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EXISTENCE THEOREMS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract. This paper concernes with the study of existence theorems for a general class of functional differential equations of the form

$$u'(t) = f(t, u \circ \gamma(t, \cdot)).$$

The obtained results generalize the retarded functional differential equations [**5, 6, 8**] and cover singular functional differential equations [**1, 2, 4, 7, 9, 12**].

1. Introduction

Let $(E, |\cdot|_E)$ be a Banach space. For a fixed $r > 0$, we define $\mathcal{C} = C([-r, 0]; E)$ to be the Banach space of continuous E-valued functions on $J := [-r, 0]$ with the usual supremum norm $\|\varphi\| = \sup_{\theta\in[-r,0]} |\varphi(\theta)|_E$.

For a continuous function $u : \mathbb{R} \to E$ and any $t \in \mathbb{R}$, we denote by u_t the element of $\mathcal{C}$, defined by

$$u_t(\theta) = u(t + \theta), \qquad \theta \in J.$$

For each $(\sigma, a) \in \mathbb{R} \times \mathbb{R}^*_+$, we consider

$$\Gamma_{\sigma,a} = \{\, \gamma : [\sigma, \sigma + a] \times [-r, 0] \to [\sigma - r, \sigma + a] \text{ continuous functions such that for all } \theta \in [-r, 0],\ s \in [0, a],\quad \gamma(\sigma, \theta) = \sigma + \theta \text{ and } \gamma(\sigma + s, 0) = \sigma + s \,\}.$$

It is clear that if $u \in C([\sigma - r, \sigma + a]; E)$ and $\gamma \in \Gamma_{\sigma,a}$, then $u \circ \gamma(t, \cdot) \in \mathcal{C}$ and $t \mapsto u \circ \gamma(t, \cdot)$ is a continuous function for $t \in [\sigma, \sigma + a]$, where $u \circ \gamma(t, \cdot)(\theta) := u(\gamma(t, \theta))$ for all $\theta \in J$, in particular, if $\gamma(t, \theta) = t + \theta$, then $u \circ \gamma(t, \cdot) = u_t \in \mathcal{C}$ and $t \mapsto u_t$ is continuous for $t \in [\sigma, \sigma + a]$.

Now we introduce a general class of functional differential equations

$$u'(t) = f(t, u \circ \gamma(t, \cdot)) \tag{1.1}$$

where f is a continuous function from $[\sigma, \sigma + a] \times \mathcal{C}$ into E.

If $\gamma(t, \theta) = t + \theta$, then the equation $R(f, \gamma)$ coincides with the classical retarded functional differential equation $u'(t) = f(t, u_t)$ (see, for example [**5, 6, 8**]).

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If $\gamma(t,\theta) = \rho(\rho^{-1}(t)+\theta)$ where $\rho : [\sigma - r, \sigma + b] \to [\sigma - r, \sigma + a]$, $(b > 0)$ is defined by

$$\rho(\tau) = \begin{cases} \sigma + \displaystyle\int_\sigma^\tau \frac{\mathrm{d}s}{\psi(s)} & \text{if} \quad \tau \in [\sigma, \sigma + b] \\ \tau & \text{if} \quad \tau \in [\sigma - r, \sigma], \end{cases}$$

$\psi : [\sigma, \sigma + b] \to \mathbb{R}^+$ is continuous, $\psi > 0$ on $(\sigma, \sigma + b]$ and $a := \int_\sigma^{\sigma+b} \frac{\mathrm{d}s}{\psi(s)} < +\infty$, then the equation $R(f,\gamma)$ coincides with the following initial value problem for the singular functional differential equation (see [**7**]):

$$\begin{cases} \psi(\tau)x'(\tau) = g(\tau, x_\tau), & \tau \in (\sigma, \sigma + b] \\ x_\sigma = \varphi, \end{cases}$$

where $g : [\sigma, \sigma + b] \times \mathcal{C} \to E$ is completely continuous and $f(t,\phi) := g(\rho^{-1}(t), \phi)$.

Also, in the Section 5, we shall study the general form

$$\begin{cases} \psi(\tau)x^{(n)}(\tau) = g(\tau, x_\tau, x'_\tau, \dots, x_\tau^{(n-1)}), & \tau \in (\sigma, \sigma + b], \quad (b > 0) \\ x_\sigma = \varphi, \end{cases}$$

or the second order delay equation of the form

$$\begin{cases} \psi(\tau)x''(\tau) = g(\tau, x(\tau), x(\tau - r_1), x'(\tau), x'(\tau - r_2)), & \tau \in (\sigma, \sigma + b], \quad (b > 0) \\ x_\sigma = \varphi, \quad x'_\sigma = \varphi' \quad \text{on } [-r, 0], \end{cases}$$

where $r = \max(r_1, r_2)$, (see [**12**]).

2. Preliminaries

Let D be a subset of $\mathbb{R} \times \mathcal{C}$ and let f be a continuous function from D into E. In the sequel, we give $(\sigma, a) \in \mathbb{R} \times \mathbb{R}_+^*$ and $\gamma \in \Gamma_{\sigma,a}$. We say that the relation

$$u'(t) = f(t, u \circ \gamma(t, \cdot)), \qquad ((t, u \circ \gamma(t, \cdot)) \in D),$$

is a functional differential equation on D and will denote this equation by $R(f,\gamma)$.

Definition 2.1. A function u is said to be a solution of the equation $R(f,\gamma)$, if there exists a real A such that $0 < A \le a$ and $u \in C([\sigma - r, \sigma + A); E)$, $(t, u \circ \gamma(t, \cdot)) \in D$ and u satisfies the equation $R(f,\gamma)$ for $t \in [\sigma, \sigma + A)$. Then, we say that u is a solution of $R(f,\gamma)$ on $[\sigma, \sigma + A)$

For $(\sigma, \varphi) \in \mathbb{R} \times \mathcal{C}$, we say $u := u(\sigma, \varphi)$ is a solution of equation $R(f,\gamma)$ through (σ, φ), if there is A such that $0 < A \le a$ and $u(\sigma, \varphi)$ is a solution of $R(f,\gamma)$ on $[\sigma - r, \sigma + A)$ and $u_\sigma(\sigma, \varphi) = \varphi$.

Let $(\sigma, \varphi) \in \mathbb{R} \times \mathcal{C}$, we consider the function $\widetilde{\varphi}$ defined by

$$\widetilde{\varphi}(t) = \begin{cases} \varphi(t - \sigma) & \text{if} \quad t \in [\sigma - r, \sigma] \\ \varphi(0) & \text{if} \quad t \ge \sigma. \end{cases}$$

We have $\widetilde{\varphi} \in C([\sigma - r, +\infty); E)$, $\widetilde{\varphi}_\sigma = \varphi$ and $\widetilde{\varphi}(t + \sigma) = \varphi(0)$ for $t \ge 0$.

It is easy to see that the following result is immediate.

