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SOME RESULTS ON INCREMENTS OF THE WIENER PROCESS

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Abstract. Let $\lambda_{(T,a_T,\alpha)} = \left\{2a_T\left[\log\frac{T}{a_T} + \alpha\log\log T + (1-\alpha)\log\log a_T\right]\right\}^{-\frac{1}{2}}$ where $0 \le \alpha \le 1$ and $\{W(t), t \ge 0\}$ be a standard Wiener process. This paper studies the almost sure limiting behaviour of $\sup_{0\le t\le T-a_T} \lambda_{(T,a_T,\alpha)}|W(t+a_T)-W(t)|$ as $T \longrightarrow \infty$ under varying conditions on a_T and $\frac{T}{a_T}$.

1. Introduction

Let $\{W(t), t \ge 0\}$ be a standard Wiener process. Suppose that a_T is a nondecreasing function of T such that $0 < a_T \le T$ and $\frac{T}{a_T}$ is nondecreasing. Csörgő and Révész [**2**], [**3**] established the following theorem.

Theorem 1.1. *Let a_T for $T \ge 0$ satisfy*

(1) a_T *is nondecreasing,*

(2) $0 < a_T \le T$,

(3) $\dfrac{a_T}{T}$ *is nonincreasing.*

Define $\beta_T = (2a_T(\log\frac{T}{a_T} + \log\log T))^{-\frac{1}{2}}$. *Then*

$$\limsup_{T\longrightarrow\infty} \sup_{0\le t\le T-a_T} \beta_T|W(T+a_T) - W(t)| = 1 \quad a.s. \tag{4}$$

$$\limsup_{T\longrightarrow\infty} \sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} \beta_T|W(t+s) - W(t)| = 1 \quad a.s. \tag{5}$$

If, in addition,

$$\lim_{T\longrightarrow\infty} \frac{\log\frac{T}{a_T}}{\log\log T} = \infty, \tag{6}$$

then "limsup" may be replaced by "lim" in both equations (4) *and* (5).

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Here and in the sequel we shall define for each $u \geq 0$ the functions

$$Lu = \log u = \log(\max(u, 1)),$$

and

$$L_2 u = \log\log(\max(u, e)).$$

ε stands for a positive number given arbitrarily, and C will be understood as a positive constant independent of n, which can take different values on each appearance.
To simplify the notation, we will set

$$A(T, a_T, \alpha) = \sup_{0 \leq t \leq T - a_T} \lambda_{(T,a_T,\alpha)} |W(t + a_T) - W(t)|,$$
$$B(T, a_T, \alpha) = \sup_{0 \leq t \leq T - a_T} \sup_{0 \leq s \leq a_T} \lambda_{(T,a_T,\alpha)} |W(t + s) - W(t)|,$$

where

$$\lambda_{(T,a_T,\alpha)} = \left\{ 2a_T \left[L\frac{T}{a_T} + \alpha L_2 T + (1 - \alpha) L_2 a_T \right] \right\}^{-\frac{1}{2}} \quad \text{and} \quad 0 \leq \alpha \leq 1.$$

2. Main result

In this section we shall investigate the analogous problem when β_T is replaced by $\lambda_{(T,a_T,\alpha)}$. Our goal is to prove the following result.

Theorem 2.1. *Under assumptions* (2) *and* (3) *of Theorem 1.1, we have*

$$\limsup_{T \longrightarrow \infty} A(T, a_T, \alpha) = 1 \quad a.s., \tag{7}$$

$$\limsup_{T \longrightarrow \infty} B(T, a_T, \alpha) = 1 \quad a.s. \tag{8}$$

If we also have

$$\lim_{T \longrightarrow \infty} \frac{L\frac{T}{a_T}}{L((LT)^\alpha (La_T)^{1-\alpha})} = \infty, \tag{$*$}$$

then

$$\lim_{T \longrightarrow \infty} A(T, a_T, \alpha) = 1 \quad a.s., \tag{9}$$

$$\lim_{T \longrightarrow \infty} B(T, a_T, \alpha) = 1 \quad a.s. \tag{10}$$

Remark 2.1. Let us mention some particular cases .

