

## REGULAR ADDITIVELY INVERSE SEMIRINGS

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ABSTRACT. In this paper we show that in a regular additively inverse semiring  $(S, +, \cdot)$  with 1 satisfying the conditions

$$(A) \quad a(a + a') = a + a';$$

$$(B) \quad a(b + b') = (b + b')a$$

and (C)  $a + a(b + b') = a$ , for all  $a, b \in S$ , the sum of two principal left ideals is again a principal left ideal. Also, we decompose  $S$  as a direct sum of two mutually inverse ideals.

### 1. INTRODUCTION

A semiring is a nonempty set  $S$  on which operations of addition,  $+$ , and multiplication,  $\cdot$ , have been defined such that the following conditions are satisfied:

- (1)  $(S, +)$  is a semigroup.
- (2)  $(S, \cdot)$  is a semigroup.
- (3) Multiplication distributes over addition from either side.

A semiring  $(S, +, \cdot)$  is called an additive inverse semiring if  $(S, +)$  is an inverse semigroup, that is, for each  $a \in S$  there exists a unique element  $a' \in S$  such that  $a + a' + a = a$  and  $a' + a + a' = a'$ . In 1974, Karvellas [3] studied additive inverse semiring and he proved the following:

(Karvellas (1974), Theorem 3(ii) and Theorem 7) Take any additive inverse semiring  $(S, +, \cdot)$ .

(i) For all  $x, y \in S$ ,  $(x \cdot y)' = x' \cdot y = x \cdot y'$  and  $x' \cdot y' = x \cdot y$

(ii) If  $a \in aS \cap Sa$  for all  $a \in S$  then  $S$  is additively commutative.

We say that an additive inverse semiring  $S$  satisfies conditions (A), (B) and (C) if for all  $a, b \in S$ ,

$$(A) \quad a(a + a') = a + a';$$

$$(B) \quad a(b + b') = (b + b')a;$$

$$(C) \quad a + a(b + b') = a.$$

Clearly, rings, distributive lattices and direct products of a distributive lattice and ring are natural examples of these types of semirings. Semirings satisfying conditions (A), (B) and (C) were first introduced and studied by Bandelt & Petrich [1].

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We consider the set  $S = \{0, a, b\}$ . On  $S$  we define addition and multiplication by the following Cayley tables:

$$\begin{array}{c|ccc} + & 0 & a & b \\ \hline 0 & 0 & a & b \\ a & a & 0 & b \\ b & b & b & b \end{array} \quad \begin{array}{c|ccc} \cdot & 0 & a & b \\ \hline 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ b & 0 & 0 & b \end{array}$$

It is easy to see that  $(S, +, \cdot)$  is an additive inverse semiring satisfying conditions (A), (B) and (C).

In the remaining part of this paper we assume that  $S$  denotes an additive inverse semiring with 1 satisfying conditions (A), (B) and (C). Also we assume that  $E^+(S) = \{a \in S : a + a = a\}$  and  $E^\bullet(S) = \{e \in S : e \cdot e = e\}$ . Note that  $E^+(S)$  is an ideal of  $S$ . For notations and terminologies not given in this paper, the reader is referred to the monograph of Golan [2] and Neumann [4].

## 2. MUTUALLY INVERSE IDEALS IN $S$

In this section we define the notion of mutually inverse ideals in  $S$ . Then we establish the actual form of two ideals such that these two ideals become mutually inverses.

**Lemma 2.1.** *If  $e \in E^\bullet(S)$  then  $1 + e' \in E^\bullet(S)$ .*

*Proof.* Now,  $(1+e')^2 = (1+e') + e'(1+e') = 1+e'+e'+e'e' = 1+e'+e'+e = 1+e'$ . Hence  $(1 + e') \in E^\bullet(S)$ .  $\square$

**Definition 2.2.** For every right ideal  $A$  of  $S$  we define

$$A^l = \{y \in S : \text{for every } z \in A, yz \in E^+(S)\}$$

and for every left ideal  $B$  of  $S$  we define

$$B^r = \{z \in S : \text{for every } y \in B, yz \in E^+(S)\}.$$

**Notation 2.3.** The sets of all left ideals and right ideals of  $S$  are denoted by  $L_S$  and  $R_S$  respectively.

From the Definition 2.2, we have the following result.

