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SYMMETRIC BOOLEAN ALGEBRAS

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Abstract. In order to study Boolean algebras in the category of vector spaces we introduce a prop whose algebras in set are Boolean algebras. A probabilistic logical interpretation for linear Boolean algebras is provided. An advantage of defining Boolean algebras in the linear category is that we are able to study its symmetric powers. We give an explicit formulae for products in symmetric and cyclic Boolean algebras of various dimensions and formulate symmetric forms of the inclusion-exclusion principle.

Introduction

Fix k a field of characteristic zero. A fundamental fact is the existence of the functor

$$\overline{(\ )} : \mathrm{Set} \longrightarrow \mathrm{Vect},$$

from the category of sets to the category of k-vector spaces, that sends x into $\overline{x}$ the free k-vector space generated by x, and sends a map $f : x \to y$ to the linear transformation $\overline{f} : \overline{x} \to \overline{y}$ whose value at $i \in x$ is $f(i)$. Notice that both Set and Vect are symmetric monoidal categories with coproducts and that $\overline{(\ )}$ is a monoidal functor that respects coproducts. The monoidal structure on Set is the Cartesian product $\times$ and the coproduct is the disjoint union $\sqcup$. The monoidal structure on Vect is the tensor product $\otimes$ and the coproduct is the direct sum $\oplus$. Notice also that the restricted functor $\overline{(\ )} : \mathrm{set} \to \mathrm{vect}$ from finite sets to finite dimensional vector spaces is such that the dimension $\dim(\overline{x})$ of $\overline{x}$ is equal to the cardinality $|x|$ of x. Using $\overline{(\ )}$ one can transform (combinatorial) set theoretical notions into (finite dimensional) linear algebra notions. For example, if x is a monoid, then $\overline{x}$ carries the structure of an associative algebra. Similarly, if x is a group, then $\overline{x}$ carries a structure of a Hopf algebra. Thus associative algebras and Hopf algebras are the linear analogues of monoids and groups, respectively.

Our main goal in this work is to uncover the linear analogue for Boolean algebras, i.e. we propose an answer to the question: what is the natural algebraic structure on $\overline{B}$ if B is a Boolean algebra? Boolean algebras [**5, 17, 20**] has been known at least since 1854 and constitute a cornerstone of modern mathematics.

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For most mathematicians the word algebra implies a linear structure, a property that is not present in the traditional definition of Boolean algebras. In this work the the presence or absence of a linear structure is the most important issue, thus we call our objects of study linear Boolean algebras to distinguish them from proper Boolean algebras. Thus by definition if B is a Boolean algebra, then $\overline{B}$ is a linear Boolean algebra. Our second goal in this work is to study the symmetric powers of linear Boolean algebras. We compute the structural constants of such algebras in various dimensions, and show that each symmetric function can be used for formulating generalization of the inclusion-exclusion principle for the symmetric powers of linear Boolean algebra. Our third goal is to propose a logical interpretation for linear Boolean algebras.

This work is organized as follows. In Section 1 we introduce the axioms for linear Boolean algebras and show that the linear span of a Boolean algebra is a linear Boolean algebra. The main difficulty lies in choosing the structural operations present in linear Boolean algebras. In Section 2 we motivate our choice of axioms for linear Boolean algebras. What we do is to construct a prop $\overline{\text{Boole}}$ such that $\overline{\text{Boole}}$-algebras in Vect are linear Boolean algebras. The prop $\overline{\text{Boole}}$ is the linear span of the prop Boole in Set, and one can show that Boole-algebras in Set are precisely Boolean algebras. Once we have a solid definition of linear Boolean algebras we proceed to study some of the properties of this kind of mathematical entities. In Section 3 we discuss the logical interpretation of linear Boolean algebras. We show that they are naturally related to probabilistic logic. The advantage of working in the linear category is that we can make use of many powerful techniques available in linear algebra. In Section 4 and 5 we apply Polya functors as defined in [**10**] to linear Boolean algebras, in particular, we study symmetric and cyclic powers of linear Boolean algebras. In Section 6 we close providing an extension of the inclusion-exclusion principle that applies to the symmetric products of Boolean algebras.

1. Linear Boolean algebras

We recall the definition of Boolean algebras for definiteness and for the reader convenience, so that he or she may contrast it with the definition of linear Boolean algebras given below. We have chosen axioms that make transparent that Boolean algebras are a particular kind of lattices. Thus a linear analogue for lattices can be readily obtained from the definition of linear Boolean algebras given below. The reader should notice that while the definition of Boolean algebras involve five structural maps, the definition of linear Boolean algebras involve seven structural maps, including quite unexpectedly, a coproduct.

A Boolean algebra is a set B together with the following data:

1. Maps $\cup : B \times B \to B$, $\cap : B \times B \to B$, and $c : B \to B$ called a union, an intersection and a complement, respectively.
2. Distinguished elements $e, t \in B$ called the empty and total elements, respectively.

This data should satisfy the following identities for $a, b, c \in B$:

- $a \cup b = b \cup a$, $a \cap b = b \cap a$.
- $a \cup (b \cup c) = (a \cup b) \cup c$, $a \cap (b \cap c) = (a \cap b) \cap c$.
- $a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$, $a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$.
- $a \cup (a \cap b) = a$, $a \cap (a \cup b) = a$.
- $a \cup e = a$, $a \cap t = a$, $a \cup a^c = t$, $a \cap a^c = e$.

To any set x one can associate the Boolean algebra $P(x) = \{a \mid a \subseteq x\}$ where the total element is x and the empty element is $\emptyset$, $a \cup b = \{i \in x \mid i \in a \text{ or } i \in b\}$, $a^c = \{i \in x \mid i \notin a\}$, and $a \cap b = \{i \in x \mid i \in a \text{ and } i \in b\}$. Let $[n] = \{1, \dots, n\}$ and S_n be the group of permutations of $[n]$. We will always write $P[n]$ instead of $P([n])$. Algebras of the form $P(x)$ are essentially the unique models of finite Boolean algebras according to the following well-known result.