Lemma 2.1. *Suppose that $f \in C(D;E)$, $\varphi \in \mathcal{C}$ and $0 < A \le a$. Then, there are equivalent statements:*

i) *u is solution of $R(f,\gamma)$ on $[\sigma - r, \sigma + A)$ through (σ,φ),*

ii) *$u \in C([\sigma - r, \sigma + A); E)$, $(t, u\circ\gamma(t,\cdot)) \in D$ for all $t \in [\sigma, \sigma + A)$ and*

$$\begin{cases} u_\sigma = \varphi \\ u(t) = \varphi(0) + \int\limits_\sigma^t f(s, u\circ\gamma(s,\cdot))\mathrm{d}s, \qquad t \in [\sigma, \sigma + A); \end{cases}$$

iii) *there exists $y \in C([-r, A); E)$ such that $(\sigma + t, y_t + \widetilde{\varphi}\circ\gamma(\sigma + t,\cdot)) \in D$ for all $t \in [0, A)$ and*

$$\begin{cases} y_0 = 0 \\ y(t) = \int\limits_0^t f(\sigma + s, y_s + \widetilde{\varphi}\circ\gamma(\sigma + s,\cdot))\mathrm{d}s, \qquad t \in [0, A). \end{cases}$$

For any real $\alpha, \beta > 0$, define

$$I_\alpha = [0,\alpha], \qquad \widehat{I_\alpha} = (0,\alpha], \qquad B_\beta = \{\psi \in \mathcal{C} : \|\psi\| \le \beta\},$$
$$A(\alpha,\beta) = \{y \in C([-r,\alpha]; E) : y_0 = 0 \ \text{ and } \ y_t \in B_\beta, \ \ t \in I_\alpha\}$$

and

$$C^0(D,E) = \{f \in C(D,E) : f \text{ is bounded on } D\}.$$

We have $A(\alpha,\beta)$ is a closed bounded convex subset of $C([-r,\alpha];E)$ and $C^0(D,E)$ is a Banach space with the norm $\|f\|_0 = \sup_{(t,\varphi)\in D} |f(t,\varphi)|_E$.

Lemma 2.2. *Suppose that $\Omega \subset \mathbb{R}\times\mathcal{C}$ is open, $W \subset \Omega$ is compact and $f^0 \in C(\Omega;E)$. Then, there exists a neighborhood $V \subset \Omega$ of W such that $f^0 \in C^0(V;E)$, there exists a neighborhood $U \subset C^0(V;E)$ of f^0 and three positive constants M, $\alpha \le a$ and β such that*

$$|f(\sigma,\varphi)|_E < M \qquad \textit{for all } (\sigma,\varphi) \in V \textit{ and } f \in U,$$

$(\sigma^0 + t, y_t + \widetilde{\varphi^0}\circ\gamma(\sigma^0 + t,\cdot)) \in V$ for any $(\sigma^0,\varphi^0) \in W$, $t \in I_\alpha$, $y \in A(\alpha,\beta)$ and $\gamma \in \Gamma_{\sigma^0,a}$.

Proof. Since $f^0(W)$ is a compact subset of the Banach space E, it is bounded, and therefore exists $M > 0$ such that

$$\left|f^0(\sigma^0,\varphi^0)\right|_E < \frac{M}{3}$$

for all $(\sigma^0,\varphi^0) \in W$. However f^0 is continuous at (σ^0,φ^0), and therefore for $0 < \varepsilon < \frac{M}{3}$, there exists $(\alpha',\beta') \in \mathbb{R}_+^*\times\mathbb{R}_+^*$ (with $\alpha' \le a$) such that for $(t,\psi) \in \left(\sigma^0 - \alpha', \sigma^0 + \alpha'\right)\times B(\varphi^0,\beta') \subset \Omega$ we have

$$\left|f^0(t,\psi)\right|_E \le \left|f^0(t,\psi) - f^0(\sigma^0,\varphi^0)\right|_E + \left|f^0(\sigma^0,\varphi^0)\right|_E < \frac{2M}{3}.$$

Consider β such that $0 < \beta < \beta'$ and $\gamma \in \Gamma_{\sigma',a}$. Since the function $s \in [\sigma^0, \sigma^0 + a] \mapsto \widetilde{\varphi^0} \circ \gamma(s, \cdot)$ is continuous at σ^0, then there exists α such that $0 < \alpha < \alpha'$ and

$$\left\| \widetilde{\varphi^0} \circ \gamma(\sigma^0 + t, \cdot) - \widetilde{\varphi^0} \circ \gamma(\sigma^0, \cdot) \right\| = \left\| \widetilde{\varphi^0} \circ \gamma(\sigma^0 + t, \cdot)) - \varphi^0 \right\| < \beta' - \beta, \qquad t \in I_\alpha.$$

Define $V = \bigcup_{(\sigma^0,\varphi^0) \in W} \left(\sigma^0 - \alpha', \sigma^0 + \alpha'\right) \times B(\varphi^0, \beta')$. Then $W \subset V \subset \Omega$, V is a neighborhood of W and $f^0 \in C^0(V; E)$. Moreover $(\sigma^0 + t, y_t + \widetilde{\varphi^0} \circ \gamma(\sigma^0 + t, \cdot)) \in V$ for all $t \in I_\alpha$, $y \in A(\alpha, \beta)$, $\gamma \in \Gamma_{\sigma^0,a}$. Indeed $\sigma^0 + t \in \left(\sigma^0 - \alpha', \sigma^0 + \alpha'\right)$ and $y_t + \widetilde{\varphi^0} \circ \gamma(\sigma^0 + t, \cdot) \in B(\varphi^0, \beta')$ because

$$\left\| y_t + \widetilde{\varphi^0} \circ \gamma(\sigma^0 + t, \cdot)) - \varphi^0 \right\| \leq \|y_t\| + \left\| \widetilde{\varphi^0} \circ \gamma(\sigma^0 + t, \cdot)) - \varphi^0 \right\| \leq \beta'.$$

Define $U = \left\{ f \in C^0(V; E) : \left\| f - f^0 \right\|_0 < \frac{M}{3} \right\}$. Then U is a neighborhood of f^0, $U \subset C^0(V; E)$ and for all $(\sigma, \varphi) \in V$, $f \in U$

$$|f(\sigma, \varphi)|_E \leq \left| f(\sigma, \varphi) - f^0(\sigma, \varphi) \right|_E + \left| f^0(\sigma, \varphi) \right|_E < M.$$

□

The next lemma will be used to apply fixed point theorems for existence of solutions of the equation $R(f, \gamma)$.

Lemma 2.3. *Suppose that $\Omega \subset \mathbb{R} \times \mathcal{C}$ is open, $W = \{(\sigma, \varphi)\} \subset \Omega$ and $f^0 \in C(\Omega; E)$ are given, the neighborhoods V, U and the constants M, α and β are the ones obtained from Lemma 2.2. Define an operator $T : U \times A(\alpha, \beta) \to C([-r, \alpha]; E)$ by*

$$T(f, y)(t) = \begin{cases} 0 & \text{if } t \in [-r, 0] \\ \displaystyle\int_0^t f((\sigma + s, y_s + \widetilde{\varphi} \circ \gamma(\sigma + s, \cdot)) \mathrm{d}s & \text{if } t \in I_\alpha. \end{cases}$$

If $M\alpha \leq \beta$, then $T : U \times A(\alpha, \beta) \to A(\alpha, \varphi)$ and T is continuous on $U \times A(\alpha, \beta)$.

Proof. It is clear that T maps $U \times A(\alpha, \beta)$ into $C([-r, \alpha]; E)$ and by Lemma 2.2, for all $t, t' \in I_\alpha$

$$|T(f, y)(t) - T(f, y)(t')|_E \leq M\, |t - t'| \qquad \text{and} \qquad |T(f, y)(t)|_E \leq M\alpha.$$

It is easy to see that for all $t, t' \in [-r, \alpha]$

$$|T(f, y)(t) - T(f, y)(t')|_E \leq M\, |t - t'| \qquad \text{and} \qquad |T(f, y)(t)|_E \leq M\alpha.$$

Hence the family $\Im = \{T(f, y) : (f, y) \in U \times A(\alpha, \beta)\}$ is bounded and uniformly equicontinuous. Also, we have $(T(f, y))_0 = 0$ and $(T(f, y))_t \in B_\beta$ if $M\alpha \leq \beta$, thus T maps $U \times A(\alpha, \beta)$ into $A(\alpha, \beta)$.