**Corollary 2.4.**  *$A^l$  is a left ideal and  $B^r$  is a right ideal. The transformation  $A \rightarrow A^l$  maps  $R_S$  on a part of  $L_S$  and the transformation  $B \rightarrow B^r$  maps  $L_S$  on a part of  $R_S$ .*

**Lemma 2.5.** *Let  $A, B$  be two left ideals. Then*

- (i)  $A \subseteq B$  implies  $B^r \subseteq A^r$ ,
- (ii)  $A \subseteq A^{rl} (\equiv (A^r)^l)$ ,
- (iii)  $A^r = A^{r^l r}$ .

(The left-right symmetric results will be denoted by (i)', (ii)', (iii)')

*Proof.* (i) If  $y \in B^r$  then for every  $z \in B$  we have  $zy \in E^+(S)$ . Then in particular for every  $z \in A$  we have  $zy \in E^+(S)$  and hence  $y \in A^r$ . Thus  $B^r \subseteq A^r$ .

(ii) Let  $u \in A$  and consider  $y \in A^r$ . Now  $z \in A$  implies  $zy \in E^+(S)$ . Hence in particular  $uy \in E^+(S)$  and  $u \in A^{rl}$ . Thus  $A \subseteq A^{rl}$ .

(iii) Since  $A \subseteq A^{rl}$ ,  $A^{rlr} \subseteq A^r$ , by (i). But by (ii)' with  $A$  replaced by  $A^r$ ,  $A^r \subseteq A^{rlr}$ . Hence  $A^r = A^{rlr}$ .  $\square$

**Definition 2.6.** An ideal  $I$  of a semiring  $S$  is called full if  $E^+(S) \subseteq I$ . The principal left ideal of  $S$  of the form  $E^+(S) + (a)_l$  is called a full principal left ideal of  $S$ .

**Lemma 2.7.**  $E^+(S)$  is a full principal left ideal.

*Proof.* Let  $e \in E^+(S)$ . Since  $S$  is an additive inverse semiring so we have  $E^+(S) + (e)_l \subseteq E^+(S)$ . Let  $f \in E^+(S)$ . Now by condition (C),  $f = f + f(e + e') = f + fe \in E^+(S) + (e)_l$ . Hence  $E^+(S) = E^+(S) + (e)_l$  and consequently  $E^+(S)$  is a full principal left ideal.  $\square$

**Remark 2.8.** We note that for any  $g \in E^+(S)$ , we have  $g = g + ga = g(1 + a) \in (1 + a)_l$ . Thus any principal left (right) ideal of the form  $(1 + a)_l$  (resp.  $(1 + a)_r$ ) is a full ideal of  $S$ . Hence in particular for any  $e \in E^\bullet(S)$ ,  $(1 + e)_l$  is also a full left ideal of  $S$ . In this connection we have the following result.

**Theorem 2.9.** Let  $e \in E^\bullet(S)$ . Then the principal left ideal  $(e)_l$  is full if and only if  $(e)_l = (1 + f')_l$  for some  $f \in E^\bullet(S)$ .

*Proof.* First suppose that  $(e)_l$  is a full ideal. Since  $e \in E^\bullet(S)$  we have  $(1 + e') \in E^\bullet(S)$ . Let  $f = 1 + e'$ . Now,  $1 + f' = 1 + 1' + e \in (e)_l$ . This leads to  $(1 + f')_l \subseteq (e)_l$ . Also,  $e = e + e' + e = e + e' + e^2 = e(1 + 1' + e) = e(1 + f') \in (1 + f')_l$ . Thus  $(e)_l \subseteq (1 + f')_l$ . Consequently,  $(e)_l = (1 + f')_l$ .  $\square$

Converse part follows from Remark 2.8.

**Lemma 2.10.** Let  $a, b \in S$  be such that  $a + b' \in E^+(S)$  and  $a + a' = b + b'$ . Then  $a = b$ .

*Proof.* Since  $a + b' \in E^+(S)$  so we have

$$a + b' = (a + b') + (a + b')' = a + b' + b + a' = a + a' + b + b' = b + b'.$$

This leads to,

$$a + b' + b = b + b' + b, \quad \text{i.e.,} \quad a + a' + a = b.$$

Hence  $a = b$ .  $\square$

We now give the following definition.

