ON THE VOLUME OF THE TRAJECTORY SURFACES UNDER THE HOMOTHETIC MOTIONS

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ABSTRACT. The volumes of the surfaces of 3-dimensional Euclidean Space which are traced by a fixed point as a trajectory surface during 3-parametric motions are given by H. R. Müller [3], [4], [5] and W. Blaschke [1].

In this paper, the volumes of the trajectory surfaces of fixed points under3-parametric homothetic motions are computed. Also, using a certain pseudo-Euclidean metric we generalized the well-known classical Holditch Theorem, [2], to the trajectory surfaces.

1. INTRODUCTION

Let R and R' be moving and fixed spaces and $\{O;\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\}$ and $\{O';\boldsymbol{e}_1{}',\boldsymbol{e}_2{}',\boldsymbol{e}_3{}'\}$ be their orthonormal coordinate systems, respectively. If $\boldsymbol{e}_j=\boldsymbol{e}_j(t_1,t_2,t_3)$ and$\boldsymbol{u}=\boldsymbol{u}(t_1,t_2,t_3)$, then a 3-parameter motion B_3 of R with respect to R' is defined, where $\boldsymbol{u}=\overrightarrow{O'O}$ and t_1,t_2,t_3 are the real parameters. For the rotation part of B_3, we have the anti-symmetric system of differentiation equations (Ableitungsgleichungen)

$$\mathrm{d}\boldsymbol{e}_i=\boldsymbol{e}_k\omega_j-\boldsymbol{e}_j\omega_k,\qquad i,j,k=1,2,3\ \text{(cyclic)}$$

with the conditions of integration (Integrierbarkeitsbedingungen)

$$\mathrm{d}\omega_i=\omega_j\wedge\omega_k,$$

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where "d" is the exterior derivative and " $\wedge$ " is the wedge product of the differential forms. For the translation part of B_3

$$\mathrm{d}\overrightarrow{O'O} = \boldsymbol{\sigma} = \sigma_1 \boldsymbol{e}_1 + \sigma_2 \boldsymbol{e}_2 + \sigma_3 \boldsymbol{e}_3,$$

where the conditions of integration are

$$\mathrm{d}\sigma_i = \sigma_j \wedge \omega_k - \sigma_k \wedge \omega_j.$$

During B_3, ω_i and σ_i are the linear differential forms with respect to t_1, t_2, t_3. We assume that $\omega_i,\ i = 1, 2, 3$ are linear independent, i.e., $\omega_1 \wedge \omega_2 \wedge \omega_3 \neq 0$.

2. The volume of the trajectory surface under the homothetic motions

I.

Now, let us consider the 3-parametric homothetic motion of the fixed point $X = (x_i)$ with respect to arbitrary moving Euclidean space. We may write

$$\boldsymbol{x}' = \boldsymbol{u} + h\boldsymbol{x},$$

where $\boldsymbol{x}$ and $\boldsymbol{x}'$ are the position vectors of the point X with respect to the moving and fixed coordinate systems, respectively, and $h = h(t_1, t_2, t_3)$ is the homothetic scale of the motion. Then, we get

$$\mathrm{d}\boldsymbol{x}' = \boldsymbol{\sigma} + \boldsymbol{x}\mathrm{d}h + h\boldsymbol{x} \times \boldsymbol{\omega},$$

where $\boldsymbol{\omega} = \sum \omega_i \boldsymbol{e}_i$ is the rotation vector and " $\times$ " denotes the vector product.

If we write $d\boldsymbol{x}' = \sum \tau_i \boldsymbol{e}_i$, we get

$$\tau_i = \sigma_i + x_i \mathrm{d}h + h(x_j \omega_k - x_k \omega_j). \tag{1}$$

The volume element of the trajectory surface of X is

$$\mathrm{d}J_X = \tau_1 \wedge \tau_2 \wedge \tau_3. \tag{2}$$

Thus, the integration of the volume element over the region G of the parameter space yields the volume of the trajectory surface, i.e., $J_X = \int_G \mathrm{d}J_X$. Let Γ be the closed and orientated edge surface of G.