It remains to show that T is continuous on $U \times A(\alpha, \beta)$.

Let (f^n, y^n) be a sequence in $U \times A(\alpha, \beta)$ that converges to a member (f, y) of $U \times A(\alpha, \beta)$.

It is clear that for each $s \in I_\alpha$

$$\begin{aligned}\|y_s^n - y_s\| &= \sup_{\theta \in [-r,0]} |y^n(s+\theta) - y(s+\theta)|_E \leq \|y^n - y\|_1 \\ &:= \sup_{t \in [-r,\alpha]} |y^n(t) - y(t)|_E\,.\end{aligned}$$

We have $(\sigma + s, y_s^n + \widetilde{\varphi} \circ \gamma(\sigma + s, \cdot))$, $(\sigma + s, y_s + \widetilde{\varphi} \circ \gamma(\sigma + s, \cdot) \in V$ because $s \in I_\alpha$ and $y^n, y \in A(\alpha, \beta)$. Since (f^n) converges uniformly to f in V, then the sequence $(f^n(\sigma + s, y_s^n + \widetilde{\varphi} \circ \gamma(\sigma + s, \cdot)))$ converges to $f(\sigma + s, y_s + \widetilde{\varphi} \circ \gamma(\sigma + s, \cdot))$ in E, but f^n and f are bounded on V (see Lemma 2.2) and by the Lebesgue dominated convergence theorem, we obtain

$$\int_0^t f^n(\sigma + s, y_s^n + \widetilde{\varphi} \circ \gamma(\sigma + s, \cdot))\mathrm{d}s \longrightarrow \int_0^t f(\sigma + s, y_s + \widetilde{\varphi} \circ \gamma(\sigma + s, \cdot))\mathrm{d}s$$

in E. Hence for all $t \in I_\alpha$, $T(f^n, y^n)(t)$ converges to $T(f, y)(t)$ in E and then $(T(f^n, y^n)(t))$ converges to $T(f, y)(t)$ in E for all $t \in [-r, \alpha]$. This implies that the set $\{T(f^n, y^n)(t) : t \in [-r, \alpha]\}$ is relatively compact in E, but the family $\{T(f^n, y^n) : n \in \mathbb{N}\}$ is bounded and uniformly equicontinuous and therefore by the Ascoli theorem [**3, 11**], the family$\{T(f^n, y^n) : n \in \mathbb{N}\}$ is relatively compact in $C([-r, \alpha]; E)$. We shall show that $T(f^n, y^n)$ converges to $T(f, y)$ in $C([-r, \alpha]; E)$. Suppose, for the sake of contradiction, that there exists $\varepsilon > 0$ such that for all $N \in \mathbb{N}$

$$\exists n > N : \ \|T(f^n, y^n) - T(f, y)\|_1 \geq \varepsilon.$$

Then for

$$N = n_0, \quad \exists n_1 > n_0 : \qquad \|T(f^{n_1}, y^{n_1}) - T(f, y)\|_1 \geq \varepsilon$$

and for $k > 1$ and

$$N = n_{k-1}, \quad \exists n_k > n_{k-1} : \quad \|T(f^{n_k}, y^{n_k}) - T(f, y)\|_1 \geq \varepsilon.$$

If necessary, passing, to a subsequence, we can assume that $(T(f^{n_k}, y^{n_k}))$ converges to $z \in A(\alpha, \beta)$ such that $\|z - T(f, y)\|_1 \geq \varepsilon$. Since $(T(f^{n_k}, y^{n_k}))$ converges to z in $C([-r, \alpha]; E\ )$, then $(T(f^{n_k}, y^{n_k}))(t)$ converges to $z(t)$ in E for each $t \in [-r, \alpha]$, but this sequence converges to $T(f, y)(t)$ in E, which is a contradiction. Therefore T is continuous on $U \times A(\alpha, \beta)$. □

3. Local existence of solutions

In this section we shall show existence theorem of solutions to $R(f, \gamma)$ by using the results obtained in the section two.

Definition 3.1. Suppose that Ω is an open set in $\mathbb{R} \times \mathcal{C}$. A function $f \in C(\Omega; E)$ is said to have the condition (l) if, for all $(\sigma, \varphi) \in \Omega$, there exists a neighborhood $V' \subset \Omega$ of (σ, φ) and a positive constant k such that for all bounded $I \times S_1 \subset V'$ with bounded $f(I \times S_1)$, then $\chi(f(I, S_1)) \leq k\chi_0(S_1)$ where χ (resp. χ_0) is the measure of noncompactness [**3, 11**] on E (resp. $\mathcal{C}$).

Theorem 3.1. *Suppose that Ω is an open set in $\mathbb{R}\times\mathcal{C}$ and $f \in C(\Omega; E)$. If f is compact or satisfying the condition (l) (resp. $f(t,\cdot)$ is locally Lipschitz), then for all $(\sigma,\varphi) \in \Omega$ and $\gamma \in \Gamma_{\sigma,a}$ there exists a positive constant $\alpha \leq a$ and a solution (resp. a unique solution) of the equation $R(f,\gamma)$ on $[\sigma - r, \sigma + \alpha]$ through (α, φ).*

Proof. By notations of Lemmas 2.2 and 2.3 with $W = \{(\sigma,\varphi)\}$, the operator $T_1 = T(f,\cdot)$ maps $A(\alpha,\beta)$ into $A(\alpha,\beta)$ if $\alpha \leq \frac{\beta}{M}$ and T_1 is continuous.

<u>First case</u>. If f is compact we shall show that T_1 is compact.

Let B be a bounded subset of $A(\alpha,\beta)$ and (z^n) a sequence of $T_1 B$, then there exists a sequence (y^n) of B such that $z^n = T_1 y^n$.

The set $\{f(\sigma + s, y^n_s + \widetilde{\varphi}\circ\gamma(\sigma+s,\cdot)) : n \in \mathbb{N},\ s \in I_\alpha\}$ is relatively compact because f is completely continuous. By Mazur theorem [**3, 11**] its closed convex hull is compact. But, for all $t \in \widehat{I_\alpha}$, we have (see [**11**, p. 25])

$$\frac{1}{t}\int_0^t f(\sigma+s, y^n_s+\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot))\mathrm{d}s \in \overline{\mathrm{Co}}\{f(\sigma+s, y^n_s+\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot)) \colon n \in \mathbb{N},\ s \in I_\alpha\}.$$

Then,

$$\{T_1 y^n(t) : n \in \mathbb{N}, s \in \widehat{I_\alpha}\} \subset \alpha\overline{\mathrm{Co}}\{f(\sigma + s, y^n_s + \widetilde{\varphi}\circ\gamma(\sigma+s,\cdot)) : n \in \mathbb{N}, s \in I_\alpha\}$$

which is compact, hence $\{T_1 y^n(t) : n \in \mathbb{N}, s \in \widehat{I_\alpha}\}$ is relatively compact. However $\{T_1 y^n(t) : n \in \mathbb{N}\}$ is bounded and uniformly equicontinuous, then by the Ascoli theorem, $\{T_1 y^n(t) : n \in \mathbb{N}\}$ is relatively compact, thus (z^n) has a subsequence that converges in $C([-r,\alpha]; E)$. By Schauder fixed-point theorem and Lemma 2.3, $R(f,\gamma)$ has a solution on $[\sigma - r, \sigma + \alpha]$ through (σ,φ).