**Definition 2.11.** Two left ideals  $A$  and  $B$  of a semiring  $S$  are said to be mutually inverses if  $A + B = S$  and  $A \cap B = E^+(S)$ . A left ideal  $B$  of  $S$  is said to be an inverse of a left ideal  $A$  of  $S$  if  $A$  and  $B$  are mutually inverses.

**Lemma 2.12.** In  $S$  the principal left ideals  $(1 + e')_l$  and  $(1 + 1' + e)_l$  where  $e \in E^\bullet(S)$  are mutually inverses.

*Proof.* First,  $(1+e')_l + (1+1'+e)_l$  contains  $1+e'+1+1'+e = 1+e+e'+1+1' = 1$ , whence  $(1+e')_l + (1+1'+e)_l = S$ . Now, if  $x \in (1+e')_l \cap (1+1'+e)_l$  then  $x = x(1+e') = x(1+1'+e)$ . Now,  $x = x(1+1'+e) = x + x(1'+e) = x + x'$  (since  $x = x(1+e')$ )  $\in E^+(S)$ . Thus,  $(1+e')_l \cap (1+1'+e)_l = E^+(S)$  and hence  $(1+e')_l$  is inverse to  $(1+1'+e)_l$ .  $\square$

We now prove the following theorem.

**Theorem 2.13.** *Two left ideals  $A$  and  $B$  of  $S$  are mutually inverses if and only if there exists  $e \in E^\bullet(S)$  such that  $A = (1+1'+e)_l$  and  $B = (1+e')_l$ .*

*Proof.* The reverse implication follows from Lemma 2.12. Let  $A$  and  $B$  be two mutually inverse left ideals. Then there exist elements  $x, y$  with  $x + y = 1$ ,  $x \in A, y \in B$ . Now,  $x + y = 1$  implies  $x = x^2 + xy$ . This leads to  $(x^2)' + x = (x^2)' + x^2 + xy \in A \cap B = E^+(S)$ . Also,  $x^2 + (x^2)' = x + x'$  (by condition (A)). Hence by Lemma 2.10, we have  $x^2 = x$  and hence  $x \in E^\bullet(S)$ . Now,  $1 + 1' + x \in A$ . This gives  $(1 + 1' + x)_l \subseteq A$ . Let  $z \in A$ . Now,  $x + y = 1$  implies that  $z = zx + zy$ . This leads to,  $zx' + z = zx' + zx + zy \in A \cap B = E^+(S)$ . Hence  $(zx' + z)' = (zx' + z) + (zx' + z)' = zx' + zx + z + z' = z + z'$  (by condition (C)) i.e.,  $z + z' = z' + zx$ . This gives

$$z = z + z' + z = z + z' + zx = z(1 + 1' + x) \in (1 + 1' + x)_l.$$

Thus  $A \subseteq (1 + 1' + x)_l$ . Consequently,  $A = (1 + 1' + x)_l$ . Similarly, we can show that  $B = (1 + e')_l$ . Thus  $e = x$  is effective in this theorem.  $\square$

### 3. PRINCIPAL IDEALS IN $S$

In this section we study the principal ideals in  $S$ . We generalize some results of regular ring to regular semiring. Finally, we prove that the set of all full principal left ideals of  $S$  is a complemented modular lattice. This is the main theorem of this section.

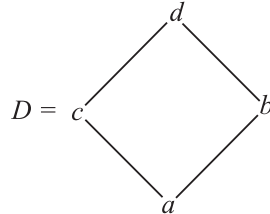
**Definition 3.1.** A semiring  $S$  is called a regular semiring if for each  $a \in S$  there exists an element  $x \in S$  such that  $a = axa$ .

A regular semiring  $S$  contains element  $e$  such that  $e \cdot e = e$ .

Note that every regular ring and every distributive lattice is regular semiring. So the direct product of a regular ring and a distributive lattice is also regular semiring.

We now consider the following example.