If we replace (1) in (2), for the volume of the trajectory surface of X we get

$$J_X = J_O + \sum_{i=1}^{3} \tilde{A}_i x_i^2 + \sum_{i \neq j} A_{ij} x_i x_j + \sum_{i=1}^{3} B_i x_i + \left(\sum_{i=1}^{3} x_i^2 \right) \left(\sum_{i=1}^{3} C_i x_i \right), \tag{3}$$

where

$$\begin{aligned} \tilde{A}_i &= \int_G \left(h^2 \sigma_i \wedge \omega_j \wedge \omega_k + h \mathrm{d}h \wedge \sigma_j \wedge \omega_j + h \mathrm{d}h \wedge \sigma_k \wedge \omega_k \right) \\ &= \frac{1}{2} \int_\Gamma \left(h^2 \sigma_j \wedge \omega_j + h^2 \sigma_k \wedge \omega_k \right), \end{aligned} \tag{4}$$

$$\begin{aligned} A_{ij} &= \int_G \left(h \mathrm{d}h \wedge \omega_i \wedge \sigma_j + h \mathrm{d}h \wedge \omega_j \wedge \sigma_i + h^2 \sigma_j \wedge \omega_j \wedge \omega_k + h^2 \sigma_i \wedge \omega_k \wedge \omega_i \right) \\ &= \frac{1}{2} \int_\Gamma \left(h^2 \omega_i \wedge \sigma_j + h^2 \omega_j \wedge \sigma_i \right), \\ B_i &= \int_G \left(h \sigma_i \wedge \sigma_k \wedge \omega_k + \mathrm{d}h \wedge \sigma_j \wedge \sigma_k + h \sigma_i \wedge \sigma_j \wedge \omega_j \right) = \int_\Gamma h \sigma_j \wedge \sigma_k, \\ C_i &= \int_G h^2 \mathrm{d}h \wedge \omega_j \wedge \omega_k = \frac{1}{3} \int_\Gamma h^3 \omega_j \wedge \omega_k \end{aligned}$$

and $J_O = \int_G \sigma_1 \wedge \sigma_2 \wedge \sigma_3$ is the volume of the trajectory surface of the origin point O.

Let us suppose that $\sigma_i \wedge \omega_i$, $i = 1, 2, 3$, have the same sign when integrated over any consistently orientated simplex from Γ. Then, using the mean-value theorem for double integrals, we obtain

$$\int_\Gamma h^2 \sigma_i \wedge \omega_i = h^2(u_i, v_i) \int_\Gamma \sigma_i \wedge \omega_i, \qquad i = 1, 2, 3, \tag{5}$$

where u_i and v_i are the parameters. If we assume that

$$h^2(u_1, v_1) = h^2(u_2, v_2) = h^2(u_3, v_3),$$

then using (4) and (5) we can find the parameters u_0 and v_0 such that

$$\begin{aligned} J_X = J_O + h^2(u_0, v_0) \sum_{i=1}^{3} A_i x_i^2 + \sum_{i \neq j} A_{ij} x_i x_j + \sum_{i=1}^{3} B_i x_i \\ + \left(\sum_{i=1}^{3} x_i^2 \right) \left(\sum_{i=1}^{3} C_i x_i \right), \end{aligned} \tag{6}$$

where

$$A_i = \frac{1}{2} \int_\Gamma (\sigma_j \wedge \omega_j + \sigma_k \wedge \omega_k).$$

Now, let us consider the plane $\boldsymbol{P}: \ C_1 x + C_2 y + C_3 z = 0$. The volumes of the trajectory surfaces of points on $\boldsymbol{P}$ are quadratic polynomial with respect to x_i. If we choose the moving coordinate system such that the coefficients of the mixture quadratic terms vanish, i.e. $A_{ij} = 0$, then we get for a point $X \in \boldsymbol{P}$

$$J_X = J_O + h^2(u_0, v_0) \sum_{i=1}^{3} A_i x_i^2 + \sum_{i=1}^{3} B_i x_i. \tag{7}$$

Hence, we may give the following theorem:

Theorem 1. *All the fixed points of* $\boldsymbol{P}$ *whose trajectory surfaces have equal volume during the homothetic motion lie on the same quadric.*

II.