1. For $a_T = T$ we obtain the law of iterated logarithm.
2. If $\alpha = 1$, we obtain Csörgő-Révész theorem (see Theorem 1.1).
3. If $\alpha = 0$, under assumptions (2) and (3) of Theorem 1.1, then we also have

$$\limsup_{T \longrightarrow \infty} A(T, a_T, 0) = 1, \quad a.s., \tag{11}$$

$$\limsup_{T \longrightarrow \infty} B(T, a_T, 0) = 1, \quad a.s. \tag{12}$$

If we also have $\lim_{T\longrightarrow\infty} \frac{\log \frac{T}{a_T}}{\log\log a_T} = \infty$, then " lim sup " in Equation (11) and (12) may be replaced by "lim".

Proof of Theorem 2.1. Our proof will be given in three steps expressed by the following three lemmas.

Lemma 2.1. *Let a_T be a nondecreasing function of T satisfying conditions* (2) *and* (3) *of Theorem 1.1. Then for any $\varepsilon > 0$ we have*

(13) $$\limsup_{T\longrightarrow\infty} A(T, a_T, \alpha) \geq 1 - \varepsilon.$$

Lemma 2.2. *Let a_T be a nondecreasing function of T satisfying conditions* (2) *and* (3) *of Theorem 1.1. Then for any $\varepsilon > 0$ we have*

(14) $$\limsup_{T\longrightarrow\infty} B(T, a_T, \alpha) \leq 1 + \varepsilon.$$

Lemma 2.3. *Let a_T be a nondecreasing function of T satisfying conditions* (2), (3) *of Theorem 1.1 and* $(*)$ *of Theorem 2.1. Then for any $\varepsilon > 0$ we have*

(15) $$\liminf_{T\longrightarrow\infty} A(T, a_T, \alpha) \geq 1 - \varepsilon.$$

Proof of Lemma 2.1. Let

$$C(T) = \lambda_{(T,a_T,\alpha)}|W(T) - W(T - a_T)|.$$

Using the well known probability inequality

(16) $$\frac{1}{\sqrt{2\pi}}\left(\frac{1}{x} - \frac{1}{x^3}\right)\exp\left(-\frac{x^2}{2}\right) \leq P(W(1) \geq x) \leq \frac{1}{\sqrt{2\pi}x}\exp\left(-\frac{x^2}{2}\right),$$

for $x \geq 0$, (see, e.g., [**4**, p.175]), it follows that

$$\begin{aligned} P(C(T) \geq 1 - \varepsilon) &\geq \left(\frac{a_T}{T(LT)^\alpha (La_T)^{1-\alpha}}\right)^{1-\varepsilon} \geq \left(\left(\frac{a_T}{TLa_T}\right)\left(\frac{La_T}{LT}\right)^\alpha\right)^{1-\varepsilon} \\ &\geq \left(\left(\frac{a_T}{TLa_T}\right)\left(\frac{La_T}{LT}\right)\right)^{1-\varepsilon} \geq \left(\frac{a_T}{TLT}\right)^{1-\varepsilon} \end{aligned}$$

if T is big enough. We define the sequence $\{T_k\}$ as follows: Let $T_1 = 1$ and define T_{k+1} by

$$T_{k+1} - a_{T_{k+1}} = T_k \qquad \text{if} \quad \rho < 1$$

and

$$T_{k+1} = \theta^{k+1} \qquad \text{if} \quad \rho = 1,$$

where $\theta > 1$ and $\lim_{T\to\infty} \frac{a_T}{T} = \rho$. The conditions (2) and (3) imply that a_T is a continuous function of T and that $\rho = 1$ if and only if $a_T = T$. Moreover $T - a_T$

is a strictly increasing function of T if $\rho < 1$. In the case $\rho = 1$ we refer to the law of the iterated logarithm. So we assume that $\rho < 1$, (13) follows from

$$\sum_{k=2}^{\infty} \frac{a_T}{T_k (LT_k)^{1-\varepsilon}} = \infty, \tag{17}$$

as was shown in Csáki, Csörgő, Földes and Révész [**1**, Lemma 3.2], and the r.v. $C(T_k)$ $(k = 1, 2, \ldots)$ are independent. □