**Example 3.2.** Let  $D$  denote the distributive lattice  $D$  given by



Let  $R = \mathbb{R} \times \mathbb{R}$  and  $I = \mathbb{R} \times \{0\}$ , where  $\mathbb{R}$  is the ring of all real numbers. Then  $S = (I \times \{a, c\}) \cup (R \times \{b, d\})$  is a regular semiring which is not the whole direct product of a regular ring and a distributive lattice.

**Theorem 3.3.** *The following statements are equivalent in  $S$ :*

- (1) *Every principal left ideal of the form  $E^+(S) + (a)_l$  has an inverse.*
- (2) *For every  $a \in S$  there exists  $e \in E^\bullet(S)$  such that  $E^+(S) + (a)_l = (1 + e')_l$ .*
- (3)  *$S$  is regular.*
- (4) *For every  $a \in S$  there exists  $e \in E^\bullet(S)$  such that  $E^+(S) + (a)_r = (1 + e')_r$ .*
- (5) *Every principal right ideal of the form  $E^+(S) + (a)_r$  has an inverse.*

*Proof.* (1)  $\implies$  (2): This follows from Theorem 2.13.

(2)  $\implies$  (1): This follows from Theorem 2.13.

(2)  $\implies$  (3): Now,  $a \in E^+(S) + (a)_l = (1 + e')_l$ . This implies that  $a = a(1 + e')$ . Again  $(1 + e') \in (1 + e')_l = E^+(S) + (a)_l$ . This leads to,  $1 + e' = g + za$  for some  $g \in E^+(S)$  and  $z \in S$ . Then  $a = ag + aza$ . This implies that

$$\begin{aligned} a &= a + a' + a = a + a' + ag + aza = a + a' + aza \quad (\text{by condition (C)}) \\ &= a(a + a' + z)a = axa \quad \text{where } x = a + a' + z \in S. \end{aligned}$$

Thus, for each  $a \in S$  there exists an element  $x \in S$  such that  $a = axa$ . Hence  $S$  is regular.

(3)  $\implies$  (2): Let  $a \in S$ . Then  $a = axa$  for some  $x \in S$ . Let  $c = xa$ . Then  $c \in E^\bullet(S)$ . Let  $e = 1 + c'$ . Now

$$a = a + a' + a = a + a' + axa = a(1 + 1' + c) = a(1 + e') \in (1 + e')_l.$$

Thus,  $E^+(S) + (a)_l \subseteq (1 + e')_l$ . Again, let  $y \in (1 + e')_l$ . Then  $y = b(1 + e')$  for some  $b \in S$ . Then

$$y = b(1 + e') = b(1 + 1' + xa) = b + b' + bxa \in E^+(S) + (a)_l.$$

Thus,

$$(1 + e')_l \subseteq E^+(S) + (a)_l \quad \text{and hence} \quad E^+(S) + (a)_l = (1 + e')_l.$$

The equivalence of (3), (4), (5) is right-left symmetric to that of (1), (2), (3). Hence the proof is completed.  $\square$

**Lemma 3.4.** *A semiring  $S$  is regular if and only if for any  $a \in S$  there exists an element  $e \in E^\bullet(S)$  such that  $Sa = Se$ .*

*Proof.* The proof is similar to ring theory and we omit the proof.

In the remaining part of the section we assume that  $S$  is regular and an additive inverse semiring with 1 satisfying conditions (A), (B) and (C).  $\square$

- Lemma 3.5.**
- (i) *If  $A = (1 + e')_l$  ( $e \in E^\bullet(S)$ ) is a full principal left ideal then  $A = C^l$  where  $C = (1 + 1' + e)_r$ .*
  - (ii) *If  $A$  is a full principal left ideal then  $A = A^{rl}$ .*
  - (iii) *If  $A$  is a full principal left ideal,  $A^r$  is a full principal right ideal.*

*Proof.*

$$\begin{aligned}
\text{(i)} \quad A &= (1 + e')_l = \{x : x = x(1 + e')\} \\
&= \{x : x + x(1 + e')' = x(1 + e') + x(1 + e')'\} \\
&\subseteq \{x : x(1 + 1' + e) \in E^+(S)\} \\
&= \{x : \text{for all } u \in S, x(1 + 1' + e)u \in E^+(S)\} \\
&= \{x : \text{for all } y \in (1 + 1' + e)_r, xy \in E^+(S)\} \\
&= C^l \quad \text{where } C = (1 + 1' + e)_r.
\end{aligned}$$