<u>Second case</u>. If f satisfies the condition (l).

Let $V = (\sigma - \alpha', \sigma + \alpha') \times B(\varphi, \beta')$ be the neighborhood obtained in the Lemma 2.2 and by the condition (l), there exist $V' = (\sigma - \alpha'', \sigma + \alpha'') \times B(\varphi, \beta'')$ and $k > 0$ such that if $f(I \times S_1)$ is bounded for all bounded $I \times S_1 \subset V'$, then $\chi(f(I \times S_1)) \leq k\chi_0(S_1)$. Take $\alpha_2 = \min(\alpha', \alpha'')$, $\beta_2 = \min(\beta', \beta'')$ and $V_1 = V \cap V'$. Let $0 < \beta_1 < \beta_2$, then there exists α_1 such that $0 < \alpha_1 < \alpha_2$ (see proof of Lemma 2.2) and for all $s \in I_{\alpha_1}$

$$\|\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot) - \widetilde{\varphi}\circ\gamma(\sigma,\cdot)\| = \|\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot) - \varphi\| < \beta_2 - \beta_1.$$

Then for every $s \in I_{\alpha_1}$ and every $y \in A(\alpha_1, \beta_1)$, $(\sigma + s, y_s + \widetilde{\varphi}\circ\gamma(\sigma+s,\cdot)) \in V_1$. Thus, if $\alpha_1 \leq \frac{\beta_1}{M}$, then T_1 maps $A(\alpha_1,\beta_1)$ into itself and T_1 is continuous. Moreover $A(\alpha_1,\beta_1)$ is closed, bounded convex subset of $C([-r,\alpha_1]; E)$. In order to apply Darboux fixed point theorem [**3, 11**], we shall show that there exists $\delta \in [0,1)$ such that $\chi^1_0(T_1 S) \leq \delta\chi^1_0(S)$ for all $S \subset A(\alpha_1,\beta_1)$ where χ^1_0 is the measure of noncompactness on $C([-r,\alpha_1]; E)$.

Let $S \subset A(\alpha_1, \beta_1)$, then for each $t \in \widehat{I_{\alpha_1}}$

$$\begin{aligned}\chi(T_1 S(t)) &= \chi\left(\left\{\int_0^t f(\sigma+s, y_s+\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot))\mathrm{d}s:\ y\in S\right\}\right)\\ &= \chi\left(\left\{t\frac{1}{t}\int_0^t f(\sigma+s, y_s+\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot))\mathrm{d}s:\ y\in S\right\}\right)\\ &\le \alpha_1\chi(\overline{\mathrm{Co}}\{f(\sigma+s, y_s+\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot)):\ y\in S,\ s\in[0,t]\})\\ &\le \alpha_1\chi(\{f(\sigma+s, y_s+\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot)):\ y\in S,\ s\in[0,t]\}).\end{aligned}$$

By definition of V_1, we have for each $t \in \widehat{I_{\alpha_1}}$

$$\{(\sigma+s, y_s+\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot)):\ y\in S,\ s\in[0,t]\}\subset V_1.$$

Take

$$I = \{\sigma+s:\ s\in[0,t]\}\subset[\sigma,\sigma+\alpha_1] \qquad \text{and}$$

$$S_1 = \{y_s+\widetilde{\varphi}\circ\gamma(\sigma+s,.):\ y\in S,\ s\in[0,t]\}.$$

Then, $I\times S_1$ and $f(I\times S_1)$ are bounded (because f is bounded on V see Lemma 2.2). Hence, for each $t \in \widehat{I_{\alpha_1}}$, $\chi(T_1S(t)) \le k\alpha_1\chi_0(S_1)$ and

$$\begin{aligned}\chi_0(S_1) &= \chi_0(\{y_s:\ y\in S,\ s\in[0,t]\}) + \chi_0(\{\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot):\ s\in[0,t]\})\\ &\le \chi_0(\{y_s: y\in S,\ s\in[0,t]\}).\end{aligned}$$

Since $\chi_0(\{\widetilde{\varphi}\circ\gamma(\sigma+s,.): s\in[0,t]\}) = 0$, the set $\{\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot): s\in[0,t]\}$ is relatively compact.

Thus, for each $t \in \widehat{I_{\alpha_1}}$

$$\chi(T_1S(t)) \le k\alpha_1\chi_0(S_s) \le k\alpha_1\chi_0^1(S) \qquad \text{(see [13])}$$

where $S_s = \{y_s: s\in[0,t],\ y\in S\}$.

But the family $\left\{t\mapsto\int_0^t f(\sigma+s, y_s+\widetilde{\varphi}\circ\gamma(\sigma+s,\cdot))\,\mathrm{d}s\right\}$ is uniformly bounded and equicontinuous, then by Ambrosetti theorem [**3, 11**], we obtain

$$\chi_0^1(T_1S) = \sup_{t\in[-r,\alpha_1]}\chi(T_1S(t)) \le k\alpha_1\chi_0^1(S).$$

Take $\alpha_1 < \min\left\{\frac{1}{k}, \frac{\beta_1}{M}\right\}$ and $\delta = k\alpha_1 \in [0,1)$.

<u>*Third case*</u>. If $f(t,\cdot)$ is locally Lipschitz.

Let $V = (\sigma-\alpha', \sigma+\alpha') \times B(\varphi,\beta')$ be the neighborhood obtained in the Lemma 2.2 and since $f(t,\cdot)$ is locally Lipschitz then there exists $V' = (\sigma-\alpha'', \sigma+\alpha'')\times B(\varphi,\beta'')$ such that $f(t,\cdot)$ is Lipschitz on V'. T_1 maps $A(\alpha_1,\beta_1)$ into $A(\alpha_1,\beta_1)$ if $\alpha_1 \le \frac{\beta_1}{M}$ (see the second case of the existence of α_1,β_1) and T_1 is a contraction strict if $\alpha_1 < \min\left\{\frac{\beta_1}{M}, \frac{1}{k}\right\}$. Indeed, for all $y, z \in A(\alpha_1, \beta_1)$

$$\begin{aligned}\|T_1 y - T_1 z\|_1 &= \sup_{t\in[-r,\alpha_1]} |T_1 y(t) - T_1 z(t)|_E \\ &\le \sup_{t\in\widehat{I_{\alpha_1}}} \int_0^t |f(\sigma+s, y_s + \widetilde{\varphi}\circ\gamma(\sigma+s,\cdot)) \\ &\qquad - f(\sigma+s, z_s + \widetilde{\varphi}\circ\gamma(\sigma+s,\cdot))|_E \mathrm{d}s \\ &\le k \sup_{t\in\widehat{I_{\alpha_1}}} \int_0^t \|y_s - z_s\| \,\mathrm{d}s,\end{aligned}$$

but for all $s \in (0, t] \subset \widehat{I_{\alpha_1}}$

$$\|y_s - z_s\| \le \sup_{\theta\in[-r,0]\cap[-s,0]} |y(s+\theta) - z(s+\theta)|_E \le \|y - z\|_1 .$$

Finally

$$\|T_1 y - T_1 z\|_1 \le k\alpha_1 \|y - z\|_1 \qquad \text{for all } y, z \in A(\alpha_1, \beta_1).$$

Thus, the equation $R(f, \gamma)$ has a unique solution on $[\sigma - r, \sigma + \alpha_1]$ through (σ, φ). □

Remark. If $f = f_1 + f_2$ where f_1 is completely continuous and f_2 is locally Lipschitz, then the condition (l) is verified.

