A REMARK ON COMMON SOLUTIONS OF A PAIR OF MATRIX EQUATION

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ABSTRACT. A new appoach for determining the general common solution to some matrix equations is proposed. This approach simplifies or generalizes relative results in [3]–[7].

Let $\mathbf{C}^{m\times n}$ be the set of all $m\times n$ matrices over the complex number field $\mathbf{C}$, and $\mathbf{C}^m$ be the set $\mathbf{C}^{m\times 1}$. The conjugate transpose and the column space of $A\in\mathbf{C}^{m\times n}$ is, respectively, denoted by A^T and $\mathcal{R}(A)$. We denote a conditional inverse of $A\in\mathbf{C}^{m\times n}$, denoted by A^-, to be any matrix $B\in\mathbf{C}^{n\times m}$ satisfying $ABA=A$. The symbol $A\otimes B$ denotes the Kronecker matrix product, $(a_{ij}B)$. We use Vec$A$ to represent the vector in $\mathbf{C}^{mn}$ formed by the vertical concatenation of the respective columns of the matrix A. That is, if $A=\begin{bmatrix} a_1 & a_2 & \cdots & a_n\end{bmatrix}$, where $a_i\in\mathbf{C}^m,\ \ i=1,2,\cdots,n$, then vec$A=\begin{bmatrix} a_1^T & a_2^T & \cdots & a_n^T\end{bmatrix}^T$. Obviously, Vec can be regard as a map from $\mathbf{C}^{m\times n}$ to $\mathbf{C}^{mn}$, and further it is linear and nonsingular. Given the vector $w\in\mathbf{C}^{mn}$, we use $\text{Invec}_{m,n}(w)$ to denote the matrix $W\in\mathbf{C}^{m\times n}$ such that Vec$W=w$. We denote by I_n and O the $n\times n$ identity matrix and the zero matrix, respectively.

Tian [6] determined solvability condition and common solutions of the pair of linear matrix equations

$$A_1X+XB_1=M,\ AXB=C, \tag{1}$$

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where A_1, B_1, M, A, B and C are known complex matrices with appropriate sizes. Moreover, Tian [6] also applied his results to theory of generalized inverses of matrices. His works depends on ranks and generalized inverses of matrices. However, when the Kronecker matrix product and the vertical concatenation are used, a more simple approach can be easily obtained. To demonstrate the simple approach, the following two lemmas are needed.

Lemma 1. [1, p. 270] *Given matrices $M \in \mathbf{C}^{m\times p}$ and $b \in \mathbf{C}^{m}$. Then the matrix equation $Mx = b$ has a solution if and only if $b \in \mathcal{R}(A)$. When this condition is satisfied, the general solution is given by*

$$x = M^{-}b + (I_p - M^{-}M)y,$$

where $y \in \mathbf{C}^{p}$ is an arbitrary parameter vector, and M^{-} is an arbitrary but fixed conditional inverse of M.

Lemma 2. [1, Problem 5.11] *The relation* $\mathrm{Vec}(DEF) = \left(F^T \otimes D\right) \mathrm{Vec}E$ *holds for any $D \in \mathbf{C}^{m\times p}$, $E \in \mathbf{C}^{p\times q}$ and $F \in \mathbf{C}^{q\times n}$.*

Based on the above two lemmas, we now present necessary and sufficient conditions for the pair of matrix equations (1) to have a common solution, and simultaneously give a new and simple representation of the general common solution to these two equations if they have a common solution.

Theorem 1. *Given matrices $A_1 \in \mathbf{C}^{k\times k}$, $B_1 \in \mathbf{C}^{l\times l}$, $M \in \mathbf{C}^{k\times l}$, $A \in \mathbf{C}^{m\times k}$, $B \in \mathbf{C}^{l\times n}$ and $C \in \mathbf{C}^{m\times n}$. Further, assume*

$$G = \begin{bmatrix} I_l \otimes A_1 + B_1^T \otimes I_k \\ B^T \otimes A \end{bmatrix}, \quad H = \begin{bmatrix} \mathrm{Vec}M \\ \mathrm{Vec}C \end{bmatrix}.$$

Then the pair of matrix equations (1) has a common solution if and only if $H \in \mathcal{R}(G)$. When the condition is satisfied, a representation of the general common solution is

$$X = \mathrm{Invec}_{k,l}\left[G^{-}H + (I_{kl} - G^{-}G)z\right],$$

where $z \in \mathbf{C}^{kl}$ is an arbitrary vector, and G^{-} is an arbitrary but fixed conditional inverse of G.