Again, let  $x \in C^l$ . Then for all  $y \in C = (1 + 1' + e)_r$ , we have  $xy \in E^+(S)$ , i.e., for all  $u \in S$ ,  $x(1 + 1' + e)u \in E^+(S)$ . This implies  $x(1 + 1' + e) \in E^+(S)$ . Hence,

$$x(1 + 1' + e) = x(1 + 1' + e) + x(1 + 1' + e)' = x + x'.$$

This leads to

$$x' = x' + x + x' = x' + xe = x'(1 + e'),$$

i.e.,

$$x = x(1 + e') \in (1 + e')_l = A.$$

Thus,  $C^l \subseteq A$  and hence  $A = C^l$ .

(ii) Since  $S$  is regular and  $A$  is a full principal left ideal so  $A = (1 + e')_l = (f)_l$  where  $f = 1 + e'$ . Then  $A = (f)_l = C^l$  where  $C = (1 + f')_r$  [by (i)]. Then  $A^{rl} = C^{lr} = C^l = A$ .

(iii)  $A = C^l$  where  $C$  is a full principal right ideal, whence  $A^r = C^{lr} = C$  [by (ii)']. Hence  $A^r$  is a full principal right ideal.  $\square$

**Theorem 3.6.** *In  $S$ , the sum of two principal left ideals of  $S$  is again a principal left ideal.*

*Proof.* Let  $Sa$  and  $Sb$  be two principal ideals in a regular semiring  $S$ . Then there exists an idempotent  $e \in S$  such that  $Sa = Se$ . Now,

$$\begin{aligned}
Sa + Sb &= Se + Sb \\
&= \{re + tb : r, t \in S\} \\
&= \{re + tb + tbe + tbe' : r, t \in S\} \quad (\text{by condition (C)}) \\
&= \{(r + tb)e + tb(1 + e') : r, t \in S\} \\
&\subseteq Se + Sb(1 + e').
\end{aligned}$$

Let  $re + tb(1 + e') \in Se + Sb(1 + e')$ . Then

$$\begin{aligned}
re + tb(1 + e') &= re + tb + tbe' \\
&= re + tb + tb(e + e') + tbe' \\
&= (r + tb' + tb)e + tb(1 + e') \\
&= (r_1 + tb)e + tb(1 + e') \quad \text{where } r_1 = r + tb' \in S.
\end{aligned}$$

Hence,  $Sa + Sb = Se + Sb(1 + e')$ . Let  $c = b(1 + e')$ . Then  $ce \in E^+(S)$ . Now by Lemma 3.4.,  $Sc = Sf$  for some idempotent  $f \in S$ . Also,  $f \in Sf = Sc$  This implies

$f = yc = yb(1 + e')$ . This leads to,  $fe = yb(1 + e')e = yb(e + e') \in E^+(S)$ . Hence,  $Sa + Sb = Se + Sf$ , where  $e^2 = e, f^2 = f$  and  $fe \in E^+(S)$ . Let  $g = (1 + e')f$ . Then  $eg = e(1 + e')f = (e + e')f$  and  $ge = (1 + e')fe$ . Thus,  $eg, ge \in E^+(S)$ . Also,

$$g^2 = g(1 + e')f = gf + ge'f = gf = g.$$

Now,

$$\begin{aligned} Sg &= S(1 + e')f \subseteq Sf = Sff \\ &= Scf = Sb(1 + e')f \\ &= Sbg \subseteq Sg. \end{aligned}$$

Hence,

$$Sf = Sg.$$

Then  $Sa + Sb = Se + Sg$ , where  $e^2 = e, g^2 = g$  and  $eg, ge \in E^+(S)$ .

We show that  $Se + Sg = S(e + g)$ . Clearly,  $S(e + g) \subseteq Se + Sg$ .