Proposition 1. *Every finite Boolean algebra is isomorphic to $P(x)$ for a finite set x.*

Indeed let B be a Boolean algebra. Define a partial order $\leq$ on B by letting $a \leq b$ if $a \cap b = a$. Let x be the set of primitive elements or atoms of B, that is, we have

$$x = \{a \in A \mid a \neq e \text{ and if } b \leq a \text{ then } b = e \text{ or } b = a\}.$$

The map $f : B \to P(x)$ given by $f(b) = \{a \in x \mid a \leq b\}$ defines the desired isomorphism.

Another interesting property of Boolean algebras is the following: if B and C are Boolean algebras, then $B \times C$ is also a Boolean algebra. Moreover one can show that $P(x)$ is isomorphic to $P[1]^{|x|}$.

For a k-vector space V we shall use the symmetry map $S : V \otimes V \to V \otimes V$ given by $S(x \otimes y) = y \otimes x$ for $x, y \in V$. We denote the identity map by $I : V \to V$. We are ready to define the linear analogue of the notion of Boolean algebras.

Definition 2. A linear Boolean algebra is a k-vector space V together with the data:

1. Linear maps $\cup : V \otimes V \to V$, $\cap : V \otimes V \to V$, and $c : V \to V$ called a union, an intersection and a complement, respectively.
2. Linear maps $T : k \to V$, $E : k \to V$ called the empty map and the total map, respectively.
3. Linear map $\triangle : V \to V \otimes V$ called a coproduct.
4. Linear map $ev : V \to k$ called the evaluation map.

The axioms below must hold:

- $\cup = \cup \circ S$, $\cap = \cap \circ S$.
- $\cup \circ (\cup \otimes I) = \cup \circ (I \otimes \cup)$, $\cap \circ (\cap \otimes I) = \cap \circ (I \otimes \cap)$.
- $\cap \circ (I \otimes \cup) = \cup \circ (\cap \otimes \cap) \circ (I \otimes S \otimes I) \circ (\triangle \otimes I \otimes I)$,
  $\cup \circ (I \otimes \cap) = \cap \circ (\cup \otimes \cup) \circ (I \otimes S \otimes I) \circ (\triangle \otimes I \otimes I)$.
- $\cap \circ (I \otimes \cup) \circ (\triangle \otimes I) = I \otimes ev$, $\cup \circ (I \otimes \cap) \circ (\triangle \otimes I) = I \otimes ev$.
- $\cup \circ (I \otimes E) = I$, $\cap \circ (I \otimes T) = I$,
  $\cap \circ (I \otimes c) \circ \triangle = E \circ ev$, $\cup \circ (I \otimes c) \circ \triangle = T \circ ev$.
- $(\triangle \otimes I) \circ \triangle = (I \otimes \triangle) \circ \triangle$.

- $S \circ \triangle = \triangle$.

Our next result guarantees the existence of infinitely many models of linear Boolean algebras, namely those naturally associated with Boolean algebras.

Proposition 3. *If B is a Boolean algebra, then $\overline{B}$ is a linear Boolean algebra.*

Proof. The structural maps on $\overline{B}$ are given as follows. The intersection, the union and the complement are the linear extensions of the corresponding maps on B. The coproduct is given by:

$$\triangle\left(\sum_{a\in B} v_a a\right) = \sum_{a\in B} v_a a \otimes a.$$

The empty and total maps are given for $s \in k$ by $E(s) = se$ and $T(s) = st$. Finally, the evaluation map is given by $ev(\Sigma_{a\in B} v_a a) = \Sigma_{a\in B} v_a$. □

Next result characterizes finite dimensional linear Boolean algebras of the form $\overline{P(x)}$.

Proposition 4. *If V and W are linear Boolean algebras, then $V \otimes W$ is a linear Boolean algebra with the Boolean operations defined componentwise. Moreover, if x is a finite set then there is a canonical isomorphism of linear Boolean algebras $\overline{P(x)} \simeq \overline{P[1]}^{\otimes|x|}$.*

2. Boolean prop

In this section we provide an explanation for our choice of axioms for linear Boolean algebras. We do so by defining a prop $\overline{\text{Boole}}$ over Vect whose algebras are linear Boolean algebras and showing that this prop actually comes from a prop Boole over Set whose algebras are Boolean algebras. Discovering the prop that defines a given family of algebras is like unveiling its genetic code [**1, 13, 14, 15, 19**]. Despite the fact that Boolean algebras have been extensively studied from a myriad of viewpoints its genetic code has not been study so far. Since the theory of props is not widely known we provide a brief overview. We define props over a symmetric monoidal category C, but the reader should bear in mind that in this work C is either Set or Vect. We assume that C is closed and that it admits finite colimits.

Definition 5.
1. A prop over C is a symmetric monoidal category P enriched over C such that $\text{Ob}(P) = \mathbb{N}$ and the monoidal structure is the addition of natural numbers.
2. Let PROP_C be the category whose objects are props over C. Morphisms in PROP_C are monoidal functors.

Explicitly we have that a prop P is given by the following data:

- Morphisms $P(n,m) \otimes_C P(m,k) \to P(n,k)$ for $n,m,k \in \mathbb{N}$.
- Morphisms $P(n,m) \otimes_C P(k,l) \to P(n+k,m+l)$ for $n,m,k,l \in \mathbb{N}$.