Proof of Lemma 2.2. Let $a_{T_k} = \theta^k$, $\theta > 1$ and $\varepsilon > 0$. Using the inequality

$$P\{\sup_{0 \le s', s \le T, 0 \le s - s' \le h} h^{-\frac{1}{2}} |W(s) - W(s')| \ge v\} \le \frac{CT}{h} \exp\left\{\frac{-v^2}{2+\varepsilon}\right\}, \tag{18}$$

where C is a positive constant depending only on ε (see in [**2**, Lemma 1*]), we have

$$\begin{aligned}
&\sum_{k=1}^{\infty} P(B(T_k, a_{T_k}, \alpha) \ge (1+\varepsilon)) \\
&\le C \sum_{k=1}^{\infty} \frac{T_k}{a_{T_k}} \exp\{-2\frac{(1+\varepsilon)^2}{2+\varepsilon} (\log \frac{T_k}{a_{T_k}} (LT_k)^{\alpha} (La_{T_k})^{(1-\alpha)})\} \\
&\le C \sum_{k=1}^{\infty} \left(\frac{a_{T_k}}{T_k}\right)^{\varepsilon} \left(\frac{1}{(LT_k)^{\alpha} (La_{T_k})^{(1-\alpha)}}\right)^{1+\varepsilon} \\
&\le C \sum_{k=1}^{\infty} \left(\frac{a_{T_k}}{T_k}\right)^{\varepsilon} \left(\left(\frac{LT_k}{La_{T_k}}\right)^{1-\alpha} \frac{1}{LT_k}\right)^{1+\varepsilon} \\
&\le C \sum_{k=1}^{\infty} \left(\frac{a_{T_k}}{T_k}\right)^{\varepsilon} \left(\left(\frac{LT_k}{La_{T_k}}\right) \frac{1}{LT_k}\right)^{1+\varepsilon} \\
&= C \sum_{k=1}^{\infty} \left(\frac{a_{T_k}}{T_k}\right)^{\varepsilon} \frac{1}{(La_{T_k})^{1+\varepsilon}} < \infty
\end{aligned}$$

and an application of Borel-Cantelli Lemma gives

$$\limsup_{k \longrightarrow \infty} B(T_k, a_{T_k}, \alpha) \le 1 \quad a.s. \tag{19}$$

Notice that

$$1 \le \frac{\lambda_{(T_k, a_{T_k}, \alpha)}}{\lambda_{(T_{k+1}, a_{T_{k+1}}, \alpha)}} \le \theta \tag{20}$$

if k is big enough. When $T_k \le T \le T_{k+1}$, we have

$$\begin{aligned}
\limsup_{T \longrightarrow \infty} B(T, a_T, \alpha) &\le \limsup_{k \longrightarrow \infty} B(T_{k+1}, a_{T_{k+1}}, \alpha) \frac{\lambda_{(T_k, a_{T_k}, \alpha)}}{\lambda_{(T_{k+1}, a_{T_{k+1}}, \alpha)}} \\
&\le \limsup_{k \longrightarrow \infty} B(T_{k+1}, a_{T_{k+1}}, \alpha) \limsup_{k \longrightarrow \infty} \frac{\lambda_{(T_k, a_{T_k}, \alpha)}}{\lambda_{(T_{k+1}, a_{T_{k+1}}, \alpha)}}.
\end{aligned}$$