Now,

$$e = e^2 + eg = e(e + g) \in S(e + g).$$

Then  $Se \subseteq S(e + g)$ . Similarly,  $Sf \subseteq S(e + g)$ . Hence,  $Se + Sg \subseteq S(e + g)$ . Consequently,

$$Sa + Sb = S(e + g).$$

Thus, the proof is completed.  $\square$

**Corollary 3.7.** *The sum of two full principal left ideals of  $S$  is again a full principal left ideal.*

**Lemma 3.8.** *If  $C, D$  are left ideals of  $S$  then  $(C + D)^r = C^r \cap D^r$ .*

*Proof.*

$$\begin{aligned} C^r \cap D^r &= \{y : \text{for all } z \in C, zy \in E^+(S) \text{ and for all } z \in D, zy \in E^+(S)\} \\ &= \{y : \text{for all } t \in C + D, ty \in E^+(S)\} \\ &\subseteq (C + D)^r. \end{aligned}$$

Let  $y \in (C + D)^r$ . Then for all  $t \in C + D, ty \in E^+(S)$ . Let  $c \in C$  and  $d \in D$ . Now by condition (C),  $c = c + c(d + d')$ . Hence

$$cy = \left( (c + c(d + d')) \right) y = cy + c(d + d')y \in E^+(S).$$

Thus  $y \in C^r$ . Similarly,  $y \in D^r$ . Hence  $y \in C^r \cap D^r$ . Consequently,

$$C^r \cap D^r = (C + D)^r.$$

$\square$

**Lemma 3.9.** *Let  $A, B$  be two full principal left ideals of  $S$ . Then  $A \cap B$  is again a full principal left ideal.*

*Proof.*

$$\begin{aligned} A \cap B &= A^{rl} \cap B^{rl} \\ &= (A^r + B^r)^l. \end{aligned}$$

But  $A^r, B^r$  are full principal right ideals by Lemma 3.5(iii). Hence again by Corollary 3.7,  $(A^r + B^r)$  is a full principal right ideal. Thus,  $A \cap B$  is a full principal left ideal by 3.5 (iii)'.  $\square$

**Theorem 3.10.** *The set  $\overline{L}_S$  of all full principal left ideals of  $S$  is a complemented modular lattice, partially ordered by set inclusion relation, the meet being  $\cap$  and the join is the sum of two ideals, its least element is  $E^+(S)$  and its greatest element is  $S$ .*

*Proof.* The fact that  $\overline{L}_S$  is a lattice follows from Corollary 3.7 and Lemma 3.9. The regularity of  $S$  and Theorem 2.13, yields that  $\overline{L}_S$  is complemented. The modularity is established as follows. Let  $A, B, C$  be full right ideals with  $A \subseteq C$ . Clearly,  $A + (B \cap C) \subseteq (A + B) \cap C$ . Let  $x \in (A + B) \cap C$ . Then  $x = a + b$  for some  $a \in A, b \in B$  and  $x \in C$ . Then

$$a' + x = a' + a + b \in B \cap C.$$

Now,  $a' + x = a' + a + b$  implies that  $a + a' + x = a + b = x$ . Hence

$$x = a + a' + x \in A + (B \cap C).$$

Thus,  $(A + B) \cap C \subseteq A + (B \cap C)$ . Therefore,

$$(A + B) \cap C = A + (B \cap C)$$

and the proof is completed.  $\square$

#### 4. DECOMPOSITION OF $S$

In this section we decompose  $S$  as direct sum of two mutually inverse ideals.

We now define the center of a semiring.

**Definition 4.1.** Let  $S$  be a semiring. The center  $Z(S)$  of  $S$  is the set  $Z(S) = \{a \in S : ax = xa \text{ for every } x \in S\}$ .

**Definition 4.2.** A subsemiring  $A$  is called a full subsemiring if  $E^+(S) \subseteq A$ .

**Lemma 4.3.** *The center  $Z(S)$  of  $S$  is a multiplicative commutative, regular and additive inverse full subsemiring of  $S$  with 1.*

*Proof.* The proof is similar to ring theory and we omit the proof.  $\square$

**Lemma 4.4.** *For every  $a \in Z(S)$  the principal left ideal  $(a)_l$  and the principal right ideal  $(a)_r$  are the same. They will be denoted by  $(a)_*$ . Moreover, if  $a \in Z(S)$ ,  $(a)_*^l = (a)_*^r$  and the common value will be denoted by  $(a)_*^*$ .*

*Proof.* Follows from [4, Lemma 2.5].  $\square$

**Lemma 4.5.** (i) *If  $A$  is both left and right ideal of the form  $(e)_l$  ( or  $(e)_r$ ) with  $e \in E^\bullet(S)$ , then  $e$  is unique,  $e \in Z(S)$  and  $A = (e)_*$ .*



- (ii) A principal left ideal  $A$  is a right ideal if and only if there exists a unique  $e \in E^\bullet(S)$  such that  $e \in Z(S)$  and  $A = (e)_*$ .