- For $n \in \mathbb{N}$ a group morphism $S_n \to P(n,n)$ such that the following diagram

$$\begin{array}{ccc} S_n \times S_m & \longrightarrow & S_{n+m} \\ \downarrow & & \downarrow \\ P(n,n) \otimes_C P(m,m) & \longrightarrow & P(n+m, n+m) \end{array}$$

  is commutative. Notice that the map $S_n \to P(n,n)$ induces a right action of S_n on $P(n,m)$ and a left action of S_m on $P(n,m)$.

Let $\mathbb{B}$ be the category whose objects are finite sets and whose morphisms are bijections. The actions constructed above can be used to define a functor $P\colon \mathbb{B}^{op} \times \mathbb{B} \to C$ given by

$$P(a,b) = \mathbb{B}(a,[a]) \times_{S_{|b|}} P(|a|,|b|) \times_{S_{|b|}} \mathbb{B}([b],b),$$

where for a finite set x we define $[x] = \{1,\dots,|x|\}$.

In order to define the free prop generated by a functor $G : \mathbb{B}^{op} \times \mathbb{B} \to C$ we need some combinatorial notions. A digraph Γ consists of the following data

- A pair of finite sets (V,E) called the set of vertices and edges of Γ, respectively.
- A map $(s,t) : E \to V \times V$. We call $s(e)$ and $t(e)$ the source and target of $e \in V$, respectively.

We use the notations $\text{in}(v) = \{e \mid t(e) = v\}$, $i(v) = |\text{in}(v)|$, $\text{out}(v) = \{e \mid s(e) = v\}$, and $o(v) = |\text{out}(v)|$. The valence of $v \in V$ is $val(v) = (i(v), o(v)) \in \mathbb{N}^2$. Also we introduce the notation $V_{\text{in}} = \{v \in V \mid i(v) = 0\}$ and $V_{\text{out}} = \{v \in V \mid o(v) = 0\}$. An oriented cycle in Γ is a sequence $e_1,\dots,e_n$ of edges in Γ such that $t(e_i) = s(e_{i+1})$ for $1 \le i \le n-1$ and $t(e_n) = s(e_1)$. Digraphs considered in this work do not have oriented cycles.

Let a and b be finite sets. An (a,b)-digraph is a triple (Γ,α,β) such that Γ is a digraph; $\alpha : a \to V_{\text{in}}$ is an injective map; $\beta : b \to V_{\text{out}}$ is an injective map.

Let $DG(a,b)$ be the groupoid of (a,b)-digraphs. A functor $G : \mathbb{B}^{op} \times \mathbb{B} \to C$ induces a functor $G : DG(a,b) \to C$ given by

$$G(\Gamma) = \bigotimes_{v \in V_{int}} G(\text{in}(v), \text{out}(v)),$$

where Γ is an object of $DG(a,b)$ and $V_{\text{int}} = V \setminus (\alpha(a) \sqcup \beta(b))$.

Definition 6. The prop P_G freely generated by $G : \mathbb{B}^{op} \times \mathbb{B} \to C$ is given for $n,m \in \mathbb{N}$ by

$$P_G(n,m) := \varinjlim G(\Gamma)$$

where the colimit is taken over the groupoid $DG([n],[m])$, where $[0] = \emptyset$ and $[n] = \{1,\dots,n\}$ for $n \ge 1$. Compositions in P_G are given by gluing of digraphs.

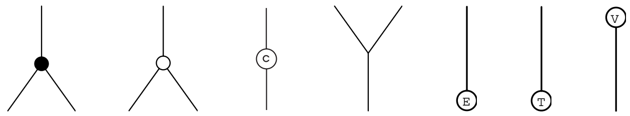
To define props via generators and relations we need to know what is the analogue of an ideal in the prop context.

Definition 7. Let P be a prop over C. A subcategory I of P is a prop ideal if $\mathrm{Ob}(I) = \mathrm{Ob}(P)$ and for $n, m, k, l \in \mathbb{N}$

$I(n,m) \otimes P(m,k) \to I(n,k)$, $\quad P(n,m) \otimes I(m,k) \to I(n,k)$,

$I(n,m) \otimes P(k,l) \to I(n \sqcup k, m \sqcup l)$, $\quad P(n,m) \otimes I(k,l) \to I(n \sqcup k, m \sqcup l)$.

We are ready to define a prop Boole over Set. Boole is a quotient by a prop ideal I_B, defined below, of the prop freely generated by vertices representing, respectively, a union, an intersection, a complement, a coproduct, the empty element, the total element and the valuation, respectively.



The prop ideal I_B is generated by the seven relations given below. Each relation corresponds with an axiom in the definition of linear Boolean algebras.

1. Commutativity for union and intersection

2. Associativity for union and intersection

3. Distributivity laws

4. Properties of the empty and total elements

5. Absorption Laws

6. Coassociativity and cocommutativity

Given an object x of a category C we let End_x^C be the prop given for $n, m \in \mathbb{N}$ by

$$\mathrm{End}_x^C(n,m) = C(x^{\otimes n}, x^{\otimes m}).$$

Let P be a prop over C. A P-algebra in C is a pair (x, r) where $r : P \to \mathrm{End}_x^C$ is a prop morphism and x is an object of C. In practice a P-algebra x is given by a family of morphisms in C

$$r : P(n,m) \to C(x^{\otimes n}, x^{\otimes m})$$

satisfying the natural compatibility conditions.

