0), (1, 0), (0, 1)\}$ if $a \in \{0, 1\}$, and similarly for $S_x(Z)$. Hence there are several possibilities, and we see that $Z$ is cartesian if and only if the third point, namely $(a, b)$, is either $\{(1, 1)\}$ or $\{(0, 0)\}$.

All these cases reflect in the regularity of the associated Lagrange problems. More precisely, by analyzing the associated determinants, we see that the Lagrange problem has unique solutions of type $P(x, y) = a_{0,0} + a_{1,0}x + a_{2,0}x^2$ unless $a \in \{0, 1\}$, has unique solutions of type $P(x, y) = a_{0,0} + a_{0,1}y + a_{0,2}y^2$ unless $b \in \{0, 1\}$, and has unique solutions of type $P(x, y) = a_{0,0} + a_{1,0}x + a_{0,1}y$ unless $a + b = 1$. Since the conditions

$$a \in \{0, 1\}, \quad b \in \{0, 1\}, \quad a + b = 1 \tag{2.2}$$

cannot be all satisfied (since $(a, b)$ was assumed distinct from $(1, 0)$ and $(0, 1)$), we see that $(Z, S)$ is regular for at least one lower set $S$. Also, if we require that this happens for a unique lower set $S$, we see that two of the three conditions (2.2) must be satisfied, and then we deduce that $(a, b)$ must be either $(1, 1)$ or $(0, 0)$. Note that these are precisely the cases for which $Z$ is cartesian, and this is in complete agreement with Theorem 1.1

## 3. Lagrange schemes and cartesian sets of nodes

In this section we study the relation between cartesian sets of nodes and the regularity of the Birkhoff-Lagrange schemes. We will show the following, whose particular case appearing in Lemma 3.1 below can be seen as an analogue of Theorem 12.3.1 of [3].

**Proposition 3.1.** *Given two lower sets $S$ and $S_0$, and a $S$-cartesian set of nodes $Z$, the following are equivalent*

(i) *$(Z, S_0)$ is solvable.*
(ii) *$S \subset S_0$.*

**Proposition 3.2.** *For any set of nodes $Z$, both schemes $(Z, S_x(Z))$ and $(Z, S_y(Z))$ are regular.*

These immediately imply the theorem stated in the introduction (see the end of the section). For the proof of Proposition 3.1, we will first need to establish the particular case of rectangular sets of nodes (for the notion of rectangular shapes, see Remark 2.1):

**Lemma 3.1.** *Given a lower set $S$ and a $(p, q)$-rectangular set of nodes $Z$, the Lagrange interpolation problem for $(Z, S)$ is solvable if and only if $R(p, q) \subset S$.*

The remaining part of this section is devoted to the proof of these results.

*Proof of Proposition 3.2.* This follows from the univariate case. To explain this, we choose $k$ and $n_i$ as in the definition of $S_y(Z)$ (see (2.1)). It follows that $Z$ can be written as

$$Z = \{(x_i, y_j^i) : 0 \le i \le k, 0 \le j \le n_i\} \tag{3.1}$$

where all the $x_i$'s are distinct, as well as all the $y^i_j$'s for each $i$. Denote by $l^x_i(x; x_0, \dots, x_p)$ the fundamental (univariate) interpolation polynomial at the node $x_i$, with respect to the Lagrange problem with the nodes $x_0, \dots, x_p$, i.e.

$$l^x_i(x; x_0, \dots, x_p) = \frac{(x - x_0)\dots\widehat{(x - x_i)}\dots(x - x_p)}{(x_i - x_0)\dots\widehat{(x_i - x_i)}\dots(x_i - x_p)},$$

where "$\hat{a}$" means that $a$ is omitted. Similarly we consider $l^y_j(y; y^i_0, \dots, y^i_{n_i})$. Then $l^{x,y}_{i,j} = l^x_i(x; x_0, \dots, x_p)$ $l^y_j(y; y^i_0, \dots, y^i_{n_i})$ will be the fundamental interpolation polynomials for the problem corresponding to the scheme $(Z, S_y(Z))$. The case $(Z, S_x(Z))$ is obtained by interchanging the role of $x$ and $y$. □