4. Global existence solutions

Definition 4.1. Let u (resp. v) be a solution of $R(f, \gamma)$ on $J_u = [\sigma - r, \sigma + A)$ (resp. $J_v = [\sigma - r, \sigma + B)$) where $0 < A, B \le a$. The solution v is said to be a continuation of u if $J_v \supset J_u$ and $v = u$ on J_u.

The solution u is said to be noncontinuable if it has no proper continuation.

The following result of the existence of noncontinuable solutions follows from Zorn lemma [**14**].

Proposition 4.1. *If u is a solution of the equation $R(f, \gamma)$ on J_u, then there exists a noncontinuable solution $\widehat{u}$ of $R(f, \gamma)$ on $J_{\widehat{u}}$ such that $\widehat{u}$ is a continuation of u.*

Theorem 3.1 gives a criterion of local existence for solutions to $R(f, \gamma)$, then we use the previous proposition to study the continuation of solutions to the equation $R(f, \gamma)$.

Theorem 4.1. *Suppose that Ω is an open subset of $\mathbb{R} \times \mathcal{C}$ and $f \in C(\Omega; E)$. If f is compact or verifies the condition (l) (resp. $f(t, \cdot)$ is locally Lipschitz), then for all $(\sigma, \varphi) \in \Omega$ and $\gamma \in \Gamma_{\sigma,a}$, there exists a noncontinuable solution (resp. a unique solution) of $R(f, \gamma)$ on $[\sigma - r, \sigma + \alpha)$ $(0 < \alpha \le a)$ through (σ, φ).*

Proof. It remains to show unicity of a noncontinuable solution if $f(t, \cdot)$ is locally Lipschitz. Suppose that there exist two noncontinuable solutions $u : [\sigma - r,$

$\sigma+\alpha_u) \to E$ and $v : [\sigma-r, \sigma+\alpha_v) \to E$ of $R(f,\gamma)$ through (σ,φ), then $u_\sigma = v_\sigma = \varphi$. If $\alpha_u < \alpha_v$, then v is a continuation of u, which is a contradiction, so $\alpha_u = \alpha_v$.

By Lemma 2.2, we associated with u (resp. v), y (resp. z) and we will see $y = z$ on $J = (0,\alpha)$. Suppose that in J there exists $t' > 0$ such that $y(t') \neq z(t')$. Define $t_0 = \inf\{t \in J : y(t) \neq z(t)\}$, by continuity of y and z, then $y(t_0) = z(t_0)$ and $y_{t_0} = z_{t_0}$. Set $\varphi^0 = y_{t_0} + \widetilde{\varphi}\circ\gamma(\sigma+t_0,\cdot) = z_{t_0} + \widetilde{\varphi}\circ\gamma(\sigma+t_0,\cdot)$, then $(\sigma+t_0,\varphi^0) \in \Omega$ and there exist a neighborhood $V \subset \Omega$ of (σ,φ) and a positive constant k such that f is k-Lipschitz on V. Let $\alpha' > 0$ such that for all $t \in J$, $0 < t - t_0 \leq \alpha'$, then $(\sigma+t, y_t + \widetilde{\varphi}\circ\gamma(\sigma+t,\cdot))$, $(\sigma+t, z_t + \widetilde{\varphi}\circ\gamma(\sigma+t,\cdot)) \in V$. However

$$\begin{aligned} y(t) - z(t) &= y(t) - y(t_0) + z(t_0) - z(t) \\ &= \int_{t_0}^{t} [f(\sigma+s, y_s + \widetilde{\varphi}\circ\gamma(\sigma+s,\cdot)) - f(\sigma+s, z_s + \widetilde{\varphi}\circ\gamma(\sigma+s,\cdot))]\mathrm{d}s, \end{aligned}$$

and thus

$$|y(t) - z(t)|_E \leq \int_{t_0}^{t} k\,\|y_s - z_s\|\,\mathrm{d}s.$$

It follows from Gronwall inequality [**5**] that $y(t) = z(t)$ for all $t_0 \leq t \leq t_0 + \alpha'$, which is a contradiction to the definition of t_0. $\square$

Now, let $\sigma \in \mathbb{R}$ and Γ_σ be a set of continuous functions $\gamma : [\sigma,+\infty) \times [-r,0] \to [\sigma-r,+\infty)$ such that $\gamma(\sigma,\theta) = \sigma+\theta$, $\gamma(\sigma+t,0) = \sigma+t$ and $\gamma([\sigma,\sigma+t]\times[-r,0]) = [\sigma-r,\sigma+t]$ for all $\theta \in [-r,0]$, $t \geq 0$.

The following theorem gives a global solutions of the equation $R(f,\gamma)$ on $[\sigma-r, +\infty)$.

Theorem 4.2. *Let $f : \mathbb{R}\times\mathcal{C} \to E$ be a continuous function. Suppose that f is compact or verifies the condition (l) (resp. $f(t,\cdot)$ is locally Lipschitz). Suppose further that there exists a continuous function $m : \mathbb{R} \to \mathbb{R}^+$ such that*

$$|f(t,\psi)|_E \leq m(t)h(\|\psi\|), \qquad (t,\psi) \in \mathbb{R}\times\mathcal{C},$$

where h is continuous nondecreasing on $\mathbb{R}^+$, positive on $\mathbb{R}^+_*$ and $\int_0^{+\infty} \frac{\mathrm{d}s}{h(s)} = +\infty$. Then, for all $(\sigma,\varphi) \in \mathbb{R}\times\mathcal{C}$ and $\gamma \in \Gamma_\sigma$, there exists a function (resp. a unique solution) $u \in C([\sigma-r,+\infty);E)$ which verifies the Cauchy problem:

$$\begin{cases} u'(t) = f(t, u\circ\gamma(t,\cdot)), & t \geq 0 \\ u_\sigma = \varphi. \end{cases} \tag{4.1}$$

Proof. By Theorem 4.1, there exists a noncontinuable solution (resp. a unique solution) u of problem (4.1) on $[\sigma-r,\beta)$ where $\beta > \sigma$. We shall show $\beta = +\infty$. Suppose that $\beta < +\infty$. By Lemma 2.1, we have for $t \in [\sigma,\beta)$

$$\begin{aligned} |u(t)|_E &\leq |\varphi(0)|_E + \int_\sigma^t |f(s, u\circ\gamma(s,\cdot))|_E\,\mathrm{d}s \\ &\leq \|\varphi\| + \int_\sigma^t m(s)h(\|u\circ\gamma(s,\cdot)\|)\mathrm{d}s. \end{aligned} \tag{$*$}$$

Consider the function v given by

$$v(t) = \sup \{|u(s)|_E : \sigma - r \le s \le t\}, \qquad t \in [\sigma, \beta).$$

It is clear that

$$(**) \qquad v(t) \le \|\varphi\| + \int_\sigma^t m(s)h(\|u \circ \gamma(s,\cdot)\|)\mathrm{d}s \le \|\varphi\| + \int_\sigma^t m(s)h(v(s))\mathrm{d}s.$$