Theorem 8. *B is a* Boole*-algebra in* Set *if and only if B is a Boolean algebra.*

Proof. Assume that (B, r) is a Boole-algebra in Set where $r : \mathrm{Boole} \to \mathrm{End}_B^{\mathrm{Set}}$ is a prop morphism. The images under r of the generators of Boole give operations $\cup, \cap, (\ \ )^c, t, e, \triangle, ev$, respectively. For example $t : \{1\} \to B$ and $e : \{1\} \to B$ are identified with elements of B. $ev : B \to \{1\}$ is the constant map and plays no essential part in this story. We also get a map $\triangle : B \to B \times B$ which does seem to fit into the definition of Boolean algebra. Assume that $\triangle$ is given by $\triangle(a) = (f(a), g(a))$ for $a \in B$. We use the relations in Boole. The cocommutativity graph implies that $f = g$. The coassociativity graph implies that $f^2 = f$. One of the absorption graphs implies the identity $f(a) \cup (f(a) \cap b) = a$ for $a, b \in B$. Thus we obtain

$$f(a) = f^2(a) \cup (f^2(a) \cap b) = f(a) \cup (f(a) \cap b) = a.$$

Thus $\triangle(a) = (a, a)$ and it is a simple check that all other relations in Boole turn B into a Boolean algebra. Assume that B is a Boolean algebra with operations $\cup, \cap, (\ \ )^c$, and distinguished elements t and e that may be thought as maps from $\{1\}$ to B. Take ev to be the constant map from B to $\{1\}$ and let $\triangle$ be given by $\triangle(a) = (a, a)$. Let r be the map assigning to each generator of the Boole prop the corresponding map from the list above. The fact that B is a Boolean algebra guarantees that all the relations defining Boole are satisfied and r is extending to a prop morphism $r : \mathrm{Boole} \to \mathrm{End}_B^{\mathrm{Set}}$. □

Notice that the functor $\overline{(\ )}$: Set $\rightarrow$ Vect induces a functor $\overline{(\ )}$: $\mathrm{PROP}_{\mathrm{Set}} \rightarrow \mathrm{PROP}_{\mathrm{Vect}}$ sending P into P given by $\overline{P}(n,m) = \overline{P(n,m)}$ and with the induced compositions.

The following result follows from the fact that each generator of the prop Boole corresponds with an operation on linear Boolean algebras and each relation in the set of generator of the prop ideal I_B corresponds with an axiom in the definition of linear Boolean algebras.

Theorem 9. *V is a $\overline{\text{Boole}}$-algebra in* Vect *if and only if V is a linear Boolean algebra.*

3. Probabilistic logic and linear Boolean algebras

It is hard to do any work on Boolean algebras and not to mention its relation with classical propositional logic at all. Indeed the motivation of Boole himself was to describe the algebraic structures underlying the laws of thought. Propositional logic deals with the deduction relation among sets of sentences of propositions S constructed recursively from a finite set of propositions connected by a fixed set of connecting symbols. There are many ways [**18**] to describe a system of propositional logic but in any of them one can imagine that there exists a sort of logical agent capable of performing the following tasks:

- Recognize when a grammatical construction is an element of S. The agent is able to translate in S expressions of the form $s \vee t$, $s \wedge t$ into sentences, and $-s$ for sentences s and t in S.
- Decide wether or not a sequence of sets of sentences $c_1, \ldots, c_n$ is a deduction. A sentence s is said to imply a sentence t if there exists a deduction $c_1, \ldots, c_n$ such that $c_1 = \{s\}$ and $c_n = \{t\}$.
- Assign a truth-value to sentences in S when provided with an assignment of truth values for propositions in P, i.e. construct an element of $\{0,1\}^S$ given an element in $\{0,1\}^P$.

The logical agent is said to be sound and complete if in addition the following property holds:

- A sentence s implies a sentence t if for any assignments of truth values to propositions in P the truth value of t is 1 if the truth value of s is 1. It is not hard to show the existence of sound and complete logical agents, for example, see [**18**].

Boolean algebras appear within the context of propositional logic as follows. We call sentences s and t in S equivalent if s implies t and t implies s. Let $B(S)$ be the quotient of S by that equivalence relation. $B(S)$ comes equipped with a natural structure of Boolean algebra with operations defined by $[s] \cup [t] = [s \vee t]$, $[s] \cap [t] = [s \wedge t]$, and $[s]^c = [-s]$. The total element is $[s \vee -s]$ and the empty element is $[s \wedge -s]$. The Boolean algebra $B(S)$ is isomorphic to the Boolean algebra $P(\{0,1\}^C)$ via the map

$$m : B(S) \rightarrow P(\{0,1\}^P)$$

that sends $[s] \in B(S)$ into the set of its models:

$$m([s]) = \{v \in \{0,1\}^P \mid \text{ the truth value of } s \text{ according to } v \text{ is } 1\}.$$

Summarizing sentences in S describes subsets of $\{0,1\}^P$ and two sentences describe the same subset if and only if they are equivalent. The expressive power of a logical description of $P(\{0,1\}^P)$ lies in the possibility of describing the same set in a variety of different ways. For example, the logical agent may be said that a subset of $\{0,1\}^P$ is described by a sentence s, another subset of $\{0,1\}^P$ is described by a sentence t and be asked to provide a sentence which describes the union of those sets. It will readily answer that $s \vee t$ is the sought sentence.