Now choosing θ near enough to one, (14) follows from (19) and (20). □

Proof of Lemma 2.3. We will set $D_T = \{A(T, a_T, \alpha) \le 1-\varepsilon\}$. Using inequality (18), for sufficiently large T, we have

$$\begin{aligned} P(D_T) &\le P(\max_{0\le i\le [\frac{T}{a_T}]-1} \lambda_{(T,a_T,\alpha)}|W(i+1)a_T - W(ia_T)| \le 1-\varepsilon) \\ &\le \left(1-\left(\frac{a_T}{T(LT)^{\alpha}(La_T)^{1-\alpha}}\right)^{1-\varepsilon}\right)^{[\frac{T}{a_T}]} \\ &\le 2\exp\left\{-\left(\frac{T}{a_T}\right)^{\varepsilon}\frac{1}{(LT)^{\alpha(1-\varepsilon)}(La_T)^{(1-\alpha)(1-\varepsilon)}}\right\}. \end{aligned}$$

Now, under condition (∗) and for all sufficiently large T,

$$\frac{T}{a_T} \ge \{(LT)^{\alpha}(La_T)^{1-\alpha}\}^{\frac{3-\varepsilon}{\varepsilon}}.$$

Define $T_k = e^{a_{T_k}} = k$.
Therefore

$$\begin{aligned} \sum_{k=2}^{\infty} P(D_{T_k}) &\le 2\sum_{k=2}^{\infty}\exp\{-(LT_k)^{2\alpha}(La_{T_k})^{2(1-\alpha)}\} \\ &= 2\sum_{k=2}^{\infty}\exp\left\{-\left(\frac{LT_k}{La_{T_k}}\right)^{2\alpha}(La_{T_k})^2\right\} \\ &\le 2\sum_{k=2}^{\infty}\exp\{-(La_{T_k})^2\} \\ &\le 2\sum_{k=2}^{\infty}a_{T_k}^{-2} \\ &= 2\sum_{k=2}^{\infty}(Lk)^{-2} \\ &< \infty \end{aligned}$$

which implies by Borel-Cantelli lemma that

$$\liminf_{k\longrightarrow\infty} A(T_k, a_{T_k}, \alpha) \ge 1-\varepsilon, a.s. \tag{21}$$

When $T_k \le T \le T_{k+1}$, we have $a_T - a_{T_k} \ge 0$ and by (3), it is easy to see that $a_T - a_{T_k} \le \frac{a_{T_k}}{T_k} \le \delta a_{T_k}$ for any $\delta > 0$. Thus

$$\begin{aligned} \liminf_{T\longrightarrow\infty} A(T, a_T, \alpha) \ge& \liminf_{k\longrightarrow\infty} \sup_{0\le t\le T_k - a_{T_k}} \lambda_{(T_{k+1}, a_{T_{k+1}}, \alpha)}|W(t+a_{T_k}) - W(t)| \\ &- \limsup_{T\longrightarrow\infty} \sup_{0\le t\le T-\delta a_T} \sup_{0\le s\le \delta a_T} \lambda_{(T,a_T,\alpha)}|W(t+s) - W(t)| \\ =& \liminf_{k\longrightarrow\infty} \sup_{0\le t\le T_k - a_{T_k}} \lambda_{(T_k, a_{T_k}, \alpha)}|W(t+a_{T_k}) - W(t)|\frac{\lambda_{(T_{k+1}, a_{T_{k+1}}, \alpha)}}{\lambda_{(T_k, a_{T_k}, \alpha)}} \\ &- \limsup_{T\longrightarrow\infty} \sup_{0\le t\le T-\delta a_T} \sup_{0\le s\le \delta a_T} \lambda_{(T,\delta a_T,\alpha)}|W(t+s) - W(t)|\frac{\lambda_{(T,a_T,\alpha)}}{\lambda_{(T,\delta a_T,\alpha)}}. \end{aligned}$$

By Lemma 2.2 we have

$$\limsup_{T\longrightarrow\infty}\sup_{0\le t\le T-\delta a_T}\sup_{0\le s\le \delta a_T}\lambda_{(T,\delta a_T,\alpha)}|W(t+s)-W(t)|\le 1, a.s. \tag{22}$$

We notice that

$$\limsup_{T\longrightarrow\infty}\frac{\lambda_{(T,a_T,\alpha)}}{\lambda_{(T,\delta a_T,\alpha)}}=\delta. \tag{23}$$

The proof of Lemma 2.3 will be completed by combining (21), (22) and (23). □

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