*Proof of Lemma 3.1.* Since $(Z, R(p, q))$ is regular (Proposition 3.2), we only have to prove that solvability implies $R(p, q) \subset S$. The solvability condition ensures the existence of a polynomial $P \in \mathcal{P}_S$ with the property that $P(x_i, y_j) = 0$ for all $(i, j) \in R(p, q)$, except for $P(x_p, y_q) = 1$. We consider the polynomials $\phi^x_0 = 1$,

$$\phi^x_i = (x - x_0)\dots(x - x_{i-1})/(x_i - x_0)\dots(x_i - x_{i-1}),$$

where we extend the sequence $x_0, \dots, x_p$ to an infinite sequence of distinct numbers (the only role of this extension is to simplify the presentation. Actually, all we need is to extend the family of linearly independent polynomials $\{\phi^x_i : 0 \le i \le p\}$ to a basis of the polynomial ring $\mathbb{R}[x]$, and using such an infinite sequence is one way of doing that). Similarly we define the polynomials $\phi^y_j$ in the variable $y$. Since $\{\phi^x_i \phi^y_j : i, j \ge 0\}$ is a basis for the space of bivariate polynomials, we can write $P = \sum_{i,j} a_{i,j}\phi^x_i\phi^y_j$, and, since $P \in \mathcal{P}_S$, we see that $(i, j) \in S$ whenever $a_{i,j} \ne 0$. Now, since $\phi^x_i(x_k) = 0$ for all $k < i$, $\phi^x_i(x_i) = 1$, and similarly for the $\phi^y_j$'s, one has the following implication:

$$a_{i,j} = 0 \;\; \forall \;\; (i, j) \in R(u, v) \setminus \{(u, v)\} \Longrightarrow P(x_u, y_v) = a_{u,v}.$$

Hence, by a simple induction, we deduce that $a_{i,j} = 0$ for $(i, j) \in R(p, q) \setminus \{p, q\}$, and $a_{p,q} = P(x_p, y_q) = 1$. Since $a_{p,q} \ne 0$, it follows that $(p, q) \in S$, hence $R(p, q) \subset S$. □

*Proof of Proposition 3.1.* By Proposition 3.2, $(Z, S)$ is regular, and this shows that $(ii) \Longrightarrow (i)$. Assume now that $(Z, S_0)$ is solvable, and, as before, write

$$S = \{(i, j) : 0 \leq i \leq k, 0 \leq j \leq n_i\},$$
$$Z = \{(x_i, y_j) : 0 \leq i \leq s, 0 \leq j \leq n_i\},$$

with $n_0 \geq \ldots \geq n_k$, and with the $x_i$'s, as well as the $y_j$'s, distinct. We now use that $(Z_0, S_0)$ is solvable for all $Z_0 \subset Z$. For any $s \leq k$, we choose $Z_0 = Z_0(s)$, where $Z_0(s) = \{x_i, y_j) : 0 \leq i \leq s, 0 \leq j \leq n_s\}$. Since $Z_0(s)$ is $(s, n_s)$-rectangular, it follows from Lemma 3.1 that $R(s, n_s) \subset S_0$. Since the sets $R(s, n_s)$ with $0 \leq s \leq k$ cover $S$ entirely, it follows that $S \subset S_0$. Hence $(i) \Longrightarrow (ii)$. □

*Proof of Theorem 1.1.* The first part is implied by Proposition 3.2. The same proposition shows that the uniqueness of $S$ implies that $S_y(Z) = S_x(Z)$ hence, by Lemma 2.1, $Z$ must be cartesian. Finally, assume that $Z$ is $S$-cartesian for some lower set $S$. Then, if $S_0$ is a lower set such that $(Z, S_0)$ is regular, on one hand we must have $|S_0| = |Z|(= |S|)$, and, on the other hand, Proposition 3.1 implies that $S_0 \subset S$; hence $S_0$ must coincide with $S$. The same proposition applied to $S_0 = S$ implies that $(Z, S)$ is indeed regular. □

**1.** Ferguson D., *The question of uniqueness for G. D. Birkhoff interpolation problems*, J. Approximation Theory **2** (1969), 1–28.
**2.** Crainic M. and Crainic N., *Birkhoff interpolation with rectangular sets of nodes*, Utrecht Univ. Preprint, 2003.
**3.** Lorentz R. A., *Multivariate Birkhoff Interpolation*, LNM **1516**, Springer-Verlag Berlin Heidelberg 1992.

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