It is natural to wonder if any logical meaning can be ascribed to the linear Boolean algebra $\overline{B(S)}$. We venture a possible answer: assume the logical agent is said that a sentence s_i describes an unknown subset of $\{0,1\}^C$ with probability p_i for $1 \leq i \leq n$ and a sentence t_j describes another unknown subset of $\{0,1\}^C$ with probability q_j for $1 \leq j \leq m$. If asked to find a sentence that describes the union of those subsets the logical agent will answer: the sentence $s_i \vee t_j$ describes the union of the unknown sets with probability $p_i q_j$. This is the only consistent answer with the product rules on $\overline{B(S)}$ which is given by

$$\left(\sum_{i=1}^{n} p_i[s_i]\right) \cup \left(\sum_{j=1}^{m} q_j[t_j]\right) = \sum_{i=1,j=1}^{n,m} p_i q_j [s_i \vee t_j].$$

This probabilistic interpretation applies as well to the linear Boolean algebra $\overline{B}$. Let v and w be a couple of vectors in $\overline{B}$ given by $v = \sum_{a\in B} v_a a$ and $w = \sum_{b\in B} v_b b$. Assume that the coefficients of v and w, respectively, are positive and add to one. This allows us to think that v_a represents the probability that the unknown subset v of x is equal to a. Similarly w_b represents the probability that w is equal to b. Under this conditions we have that

- The probability that $v \cup w$ is equal to c is given by $(v \cup w)_c = \sum\limits_{a\cup b=c} v_a w_b$.
- The probability that $v \cap w$ is equal to c is given by $(v \cap w)_c = \sum\limits_{a\cap b=c} v_a w_b$.
- The probability that v^c is equal to a is v_{a^c}.

We invite the reader to take a look at the structural coefficients of the algebras $Sym^2\overline{P[1]}$ and $\overline{P[1]}^{\otimes 3}/\mathbb{Z}_3$ given in Section 4 and Section 5 below, and check that they are indeed consistent with the probabilistic interpretation just outlined.

4. Symmetric powers of Boolean algebras

The following ideas introduced in [**10**] and further applied in [**7, 8, 11**] are useful for studying the symmetric powers of algebras. After a brief review of the general theory we shall apply it to study the symmetric powers of linear Boolean algebras.

Suppose that a group G acts by automorphisms on the k-algebra A. The space of co-invariants

$$A/G = A/\overline{\{ga - a \mid g \in G \text{ and } a \in A\}}$$

is a k-algebra with the product given by

$$\overline{a}\overline{b} = \frac{1}{|G|}\sum_{g\in G}\overline{a(gb)}.$$

For each subgroup $K \subset S_n$ the Polya functor P_K : k-alg$\rightarrow$ k-alg from the category of associative k-algebras into itself is defined as follows. If A is a k-algebra then $P_K A$ denotes the k-algebra whose underlying vector space is

$$P_K A = A^{\otimes n}/\langle a_1 \otimes \cdots \otimes a_n - a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(n)} : a_i \in A, \sigma \in K\rangle.$$

The rule for the product of m elements in $P_K A$ is provided by our next result.

Theorem 10. *For any* $\{a_{ij}\}_{i=1,j=1}^{m,n} \subseteq A$ *the following identity holds in* $P_K A$*:*

$$|K^{m-1}|\prod_{i=1}^{m}\left(\overline{\bigotimes_{j=1}^{n} a_{ij}}\right) = \sum_{\sigma\in\{id\}\times K^{m-1}}\overline{\bigotimes_{j=1}^{n}\left(\prod_{i=1}^{m} a_{i\sigma_i^{-1}(j)}\right)}.$$

In particular for each algebra A and each positive integer m the Polya functor P_{S_n} yields an algebra $P_{S_n}A$ which we denote by $S^n A$. Recall that $\overline{P[k]}$ denotes the k-vector space generated by the subsets of $[k]$. The structural maps $\cup, \cap$, and $(\ )^c$ for $\overline{P[k]}$ are the linear extensions of the union, intersection, and complement on $P[k]$.

Definition 11. We call $S^m\overline{P[k]}$ the symmetric Boolean algebra of type (m,k). The structural operations on $S^m\overline{P[k]}$ are induced from the corresponding operations on $\overline{P[k]}$.

The group S_x acts by automorphisms on $\overline{P(x)}$ for any finite set x. The next result gives a characterization of the algebra of co-invariants $\overline{P(x)}/S_x$.

Proposition 12. *We have that* $\overline{P(x)}/S_x \simeq S^{|x|}\overline{P[1]}$ *and therefore*

$$\dim(\overline{P(x)}/S_x) = |x| + 1.$$

A basis for $\overline{P[k]}/S_k$ is given by $\mathbf{0},\ldots,\mathbf{k}$ where $\mathbf{i}$ denotes the equivalence class of $[i] \subseteq [k]$. Let us study the operation of union, intersection, and complements on the space $\overline{P[k]}/S_k$ in details. Below we use the notation $P(x,k) = \{c \in P(x) \mid |c| = k\}$ for any set x.

Theorem 13. *For* $0 \leq a, b, \leq k$*, the following identities hold in* $\overline{P[k]}/S_k$*. Let* $m = \min(k-a, b)$*, then*

$$\mathbf{a}\cup\mathbf{b} = \frac{1}{\binom{k}{b}}\sum_{l=0}^{m}\binom{a}{b-l}\binom{k-a}{l}(\mathbf{a}+\mathbf{l}).$$

Let $m = \min(a,b)$*, then*

$$\mathbf{a}\cap\mathbf{b} = \frac{1}{\binom{k}{b}}\sum_{l=0}^{m}\binom{a}{l}\mathbf{l}.$$

Also we have that $\mathbf{a^c} = \mathbf{k} - \mathbf{a}$.