*Proof.* (i) Let  $A = (e)_l$  be a right ideal,  $e \in E^\bullet(S)$ . For every  $y \in S$ ,  $ey \in (e)_l$  whence  $eye = ey$ , i.e.,  $ey(1 + e') \in E^+(S)$ . Now for every  $x \in S$ , there is some  $y \in S$  with

$$(1 + e')xey(1 + e')xe = (1 + e')xe.$$

Since  $ey(1 + e') \in E^+(S)$ , it follows that

$$(1 + e')xe = xe + e'xe \in E^+(S).$$

Also

$$\begin{aligned} exe + exe' &= (ex + ex')e = e(ex + ex') \quad (\text{by condition (C)}) \\ &= ex + ex'. \end{aligned}$$

Hence by Lemma 2.10, we have  $ex = exe$  and hence  $ex = xe$ . This shows that  $e \in Z(S)$  and  $a = (e)_*$ . The uniqueness of  $e$  follows from [4, Lemma 2.6(i)].

(ii) The reverse implication is trivial. suppose  $A$  is a principal left ideal and also a right ideal. Then there exists an element  $e \in E^\bullet(S)$  such that  $A = (e)_l$ , since  $S$  is regular. Then by part (i),  $e$  is unique,  $e \in Z(S)$  and  $A = (e)_*$ .  $\square$

**Definition 4.6.** A semiring  $S$  is said to be the direct sum of two full subsemirings  $S_1$  and  $S_2$  if every element  $x \in S$  is expressible in the form  $y + z$ ,  $y \in S_1$ ,  $z \in S_2$  and  $yz, zy \in E^+(S)$  for every  $y \in S_1, z \in S_2$ .

**Theorem 4.7.** If  $S$  is the direct sum of  $S_1$  and  $S_2$ , then  $S_1$  and  $S_2$  are mutually inverse ideals (both left or right ideals). Conversely, any two mutually inverse left ideals yields a direct sum decomposition of  $S$ .

*Proof.* If  $y \in S_1$ ,  $x \in S$ , then  $x = y' + z'$ ,  $y' \in S_1, z' \in S_2$ , and  $yx = y(y' + z') = yy' + yz' \in S_1$ , whence  $S_1$  is a right ideal. Likewise  $xy = (y' + z')y = y'y + z'y \in S_1$ , whence  $S_1$  is a left ideal. Thus  $S_1$  is an ideal. Similarly,  $S_2$  is an ideal. Let  $x \in S_1 \cap S_2$ . Now,  $1 = y_1 + z_1, y_1 \in S_1, z_1 \in S_2$ . Then

$$x = x \cdot 1 = x(y_1 + z_1) = xy_1 + xz_1 \in E^+(S).$$

Moreover,  $S_1 + S_2 = S$ . Hence  $S_1$  and  $S_2$  are inverses to each other.

Converse part is obvious.  $\square$

**Theorem 4.8.** The only direct sum decompositions of  $S$  are those of the form

$$(1) \quad S = (1 + e')_* + (1 + 1' + e)_*,$$

where  $e \in E^\bullet(S)$  and in  $Z(S)$ .

*Proof.* Clearly, any decomposition of the form (1) a direct sum decomposition. Let  $S = A + B$ ,  $A \cap B = E^+(S)$ , with  $A, B$  ideals. Then by Theorem 2.13, there exists an element  $e \in E^\bullet(S)$  such that  $A = (1 + e')_r, B = (1 + 1' + e)_r$ . But since  $A, B$  are also left ideals, Lemma 4.5(i), yields that  $e \in Z(S), A = (1 + e')_*$ ,  $B = (1 + 1' + e)_*$ .  $\square$

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