Denoting by $w(t)$ the right-hand side of the above inequality $(**)$, we obtain $w(\sigma) = \|\varphi\|$ and

$$v(t) \le w(t), \qquad w'(t) = m(t)h(v(t)) \le m(t)h(w(t)), \qquad t \in [\sigma, \beta).$$

Integrating over $[\sigma, t]$, we obtain

$$\int_\sigma^t \frac{w'(s)}{h(w(s))}\mathrm{d}s = \int_{w(\sigma)}^{w(t)} \frac{\mathrm{d}s}{h(s)} \le \int_\sigma^t m(s)\mathrm{d}s < +\infty.$$

This inequality implies that there is a positive constant c such that for all $t \in [\sigma, \beta)$, $w(t) \le c$, then $v(t) \le c$. This majoration implies $|u'(t)|_E$ is bounded, hence u is uniformly continuous on $[\sigma - r, \beta)$, then there exists a unique continuous function $\overline{u} : [\sigma - r, \beta] \to E$ defined by

$$\overline{u}(t) = \begin{cases} u(t) & \text{if } t < \beta \\ \lim\limits_{s\to\beta} u(s) & \text{if } t = \beta. \end{cases}$$

Since $\gamma(s, \theta) \in [\sigma - r, s]$, then $\overline{u} \circ \gamma(s, \cdot) = u \circ \gamma(s, \cdot)$ and

$$\begin{aligned} \overline{u}(\beta) = \lim_{s\to\beta} u(s) &= \varphi(0) + \lim_{s\to\beta} \int_\sigma^s f(s', u \circ \gamma(s', \cdot))\mathrm{d}s' \\ &= \varphi(0) + \lim_{s\to\beta} \int_\sigma^s f(s', \overline{u} \circ \gamma(s', \cdot))\mathrm{d}s' \\ &= \varphi(0) + \int_\sigma^\beta f(s', \overline{u} \circ \gamma(s', \cdot))\mathrm{d}s'. \end{aligned}$$

This implies $\overline{u}$ is a solution to (4.1) on $[\sigma - r, \beta]$, which is a contradiction, and thus $\beta = +\infty$. □

Remark. By the fixed-point theorem for a strict contraction, we obtain the following result easily.

Theorem 4.3. *Let* Γ *be the set of continuous functions* $\gamma : \mathbb{R} \times [-r, 0] \to \mathbb{R}$ *such that*

$$\gamma([\sigma, \sigma + T] \times [-r, 0]) \subset [\sigma - r, \sigma + T] \qquad \textit{for all} \;\; (\sigma, T) \in \mathbb{R} \times \mathbb{R}_+^*$$

and $f : \mathbb{R} \times \mathcal{C} \to E$ *be a continuous function such that* $f(t, \cdot)$ *is Lipschitz. Then, for all* $(\sigma, \varphi) \in \mathbb{R} \times \mathcal{C}$ *and* $\gamma \in \Gamma$*, there exists a unique function* $u \in C([\sigma - r, +\infty); E) \cap C^1([\sigma, +\infty); E)$ *which verifies the Cauchy problem:*

$$(4.2) \qquad \begin{cases} u'(t) = f(t, u \circ \gamma(t, \cdot)), & t \ge 0 \\ u_\sigma = \varphi. \end{cases}$$

5. Applications

Retarded functional differential equations

Let $(\sigma, a) \in \mathbb{R} \times \mathbb{R}^*_+$. Consider $\gamma : [\sigma, \sigma + a] \times [-r, 0] \to [\sigma - r, \sigma + a]$ defined by $\gamma(t, \theta) = t + \theta$. Then $\gamma \in \Gamma_{\sigma,a}$ and the equation $R(f, \gamma)$ coincides with the classical retarded functional differential equations $u'(t) = f(t, u_t)$ (see, for example [**5, 6, 8**]).

Singular functional differential equations

Singular functional differential equations have been studied by many authors, for instance, Baxely [**1**], Bobisud and O'Regan [**2**], Gatica and al [**4**], Huaxing and Tadeusz [**7**], Labovskii [**9**] and O'Regan [**12**].

Our purpose in this section is to apply the previous results to give some theorems of existence for singular functional differential equations.

Theorem 5.1. *Consider the initial value problem for singular functional differential equations*

$$\left\{ \begin{array}{ll} \psi(\tau)x'(\tau) = g(\tau, x_\tau), & \tau \in (\sigma, \sigma + b], \ (b > 0) \\ x_\sigma = \varphi \end{array} \right. \tag{5.1}$$

where $g : [\sigma, \sigma + b] \times \mathcal{C} \to E$ is completely continuous, $\psi : [\sigma, \sigma + b] \to \mathbb{R}^+$ is continuous, $\psi > 0$ on $(\sigma, \sigma + b)$ and $a := \int_\sigma^{\sigma+b} \frac{\mathrm{d}s}{\psi(s)} < +\infty$. Then, (5.1) *has at least one noncontinuable solution.*

Proof. Let $\rho : [\sigma - r, \sigma + b] \to [\sigma - r, \sigma + a]$ defined by

$$\rho(\tau) = \left\{ \begin{array}{ll} \sigma + \displaystyle\int_\sigma^\tau \frac{\mathrm{d}s}{\psi(s)} & \text{if } \ \tau \in [\sigma, \sigma + b] \\ \tau & \text{if } \ \tau \in [\sigma - r, \sigma], \end{array} \right.$$

then ρ is bijective and continuous.

For all $\tau \in [\sigma, \sigma + b]$ and $\theta \in [-r, 0]$, put

$$u(\rho(\tau + \theta)) = x(\tau + \theta).$$

Then, for all $\tau \in (\sigma, \sigma + b]$,

$$x'(\tau) = u'(\rho(\tau))\rho'(\tau) = u'(\rho(\tau))\frac{1}{\psi(\tau)}.$$

Hence $u'(\rho(\tau)) = \psi(\tau)x'(\tau) = g(\tau, x_\tau)$, and thus

$$u'(t) = g(\rho^{-1}(t), x_{\rho^{-1}(t)}), \qquad t \in (\sigma, \sigma + a], \qquad \text{and}$$

$$x_{\rho^{-1}(t)}(\theta) = x(\rho^{-1}(t) + \theta) = u(\rho(\rho^{-1}(t) + \theta)).$$

Consider the function $\gamma : [\sigma, \sigma + a] \times [-r, 0] \to [\sigma - r, \sigma + a]$ defined by $\gamma(t, \theta) = \rho(\rho^{-1}(t) + \theta)$. It is clear that $\gamma \in \Gamma_{\sigma,a}$, $x_{\rho^{-1}(t)} = u \circ \gamma(t, \cdot)$ and

$$u_\sigma(\theta) = u(\sigma + \theta) = u(\gamma(\sigma, \theta)) = u(\rho(\rho^{-1}(\sigma) + \theta)) = x_\sigma(\theta) = \varphi(\theta).$$

□

Finally, the initial value problem (5.1) is equivalent to following problem

$$\begin{cases} u'(t) = f(t, u \circ \gamma(t, \cdot)), & t \in (\sigma, \sigma + a] \\ u_\sigma = \varphi, \end{cases} \tag{5.2}$$

where $f(t, \phi) = g(\rho^{-1}(t), \phi)$, f is also completely continuous and by Theorem 4.1, the problem (5.2) has at least one noncontinuable solution u on $[\sigma - r, \sigma + \alpha)$ with $\alpha \leq a$. It easy to see that the problem (5.1) has also a noncontinuable solution x on $[\sigma - r, \sigma + \beta)$ where α, β are fasten by the relation $\alpha = \int_\sigma^{\sigma+\beta} \frac{\mathrm{d}s}{\psi(s)}$.