Proof. For the first identity we have that

$$\begin{aligned}\mathbf{a} \cup \mathbf{b} &= \frac{1}{k!} \sum_{\sigma \in S_k} \overline{[a] \cup \sigma[b]} = \frac{1}{\binom{k}{b}} \sum_{c \in P([k],b)} \overline{[a] \cup c} \\ &= \frac{1}{\binom{k}{b}} \sum_{\substack{c_0 \in P([k]\setminus[a],l) \\ c_1 \in P([a],b-l)}} \overline{[a] \cup c} = \frac{1}{\binom{k}{b}} \sum_{\substack{c_0 \in P([k]\setminus[a],l) \\ c_1 \in P([a],b-l)}} \overline{[a] \cup c_0} \\ &= \frac{1}{\binom{k}{b}} \sum_{l=0}^{m} \binom{a}{b-l} \binom{k-a}{l} (\mathbf{a} + \mathbf{l}).\end{aligned}$$

The second identity follows from the fact that the number of permutations $\sigma \in S_k$ such $|[a] \cap \sigma[b]| = l$ is given by

$$\binom{a}{l}\binom{b}{l} l! \binom{k-a}{b-l} (b-l)!(k-b)!$$

The third identity is obvious. □

Let $\pi = \{b_1, \ldots, b_k\}$ be a partition of x and $S_\pi \subseteq S_x$ the Young subgroup consisting of block preserving permutations of x. Our next result characterizes algebras of the form $\overline{P(x)}/S_\pi$.

Proposition 14. *There is an isomorphism* $\overline{P(x)}/S_\pi \simeq \bigotimes_{i=1}^{k} S^{|b_i|}\overline{P[1]}$, *thus we have that*

$$\dim(\overline{P(x)}/S_\pi) = \prod_{i=1}^{k} (|b_i| + 1).$$

5. Cyclic Boolean Algebras

In this section we consider another application of Polya functors in the context of linear Boolean algebra, namely, we consider the cyclic powers of the linear Boolean algebra $\overline{P[1]}$, i.e. the algebras

$$\overline{P[1]}^{\otimes m}/\mathbb{Z}_m.$$

For $m = 2$ one gets the space $\overline{P[1]}^{\otimes 2}/\mathbb{Z}_2$ which has a basis given by $\overline{(0,0)}, \overline{(1,0)}$ and $\overline{(1,1)}$.

The union map $\cup : S^2\overline{P[1]} \otimes S^2\overline{P[1]} \to S^2\overline{P[1]}$ is given by

$\cup$	$\overline{(0,0)}$	$\overline{(1,0)}$	$\overline{(1,1)}$
$\overline{(0,0)}$	1,0,0	0,1,0	0,0,1
$\overline{(1,0)}$	0,1,0	$0, \frac{1}{2}, \frac{1}{2}$	0,0,1
$\overline{(1,1)}$	0,0,1	0,0,1	0,0,1

The intersection $\cap : S^2\overline{P[1]} \otimes S^2\overline{P[1]} \to S^2\overline{P[1]}$ is given by

$\cap$	$\overline{(0,0)}$	$\overline{(1,0)}$	$\overline{(1,1)}$
$\overline{(0,0)}$	1,0,0	1,0,0	1,0,0
$\overline{(1,0)}$	1,0,0	$\frac{1}{2}, \frac{1}{2}, 0$	0,1,0
$\overline{(1,1)}$	1,0,0	0,1,0	0,0,1

The complement $(\ \ )^c : S^2\overline{P[1]} \to S^2\overline{P[1]}$ is given by

$\cap$	$\overline{(0,0)}$	$\overline{(1,0)}$	$\overline{(1,1)}$
	0,0,1	0,1,0	1,0,0

Although the algebra $S^2\overline{P[1]}$ does not satisfy all the axioms required to make it into a linear Boolean algebra (the absorption laws fail!) it does share many properties of linear Boolean algebras, and in any case it is a mathematical object of great interest. For $m = 3$ the space $\overline{P[1]}^{\otimes 3}/\mathbb{Z}_3$ has the basis

$$\overline{(0,0,0)}, \quad \overline{(1,0,0)}, \quad \overline{(1,1,0)} \quad \text{and} \quad \overline{(1,1,1)}.$$

The union map $\cup : \overline{P[3]}^{\otimes 3}/\mathbb{Z}_3 \otimes \overline{P[3]}^{\otimes 3}/\mathbb{Z}_3 \to \overline{P[3]}^{\otimes 3}/\mathbb{Z}_3$ is given by

$\cup$	$\overline{(0,0,0)}$	$\overline{(1,0,0)}$	$\overline{(1,1,0)}$	$\overline{(1,1,1)}$
$\overline{(0,0,0)}$	1,0,0,0	0,1,0,0	0,0,1,0,	0,0,0,1
$\overline{(1,0,0)}$	0,1,0,0	$0, \frac{1}{3}, \frac{2}{3}, 0$	$0, 0, \frac{2}{3}, \frac{1}{3}$	0,0,0,1
$\overline{(1,1,0)}$	0,0,1,0	$0, 0, \frac{2}{3}, \frac{1}{3}$	$0, 0, \frac{1}{3}, \frac{2}{3}$	0,0,0,1
$\overline{(1,1,1)}$	0,0,0,1	0,0,0,1	0,0,0,1	0,0,0,1

The intersection map $\cap : \overline{P[3]}^{\otimes 3}/\mathbb{Z}_3 \otimes \overline{P[3]}^{\otimes 3}/\mathbb{Z}_3 \to \overline{P[3]}^{\otimes 3}/\mathbb{Z}_3$ is given by

$\cap$	$\overline{(0,0,0)}$	$\overline{(1,0,0)}$	$\overline{(1,1,0)}$	$\overline{(1,1,1)}$
$\overline{(0,0,0)}$	1,0,0,0	1,0,0,0	1,0,0,0	1,0,0,0
$\overline{(1,0,0)}$	1,0,0,0	$\frac{2}{3},\frac{1}{3},0,0$	$\frac{1}{3},\frac{2}{3},0,0$	0,1,0,0
$\overline{(1,1,0)}$	1,0,0,0	$\frac{1}{3},\frac{2}{3},0,0$	$0,\frac{2}{3},\frac{1}{3},0$	0,0,1,0
$\overline{(1,1,1)}$	1,0,0,0	0,1,0,0	0,0,1,0	0,0,0,1