Proof. It is clear from Lemma 2 that X is a common solution to the pair of matrix equations (1) if and only if Vec$X$ is a common solution to the pair of matrix equations

$$\left(I_l \otimes A_1 + B_1^T \otimes I_k\right) \mathrm{Vec}X = \mathrm{Vec}M, \ \left(B^T \otimes A\right) \mathrm{Vec}X = \mathrm{Vec}C,$$

i.e., Vec$X$ is solution of the matrix equation

$$\begin{bmatrix} I_l \otimes A_1 + B_1^T \otimes I_k \\ B^T \otimes A \end{bmatrix} \mathrm{Vec}X = \begin{bmatrix} \mathrm{Vec}M \\ \mathrm{Vec}C \end{bmatrix}.$$

This, together with Lemma 1, complete the proof. □

Obviously, the solvability condition and the representation of the general common solution in the above theorem are also more simple than those in [6, Theorem 2.1 and Theorem 2.3].

The following corollaries can be derived from the above theorem. i.e., the above theorem can be applied to theory of generalized inverses of matrices.

Corollary 1. *Let $A, M \in \mathbf{C}^{m\times m}$. Then there is A^- such that $M = AA^- \pm A^-A$ if and only if A and M satisfy*

$$\begin{bmatrix} \mathrm{Vec}M \\ \mathrm{Vec}A \end{bmatrix} \in \mathcal{R}\left(\begin{bmatrix} I_m \otimes A \pm A^T \otimes I_m \\ A^T \otimes A \end{bmatrix}\right).$$

Corollary 2. *Let $A, B \in \mathbf{C}^{m\times m}$. Then there is A^- such that $BAA^- = A^-AB$ if and only if*

$$\begin{bmatrix} O \\ \mathrm{Vec}A \end{bmatrix} \in \mathcal{R}\left(\begin{bmatrix} I_m \otimes BA - B^TA^T \otimes I_m \\ A^T \otimes A \end{bmatrix}\right).$$

It should be pointed out that, by a similar argument, the above Theorem 1 can be extended to the following theorem.

Theorem 2. *Given positive integers* s_v, $v = 1, 2, \cdots, t$, *and matrices* $A_{uv} \in \mathbf{C}^{m_v \times k}$, $B_{uv} \in \mathbf{C}^{l \times n_v}$, *and* $C_v \in \mathbf{C}^{k \times l}$, $v = 1, 2, \cdots, t$, $u = 1, 2, \cdots, s_v$. *Further, assume*

$$\hat{G} = \begin{bmatrix} \sum\limits_{u=1}^{s_1} \left(B_{u1}^T \otimes A_{u1}\right) \\ \vdots \\ \sum\limits_{u=1}^{s_t} \left(B_{ut}^T \otimes A_{ut}\right) \end{bmatrix}, \quad \hat{H} = \begin{bmatrix} \mathrm{Vec} C_1 \\ \vdots \\ \mathrm{Vec} C_t \end{bmatrix}.$$

Then the t *matrix equations*

$$\sum_{u=1}^{s_v} A_{uv} X B_{uv} = C_v, \; v = 1, 2, \cdots, t. \tag{2}$$

has a common solution if and only if $\hat{H} \in \mathcal{R}\left(\hat{G}\right)$. *When the condition is satisfied, a representation of the general common solution is*

$$X = \mathrm{Invec}_{k,l}\left[\hat{G}^- \hat{H} + (I_{kl} - \hat{G}^- \hat{G})z\right],$$

where $z \in \mathbf{C}^{kl}$ *is an arbitrary vector, and* $\hat{G}^-$ *is an arbitrary but fixed conditional inverse of* $\hat{G}$.

We end this paper by remarking the above two theorems as follows.

Remark 1. Theorems 1 and 2 simplify or generalize relative results in [3]–[7].

Remark 2. Note that

$$\|X\| = \|\mathrm{Vec} X\|, \quad \forall X \in \mathbf{C}^{m \times n},$$

where $\|\cdot\|$ denotes the Frobenius norm of matrix. By a similar argument to Theorems 1 and 2, we can easily give the least-squares solutions, the minimal least-squares solution and the minimum norm solution (for these definitions see [2] and [1]) of (1) and (2).

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