An important criteria given by the following theorem assure the existence of global solutions of (5.1).

Theorem 5.2. *Assume the conditions of Theorem 5.1 are satisfied. Suppose further that*

(1) *for all* $(\tau, \phi) \in [\sigma, \sigma + b] \times \mathcal{C}$
$$|g(\tau, \phi)|_E \leq m(\tau) h(\|\phi\|),$$
where $m : [\sigma, \sigma + b] \to \mathbb{R}^+$ *and* $h : \mathbb{R}^+ \to \mathbb{R}^+_*$ *are continuous,* h *nondecreasing on* $\mathbb{R}^+$ *and* $\int_\sigma^{\sigma+b} \frac{m(s)}{\psi(s)}\, \mathrm{d}s < \int_{\|\varphi\|}^{+\infty} \frac{\mathrm{d}s}{h(s)}$, *or*

(2) *for all* $(\tau, \phi) \in [\sigma, \sigma + b] \times \mathcal{C}$
$$|g(\tau, \phi)|_E \leq h(\tau, |\phi(0)|_E),$$
where $h : [\sigma, \sigma + b] \times \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous function and the set of solutions to the singular ordinary differential equation*
$$\begin{cases} \psi(\tau) y'(\tau) = h(\tau, y(\tau)) \\ y(\lambda) = \mu \end{cases}$$
is bounded on $C([\lambda, \sigma + b]; \mathbb{R})$, *or*

(3) (ξ) *there are three functions* $V \in C([\sigma, \sigma + b] \times \mathcal{C}; \mathbb{R}^+)$, $a_1, a_2 \in C(\mathbb{R}^+; \mathbb{R}^+)$ *with* $\lim_{s \to +\infty} a_1(s) = +\infty$ *and* $a_1(\|\phi\|) \leq V(t, \phi) \leq a_2(\|\phi\|)$ *for all* $(t, \phi) \in [\sigma, \sigma + b] \times \mathcal{C}$
(ξ') *for any* $0 < \beta \leq b$ *and for any solution* x *of* (5.1) *on* $[\sigma - r, \sigma + \beta)$, *we have for all* $t \in (\sigma, \sigma + \beta)$
$$\begin{aligned} D^+ V(t, x_t(\sigma, \varphi)) := {} & \limsup_{k \to 0^+} \frac{1}{k} [V(t + k, x_{t+k}(t, \varphi)) - V(t, x_t(\sigma, \varphi))] \\ & \leq [\psi(t)]^{-1} h(t, V(t, x_t(\sigma, \varphi)) \end{aligned}$$
where $h : [\sigma, \sigma + b] \times \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous function and the set of solutions of the singular ordinary differential equation*
$$\begin{cases} \psi(t) y'(t) = h(t, y(t)) \\ y(\lambda) = \mu \end{cases}$$
is bounded on $C([\lambda, \sigma + b]; \mathbb{R})$.
Then (5.1) *has at least one global solution on* $[\sigma - r, \sigma + b]$.

Proof.

– Suppose that the condition (1) is verified.
Let x (resp. u) be a solution of (5.1) (resp. (5.2)) on $[\sigma - r, \sigma + \beta_1)$ (resp. on $[\sigma - r, \sigma + \alpha_1)$) with $\alpha_1 = \int_{\sigma}^{\sigma+\beta_1} \frac{\mathrm{d}s}{\psi(s)}$. By using the same argument seen in the Theorem 4.2, we obtain

$$\lim_{\tau \to \sigma+\beta_1} x(\tau) = \lim_{t \to \sigma+\alpha_1} u(t) \text{ exist.}$$

Take

$$x^1(\tau) = \begin{cases} x(\tau) & \text{if } \tau \in [\sigma - r, \sigma + \beta_1) \\ \lim\limits_{\tau' \to \sigma+\beta_1} x(\tau') & \text{if } \tau = \sigma + \beta_1. \end{cases}$$

Then x^1 is a solution to (5.1) on $[\sigma - r, \sigma + \beta_1]$. If $\beta_1 < b$, consider the problem

$$\begin{cases} \psi(\tau)x'(\tau) = g(\tau, x_\tau), & \tau \in [\sigma + \beta_1, \sigma + b] \\ x_{\sigma+\beta_1} = x^1_{\sigma+\beta_1}, \end{cases}$$

then this problem has a solution x^2 on $[\sigma + \beta_1 - r, \sigma + \beta_1 + \beta_2]$. Define

$$z(t) = \begin{cases} x^1(t) & \text{if} \quad t \in [\sigma - r, \sigma + \beta_1] \\ x^2(t) & \text{if} \quad t \in [\sigma + \beta_1, \sigma + \beta_1 + \beta_2]. \end{cases}$$

Then z is a solution of (5.1) on $[\sigma - r, \sigma + \beta_1 + \beta_2]$. Repeating this method, we can get a global solution of (5.1) on $[\sigma - r, \sigma + b]$.

– Suppose that the condition (2) is verified.
Let x be a solution of (5.1) on $[\sigma - r, \sigma + \beta_1)$. Take $m(\tau) = |x(\tau)|_E$, $\tau \in [\sigma, \sigma + \beta)$, then $m(\sigma) \leq \|\varphi\| = y(\sigma)$ (with $\mu = \|\varphi\|$) and for all $\tau \in (\sigma, \sigma + \beta)$

$$\psi(\tau)D^+m(\tau) := \psi(\tau)\limsup_{k \to 0^+} \frac{m(\tau + k) - m(\tau)}{k} \leq \psi(\tau)\,|x'(\tau)|_E \leq h(\tau, m(\tau)).$$

Consequently, by [**10**] we obtain $m(\tau) \leq y_{\max}(\tau)$ (where $y_{\max}$ is a maximal solution of singular the ordinary differential equation). Let $M = \sup\{y_{\max}(\tau) : \tau \in [\sigma, \sigma + b]\}$. Note that $|x(\tau)|_E \leq M,\ \tau \in [\sigma, \sigma + \beta_1)$. We shall prove that $\lim\limits_{\tau \to \sigma+\beta_1} x(\tau)$ exists. For $\sigma < \tau < \tau' < \sigma + \beta$ we have

$$\begin{aligned} |x(\tau) - x(\tau')|_E &\leq \int_{\tau}^{\tau'} [\psi(s)]^{-1}\,|g(s, x_s)|_E\,\mathrm{d}s \\ &\leq \int_{\tau}^{\tau'} [\psi(s)]^{-1} h(s, |x(s)|_E)\mathrm{d}s \leq M' \int_{\tau}^{\tau'} [\psi(s)]^{-1}\mathrm{d}s, \end{aligned}$$

where $M' = \max\{h(s,t)\ : s \in [\sigma, \sigma + b],\ t \leq M\}$.

For any $\varepsilon > 0$, we can find $\eta > 0$ such that

$$\left| \int_{\tau}^{\tau'} \frac{\mathrm{d}s}{\psi(s)} \right| < \frac{\varepsilon}{M'}$$

whenever $|\tau-\tau'|<\eta$, now for any $\tau<\tau'$ such that $|\tau-(\sigma+\beta)|<\frac{\eta}{2}$ and $|\tau'-(\sigma+\beta)|<\frac{\eta}{2}$, then

$$|x(\tau)-x(\tau')|_E \leq M' \int_{\tau}^{\tau'} [\psi(s)]^{-1}\mathrm{d}s < \varepsilon.$$

The rest of the proof is identical to the condition (1).