The complement map $(\ \ )^c : \overline{P[3]}^{\otimes 3}/\mathbb{Z}_3 \to \overline{P[3]}^{\otimes 3}/\mathbb{Z}_3$ is given by

$(\ \ )^c$	$\overline{(0,0,0)}$	$\overline{(1,0,0)}$	$\overline{(1,1,0)}$	$\overline{(1,1,1)}$
	0,0,0,1	0,0,1,0	0,1,0,0	1,0,0,0

6. Symmetric inclusion-exclusion principles

Perhaps the most fundamental elementary result concerning Boolean algebras is the inclusion-exclusion principle. In this section we consider the extensions of this principle for linear Boolean algebras. The reader will find interesting information on the inclusion-exclusion principle and its generalizations in several works by Rota and his collaborators [**16**]. We use the inclusion-exclusion principle in the following form:

Proposition 15. *Let $a_1, \ldots, a_n \in P(x)$, then*

$$\left|\bigcup_{i=1}^{n} a_i\right| = \sum_{I \subseteq [n]} (-1)^{|I|+1} \left|\bigcap_{i \in I} a_i\right|.$$

In this section we consider vector spaces over the complex numbers and we write $\{a_1, \ldots, a_m\}$ for the basis element

$$\overline{a_1 \otimes \cdots \otimes a_m} \in S^m\overline{P[k]} = \overline{P[k]}^{\otimes m}/S_m.$$

The following result follows from Theorem 10.

Theorem 16. *Let $\{a_1^i, \ldots, a_m^i\}$ be in the basis of $S^m\overline{P[k]}$ for $1 \leq i \leq n$. The union map on $S^m\overline{P[k]}$ is given by*

$$\bigcup_{i=1}^{n}\{a_1^i, \ldots, a_m^i\} = \frac{1}{(m!)^{n-1}} \sum_{\sigma \in \{1\} \times S_m^{(n-1)}} \left\{\bigcup_{i=1}^{n} a^i_{\sigma_i(1)}, \ldots, \bigcup_{i=1}^{n} a^i_{\sigma_i(m)}\right\}.$$

For example for $m, n = 2$, one gets

$$\{a,b\} \cup \{c,d\} = \frac{1}{2}\{a \cup c, b \cup d\} + \frac{1}{2}\{a \cup d, b \cup c\}.$$

Recall that a measure on a finite set x is a map $\mu : P(x) \to \mathbb{C}$ such that

$$\mu(a \cup b) = \mu(a) + \mu(b)$$

for $a, b \subseteq x$ disjoint. Let us fix a measure μ on $[k]$. An element $\{a_1, ..., a_m\}$ in the basis of $S^m\overline{P[k]}$ determines a vector $(\mu(a_1), \dots, \mu(a_m)) \in \mathbb{C}^m/S_m$. Functions on $\mathbb{C}^m/S_m$ are known as symmetric functions. There are many interesting examples of polynomial symmetric functions such as the power functions, the elementary symmetric functions, the homogeneous functions, the Schur functions and so on. For example, the polynomial $x_1^l + \cdots + x_m^l$ is S_m-invariant. Each symmetric function can be used to obtain a symmetric form of the inclusion-exclusion principle. We consider explicitly the symmetric inclusion-exclusion principles derived from the power, elementary, and homogeneous symmetric functions; other symmetric functions may be considered as well but we shall not do so here. Notice that Gessel [**12**] uses the name symmetric inclusion-exclusion to refer to a different mathematical gadget.

The power function $p_l : S^m\overline{P[k]} \to \mathbb{C}$ is given on the basis elements by:

$$p_l(\{a_1, \dots, a_m\}) = \sum_{i=1}^{m} \mu(a_i)^l.$$

We use the power functions p_l to get a symmetric form of the inclusion-exclusion principle.

Theorem 17. *Let $\{a_1^i, \dots, a_m^i\}$ be in the basis of $S^m\overline{P[k]}$ for $1 \leq i \leq n$. Then*

$$p_l(\bigcup_{i=1}^{n}\{a_1^i, \dots, a_m^i\}) = \frac{1}{(m!)^{n-1}} \sum_{\substack{\sigma\in\{1\}\times S_m^{(n-1)} \\ j\in\{1,\dots,m\} \\ \Sigma c_I = l}} \binom{l}{\{c_I\}} \prod_{I\subseteq[n]} (-1)^{(|I|+1)c_I} \mu(\bigcap_{i\in I} a^i_{\sigma_i(j)})^{c_I}.$$

Proof.

$$\begin{aligned}
p_l(\bigcup_{i=1}^{n}\{a_1^i, \dots, a_m^i\}) &= \frac{1}{(m!)^{n-1}} \sum_{\substack{\sigma\in\{1\}\times S_m^{(n-1)} \\ j\in\{1,\dots,m\}}} \mu(\bigcup_{i=1}^{n} a^i_{\sigma_i(j)})^l \\
&= \frac{1}{(m!)^{n-1}} \sum_{\substack{\sigma\in\{1\}\times S_m^{(n-1)} \\ j\in\{1,\dots,m\}}} \Big(\sum_{I\subseteq[n]} (-1)^{|I|+1} \mu(\bigcap_{i\in I} a^i_{\sigma_i(j)})\Big)^l \\
&= \frac{1}{(m!)^{n-1}} \sum_{\substack{\sigma\in\{1\}\times S_m^{(n-1)} \\ j\in\{1,\dots,m\}}} \Big(\sum_{\Sigma c_I = l} \binom{l}{\{c_I\}} \prod_{I\subseteq[n]} [(-1)^{|I|+1} \mu(\bigcap_{i\in I} a^i_{\sigma_i(j)})]^{c_I}\Big) \\
&= \frac{1}{(m!)^{n-1}} \sum_{\substack{\sigma\in\{1\}\times S_m^{(n-1)} \\ j\in\{1,\dots,m\} \\ \Sigma c_I = l}} \binom{l}{\{c_I\}} \prod_{I\subseteq[n]} (-1)^{(|I|+1)c_I} \mu(\bigcap_{i\in I} a^i_{\sigma_i(j)})^{c_I}.
\end{aligned}$$