Let X and Y be two fixed points on $\boldsymbol{P}$ and Z be another point on the line segment XY, that is,

$$z_i = \lambda x_i + \mu y_i, \qquad \lambda + \mu = 1.$$

Using (7), we get

$$J_Z = \lambda^2 J_X + 2\lambda\mu J_{XY} + \mu^2 J_Y, \tag{8}$$

where

$$J_{XY} = J_{YX} = J_O + h^2(u_0, v_0)\sum_{i=1}^{3} A_i x_i y_i + \frac{1}{2}\sum_{i=1}^{3} B_i(x_i + y_i)$$

is called the *mixture trajectory surface volume*. It is clearly seen that $J_{XX} = J_X$. Since

$$J_X - 2J_{XY} + J_Y = h^2(u_0, v_0)\sum_{i=1}^{3} A_i(x_i - y_i)^2, \tag{9}$$

we can rewrite (8) as follows:

$$J_Z = \lambda J_X + \mu J_Y - h^2(u_0, v_0)\lambda\mu\sum_{i=1}^{3} A_i(x_i - y_i)^2. \tag{10}$$

We will define the distance $D(X, Y)$ between the points X, Y of $\boldsymbol{P}$ by

$$D^2(X, Y) = \varepsilon\sum_{i=1}^{3} A_i(x_i - y_i)^2, \qquad \varepsilon = \pm 1, \quad [4]. \tag{11}$$

By the orientation of the line XY we will distinguish $D(X,Y) = -D(Y,X)$. Therefore, from (10) we have

$$J_Z = \lambda J_X + \mu J_Y - \varepsilon h^2(u_0, v_0)\lambda\mu D^2(X,Y). \tag{12}$$

Since X, Y and Z are collinear, we may write

$$D(X,Z) + D(Z,Y) = D(X,Y).$$

Thus, if we denote

$$\lambda = \frac{D(Z,Y)}{D(X,Y)}, \qquad \mu = \frac{D(X,Z)}{D(X,Y)},$$

from (12) we get

$$\begin{aligned} J_Z = \; & \frac{1}{D(X,Y)}\left[D(Z,Y)J_X + D(X,Z)J_Y\right] \\ & - \varepsilon h^2(u_0, v_0)D(X,Z)D(Z,Y). \end{aligned} \tag{13}$$

Now, we consider that the points X and Y trace the same trajectory surface. In this case, we get $J_X = J_Y$. Then, from (13) we obtain

$$J_X - J_Z = \varepsilon h^2(u_0, v_0)D(X,Z)D(Z,Y) \tag{14}$$

which is the generalization of Holditch's result, [2], for trajectory surfaces during the homothetic motions. (14) is also equivalent to the result given by [6]. We may give the following theorem:

Theorem 2. *Let XY be a line segment with the constant length on $\boldsymbol{P}$ and the endpoints of this line segment have the same trajectory surface. Then, the point Z on this line segment traces another trajectory surface. The volume between these trajectory surfaces depends on the distances (in the sense of (11)) of Z from the endpoints and the homothetic scale h.*

Special case: In the case of $h \equiv 1$, we have the result given by H. R. Müller, [3].

III.

Let $X_1 = (x_i), X_2 = (y_i)$ and $X_3 = (z_i)$, i=1,2,3 be noncollinear points on $\boldsymbol{P}$ and $Q = (q_i)$ be another point on $\boldsymbol{P}$ (Fig. 1). Then, we may write

$$q_i = \lambda_1 x_i + \lambda_2 y_i + \lambda_3 z_i, \qquad \lambda_1 + \lambda_2 + \lambda_3 = 1.$$

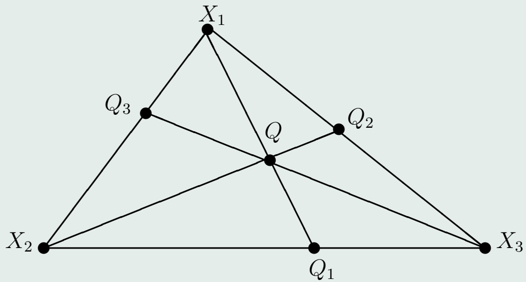

Figure 1.