– Suppose that the condition (3) is verified.
  Let $m(t)=V(t,x_t)$. By (ξ) we obtain

$$m(\sigma) \leq a_2(\|x_\sigma\|) = a_2(\|\varphi\|) := y(\sigma)$$

and by (ξ') we have

$$\psi(t)D^+m(t) \leq h(t,m(t)), \qquad t\in(\sigma,\sigma+\beta).$$

Then, $m(t)\leq y_{\max}(t)$ where $y_{\max}$ is a maximal solution of the singular ordinary differential equation. Hence

$$a_1(\|x_t\|) \leq V(t,\phi) = m(t) \leq y_{\max}(t) \leq M.$$

But $\lim\limits_{s\to+\infty} a_1(s)=+\infty$, there exists $M'>0$ such that $M<a_1(M')$, so $\|x_t\|\leq M'$. Let $M_1=\sup\{|g(t,\varphi)|_E : t\in(\sigma,\sigma+\beta),\ \|\varphi\|<M'\}$, then the rest of the proof is similar to the condition (2). ☐

Theorem 5.3. *Consider the initial value problem for singular functional differential equations*

$$\text{(5.3)} \qquad \begin{cases} \psi(\tau)x^{(n)}(\tau) = g(\tau,x_\tau,x'_\tau,\ldots,x_\tau^{(n-1)}), & \tau\in(\sigma,\sigma+b],\ (b>0) \\ x_\sigma=\varphi\in C^{(n-1)}([-r,0];E), \end{cases}$$

where $g:[\sigma,\sigma+b]\times\mathcal{C}^n\to E$ is completely continuous, $\psi:[\sigma,\sigma+b]\to\mathbb{R}^+$ is continuous, $\psi>0$ on $(\sigma,\sigma+b]$ and $a:=\int_\sigma^{\sigma+b}\frac{\mathrm{d}s}{\psi(s)}<+\infty$. Then, (5.3) has at least one noncontinuable solution.

Proof. Let ρ and γ be the functions defined in Theorem 5.1. For all $\tau\in[\sigma,\sigma+b]$ and $\theta\in[-r,0]$, put

$$u(\rho(\tau+\theta)) = \begin{pmatrix} u^1(\rho(\tau+\theta)) \\ u^2(\rho(\tau+\theta)) \\ \vdots \\ u^n(\rho(\tau+\theta)) \end{pmatrix} = \begin{pmatrix} x(\tau+\theta) \\ x'(\tau+\theta) \\ \vdots \\ x^{(n-1)}(\tau+\theta) \end{pmatrix}.$$

Using the same technique as in the proof of Theorem 5.1, the problem (5.3) becomes equivalent to the following problem

$$\text{(5.4)} \qquad \begin{cases} u'(t) = F_1(t,u\circ\gamma(t,\cdot)) + F_2(t,u\circ\gamma(t,\cdot)), & t\in(\sigma,\sigma+a] \\ u_\sigma=(\varphi,\varphi',\ldots,\varphi^{(n-1)}), \end{cases}$$

where $F_1, F_2 : [\sigma, \sigma + a] \times C([-r, 0]; E^n) \to E^n$ are defined by

$$F_1(t,\phi) = \begin{pmatrix} \psi(\rho^{-1}(t))\phi_2(0) \\ \vdots \\ \psi(\rho^{-1}(t))\phi_n(0) \\ 0 \end{pmatrix}, \qquad F_2(t,\phi) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ g(\rho^{-1}(t), \phi_1, \phi_2, \dots, \phi_n) \end{pmatrix}$$

with $\phi_1, \phi_2, \dots, \phi_n$ which are the components of $\phi \in C([-r, 0]; E^n)$. It is easy to see that $F_1(t, \cdot)$ is Lipschitz and F_2 is completely continuous, then $F = F_1 + F_2$ verifies the condition (l) and by Theorem 4.1, the problem (5.4) has at least one noncontinuable solution and therefore, the problem (5.3) has also at least a noncontinuable solution. □

Theorem 5.4. *Consider the initial value problem for singular functional equations*

$$(5.5) \qquad \begin{cases} \psi(\tau)x'(\tau) = g(\tau, x(\tau - r_1), \dots, x(\tau - r_m)), & \tau \in]\sigma, \sigma + b], \ (b > 0) \\ x_\sigma = \varphi & \text{on } [-r, 0] \text{ with} \\ & r = \max_{1 \le i \le m} (r_i), \ r_i \ge 0 \end{cases}$$

where $g : [\sigma, \sigma + b] \times E^2 \to E$ is completely continuous, $\psi : [\sigma, \sigma + b] \to \mathbb{R}^+$ is continuous, $\psi > 0$ on $(\sigma, \sigma + b]$ and $a := \int_\sigma^{\sigma+b} \frac{ds}{\psi(s)} < +\infty$.
Then, (5.5) has at least one noncontinuable solution.

Proof. Using the same argument as in the proof of Theorem 4.1, we can see that the problem (5.5) is equivalent to the following problem

$$(5.6) \qquad \begin{cases} u'(t) = f(t, u \circ \gamma(t, \cdot)), & t \in (\sigma, \sigma + a] \\ u_\sigma = \varphi, \end{cases}$$

where $f(t, \phi) = g(\rho^{-1}(t), \phi(-r_1), \dots, \phi(-r_m))$. □

Theorem 5.5. *Consider the initial value problem for singular functional equations*

$$(5.7) \qquad \begin{cases} \psi(\tau)x''(\tau) = g(\tau, x(\tau), x(\tau - r_1), x'(\tau), x'(\tau - r_2)), & \tau \in (\sigma, \sigma + b] \\ x_\sigma = \varphi, \ x'_\sigma = \varphi', & \text{on } [-r, 0] \text{ with} \\ & r = \max(r_1, r_2) \end{cases}$$

where $g : [\sigma, \sigma + b] \times E^2 \to E$ is completely continuous, $\psi : [\sigma, \sigma + b] \to \mathbb{R}^+$ is continuous, $\psi > 0$ on $(\sigma, \sigma + b]$ and $a := \int_\sigma^{\sigma+b} \frac{ds}{\psi(s)} < +\infty$.
Then, (5.7) has at least one noncontinuable solution.

Proof. The problem (5.7) is equivalent to the following problem

$$(5.8) \qquad \begin{cases} u'(t) = F_1(t, u \circ \gamma(t, \cdot)) + F_2(t, u \circ \gamma(t, \cdot)), & t \in (\sigma, \sigma + a] \\ u_\sigma = \Phi & \text{with } \Phi = (\varphi, \varphi'), \end{cases}$$

where $F_1, F_2 : [\sigma, \sigma + a] \times C([-r, 0]; E^2) \to E^2$ are defined by

$$F_1(t, \phi) = \begin{pmatrix} \psi(\rho^{-1}(t))\phi_2(0) \\ 0 \end{pmatrix}$$

and

$$F_2(t, \phi) = \begin{pmatrix} 0 \\ g(\rho^{-1}(t), \phi_1(0), \phi_1(-r_1)\phi_2(0), \phi_2(-r_2)) \end{pmatrix}$$

with $\phi : \theta \in [-r, 0] \to \phi(\theta) = (\phi_1(\theta), \phi_2(\theta))$, $\phi_1, \phi_2 \in \mathcal{C}$. $\square$

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