□

For example, for $l = 1$, one gets

$$p_1(\bigcup_{i=1}^{n}\{a_1^i, \ldots, a_m^i\}) = \frac{1}{(m!)^{n-1}} \sum_{\substack{\sigma\in\{1\}\times S_m^{(n-1)}\\ j\in\{1,\ldots,m\}\\ I\subseteq[n]}} (-1)^{|I|+1}\mu(\bigcap_{i\in I} a^i_{\sigma_i(j)}).$$

For $l = 1$, $n = 2$, we get

$$p_1(\{a_1^1, \ldots, a_m^1\} \cup \{a_1^1, \ldots, a_m^2\}) = \frac{1}{m!} \sum_{\substack{\sigma\in S_m\\ j\in[m]}} \{\mu(a_j^1) + \mu(a^2_{\sigma(j)}) - \mu(a_j^1 \bigcap a^2_{\sigma(j)})\}.$$

Next we consider a generalized inclusion-exclusion principle using the elementary symmetric functions

$$e_l(x_1, \ldots, x_m) = \sum_{1\le t_1<t_2<\cdots<t_l\le m} \prod_{j=1}^{l} x_{t_j}.$$

Theorem 18. *Let $\{a_1^i, \ldots, a_m^i\}$ be in the basis of $S^m\overline{P[k]}$ for $1 \le i \le n$. Then*

$$e_l\big(\bigcup_{i=1}^{n}\{a_1^i, \ldots, a_m^i\}\big) = \frac{1}{(m!)^{n-1}} \sum_{\substack{\sigma\in\{1\}\times S_m^{n-1}\\ 1\le t_1<t_2<\cdots<t_l\le m\\ f:[l]\to P([n])}} \prod_{j=1}^{l}(-1)^{|f(j)|+1}\mu\big(\bigcap_{i\in f(j)} a^i_{\sigma_i(t_j)}\big).$$

For $n = 2$, $m = 2$ and $l = 2$, the map $e_2 : S^2\overline{P[k]} \to \mathbb{R}$ is given by $e_2(\{a, b\}) = \mu(a)\mu(b)$ and Theorem 18 implies that

$$\begin{aligned}
2e_2(\{a, b\} \cup \{c, d\}) \;=\;& 2\mu(a)\mu(b) + 2\mu(c)\mu(d) + \mu(a)\mu(d) + \mu(c)\mu(b)\\
+\;& \mu(a)\mu(c) + \mu(d)\mu(b) - \mu(a)\mu(b \cap d) + \mu(c)\mu(b \cap d)\\
+\;& \mu(b)\mu(a \cap c) + \mu(d)\mu(a \cap c) + \mu(a)\mu(b \cap c)\\
+\;& \mu(d)\mu(b \cap c) + \mu(b)\mu(a \cap d) + \mu(c)\mu(a \cap d).
\end{aligned}$$

Next we describe the generalization of the inclusion-exclusion principle using the homogenous symmetric functions

$$h_l(x_1, \ldots, x_m) = \sum_{1\le t_1\le t_2\le\cdots<t_l\le m} \prod_{j=1}^{l} x_{t_j}.$$

Theorem 19. *Let $\{a_1^i, \ldots, a_m^i\}$ be in the basis of $S^m\overline{P[k]}$ for $1 \le i \le n$. Then*

$$h_l\big(\bigcup_{i=1}^{n}\{a_1^i, \ldots, a_m^i\}\big) = \frac{1}{(m!)^{n-1}} \sum_{\substack{\sigma\in\{1\}\times S_m^{n-1}\\ 1\le t_1\le t_2\le\cdots\le t_l\le m\\ f:[l]\to P([n])}} \prod_{j=1}^{l}(-1)^{|f(j)|+1}\mu\big(\bigcap_{i\in f(j)} a^i_{\sigma_i(t_j)}\big).$$

For $n = 2$, $m = 2$ and $l = 2$, the map $h_2 : S^2\overline{P[k]} \to \mathbb{R}$ is given by

$$h_2(\{a, b\}) = \mu(a)^2 + \mu(a)\mu(b) + \mu(b)^2.$$

Theorem 19 implies that

$$\begin{aligned} 2h_2(\{a,b\} \cup \{c,d\}) = {} & [\mu(a) + \mu(c) - \mu(a \cap c)]^2 + [\mu(b) + \mu(d) - \mu(b \cap d)]^2 \\ & + [\mu(a) + \mu(d) - \mu(a \cap d)]^2 + [\mu(b) + \mu(c) - \mu(b \cap c)]^2 \\ & + 2\mu(a)\mu(b) + 2\mu(c)\mu(a) + \mu(a)\mu(d) + \mu(c)\mu(b) + \mu(a)\mu(c) \\ & + \mu(d)\mu(a) - \mu(a)\mu(b \cap d) + \mu(c)\mu(b \cap d) + \mu(b)\mu(a \cap c) \\ & + \mu(d)\mu(a \cap c) + \mu(a)\mu(b \cap c) + \mu(d)\mu(b \cap c) + \mu(b)\mu(a \cap d) \\ & + \mu(c)\mu(a \cap d). \end{aligned}$$

Notice that the structural constants of the symmetric and cyclic powers of Boolean algebras are rational numbers. It would be interesting to study the combinatorics of those numbers along the lines of [**2, 3, 4, 9**].

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