If we use (7), we obtain

$$J_Q = \lambda_1^2 J_{X_1} + \lambda_2^2 J_{X_2} + \lambda_3^2 J_{X_3} + 2\lambda_1\lambda_2 J_{X_1X_2} + 2\lambda_1\lambda_3 J_{X_1X_3} + 2\lambda_2\lambda_3 J_{X_2X_3}.$$

After eliminating the mixture trajectory surface volumes by using (9), we get

$$\begin{aligned} J_Q =& \lambda_1 J_{X_1} + \lambda_2 J_{X_2} + \lambda_3 J_{X_3} - h^2(u_0, v_0) \\ & \cdot \left\{\varepsilon_{12}\lambda_1\lambda_2 D^2(X_1, X_2) + \varepsilon_{13}\lambda_1\lambda_3 D^2(X_1, X_3) + \varepsilon_{23}\lambda_2\lambda_3 D^2(X_2, X_3)\right\}. \end{aligned} \tag{15}$$

On the other hand, if we consider the point $Q_1 = (a_i)$, we may write

$$a_i = \mu_1 y_i + \mu_2 z_i, \qquad q_i = \mu_3 x_i + \mu_4 a_i, \qquad \mu_1 + \mu_2 = \mu_3 + \mu_4 = 1.$$

Thus, we have $\lambda_1 = \mu_3,\ \lambda_2 = \mu_1\mu_4,\ \lambda_3 = \mu_2\mu_4$ i.e.

$$\lambda_1 = \frac{D(Q,Q_1)}{D(X_1,Q_1)}, \quad \lambda_2 = \frac{D(X_1,Q)D(Q_1,X_3)}{D(X_1,Q_1)D(X_2,X_3)}, \quad \lambda_3 = \frac{D(X_1,Q)D(X_2,Q_1)}{D(X_1,Q_1)D(X_2,X_3)}.$$

Similarly, considering the points Q_2 and Q_3, respectively, we find

$$\begin{aligned}\lambda_i &= \frac{D(Q,Q_i)}{D(X_i,Q_i)} = \frac{D(X_j,Q)D(X_k,Q_j)}{D(X_j,Q_j)D(X_k,X_i)} \\ &= \frac{D(X_k,Q)D(Q_k,X_j)}{D(X_k,Q_k)D(X_i,X_j)}, \qquad i,j,k = 1,2,3 \text{ (cyclic)}.\end{aligned}$$

Then, from (15) the generalization of (12) is found as

$$J_Q = \sum \frac{D(Q,Q_i)}{D(X_i,Q_i)} J_{X_i} - h^2(u_0,v_0) \sum \varepsilon_{ij} \left(\frac{D(X_k,Q)}{D(X_k,Q_k)}\right)^2 D(Q_k,X_j)D(X_i,Q_k).$$

If X_1, X_2, X_3 trace the same trajectory surface, then the difference between the volumes is

$$J_{X_1} - J_Q = h^2(u_0,v_0) \sum \varepsilon_{ij} \left(\frac{D(X_k,Q)}{D(X_k,Q_k)}\right)^2 D(Q_k,X_j)D(X_i,Q_k).$$

Then, we can give the following theorem:

Theorem 3. *Let us consider a triangle on the plane* $\boldsymbol{P}$*. If the vertices of this triangle trace the same trajectory surface, then a different point on* $\boldsymbol{P}$ *traces another surface. The volume between these trajectory surfaces depends on the distances (in the sense of (11)) of the moving triangle and the homothetic scale* h*.*

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