

COMPARISON THEOREMS FOR HALF-LINEAR DIFFERENTIAL EQUATIONS OF THE FOURTH ORDER

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ABSTRACT. An identity of the Picone type for fourth-order half-linear ordinary differential operators of the form

$$l_\alpha[x] \equiv (p\varphi(x''))'' - (r\varphi(x'))' + q\varphi(x)$$

and

$$L_\alpha[y] \equiv (P\varphi(y''))'' - (R\varphi(y'))' + Q\varphi(y).$$

where $\varphi(u) := |u|^{\alpha-1}u$, $\alpha > 0$, $u \in R$, and p, q, r, P, Q and R are continuous functions on a given interval I is derived and then Sturmian comparison theory for the corresponding fourth-order equations $l_\alpha[x] = 0$ and $L_\alpha[y] = 0$ based on this identity is developed.

1. INTRODUCTION

The classical Picone identity (see [10]) associated with a pair of Sturm-Liouville differential equations of the form

$$(1) \quad (p(t)u')' + q(t)u = 0$$

and

$$(2) \quad (P(t)v')' + Q(t)v = 0$$

where p, q, P and Q are continuous functions on a given interval I with $p(t) > 0$ and $P(t) > 0$ on I , says that if u and v satisfy (1) and (2), respectively, and $v(t) \neq 0$ on I , then

$$(3) \quad \frac{d}{dt} \left[\frac{u}{v} (pu'v - Pv'u) \right] = (Q - q)u^2 + (p - P)u'^2 + P \left(u' - u \frac{v'}{v} \right)^2.$$

The Sturm-Picone comparison theorem readily follows from (3). Indeed, if we assume that Eq. (1) has a nontrivial solution u with consecutive zeros a and b , $a < b$, and

$$(4) \quad p(t) \geq P(t), \quad Q(t) \geq q(t)$$

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on $[a, b]$, then integrating (3) on $[a, b]$ we get that Eq. (2) cannot possess a solution v which is nonzero in (a, b) , except in the special case where $p(t) \equiv P(t)$ and $q(t) \equiv Q(t)$ and v is a constant multiple of u on $[a, b]$.

In [3] (see also [4]), the identity (3) was generalized to the case of the half-linear differential equations

$$(5) \quad (p(t)\varphi(u'))' + q(t)\varphi(u) = 0$$

and

$$(6) \quad (P(t)\varphi(v'))' + Q(t)\varphi(v) = 0,$$

where $\varphi(u) := |u|^{\alpha-1}$, $u \in \mathbb{R}$, $\alpha > 0$, and p, q, P and Q are continuous functions on an interval I with $p(t) > 0$ and $P(t) > 0$ on I .

If u and v satisfy (5) and (6), respectively, with $v(t) \neq 0$ on I , then

$$(7) \quad \begin{aligned} & \frac{d}{dt} \left\{ \frac{u}{\varphi(v)} [\varphi(v)p\varphi(u') - \varphi(u)P\varphi(v')] \right\} \\ & = (Q - q)|u|^{\alpha+1} + (p - P)|u'|^{\alpha+1} \\ & \quad + P \left[|u'|^{\alpha+1} + \alpha \left| \frac{uv'}{v} \right|^{\alpha+1} - (\alpha + 1)u'\varphi\left(\frac{uv'}{v}\right) \right]. \end{aligned}$$

The half-linear generalization of Sturm-Picone comparison principle obtained previously in [1], [9] and [11] by different methods, now easily follows from (7) if we assume that the inequalities (4) hold on $[a, b]$, where a and b are consecutive zeros of u , and use the Young inequality to show that the last expression in (7) is nonnegative with the equality holding if and only if u and v are proportional on $[a, b]$. Actually, the following more general result is true.

Theorem A (Leighton-type comparison). *If there exists a nontrivial solution u of (5) such that $u(a) = u(b) = 0$ and*

$$(8) \quad \int_a^b [(p(t) - P(t))|u'(t)|^{\alpha+1} + (Q(t) - q(t))|u(t)|^{\alpha+1}] dt \geq 0,$$

then every solution v of (7) has at least one zero in (a, b) except in the special case when $p(t) \equiv P(t)$, $q(t) \equiv Q(t)$ and $u(t) = cv(t)$ on $[a, b]$ for some constant c .

The situation in the case of fourth-order linear differential equations of the form

$$(9) \quad (p(t)u'')'' + q(t)u = 0$$

and

$$(10) \quad (P(t)v'')'' + Q(t)v = 0$$

is more complicated. If u is a nontrivial solution of [9] on an interval $[a, b]$ satisfying

$$(11) \quad u(a) = u'(a) = u(b) = u'(b) = 0$$

and if

$$(12) \quad p(t) \geq P(t), \quad q(t) \geq Q(t) \quad \text{for } t \in [a, b]$$

then, in general, it is not true that an arbitrary solution v of [10] (or any of its derivatives) has a zero in $[a, b]$. This is the consequence of the result of Leighton and Nehari (see [8]) which asserts that if $Q(t) < 0$ for $t \geq a$ and v is a solution of [10] generated by the initial conditions

$$v(a) \geq 0, \quad v'(a) \geq 0, \quad v''(a) \geq 0 \quad \text{and} \quad (Pv'')'(a) \geq 0$$

(but not all zero), then

$$v(t) > 0, \quad v'(t) > 0, \quad v''(t) > 0 \quad \text{and} \quad (Pv'')'(t) > 0$$

for all $t > a$. Thus, neither the solution v itself nor any of its derivatives v', v'' and (Pv'') ' can vanish at the point greater than a .

However, a sort of the Sturm-Picone comparison result can be obtained for [9] and [10] if we consider only solutions v of [10] for which v' and (Pv'') ' have opposite signs.

Theorem B. *Let u be a nontrivial solution of [9] satisfying (11). If v is a solution of [10] for which v' and (Pv'') ' have opposite signs and if the inequalities (12) hold on $[a, b]$, then v, v' or (Pv'') ' has a zero in $[a, b]$.*

(See [5].) The key tool in proving the above theorem was the Picone-type identity which asserts that if u and v are solutions of [9] and [10], respectively, and none of v and v' vanish in I , then

$$\begin{aligned} & \frac{d}{dt} \left\{ \frac{u'}{v'} [v'pu'' - u'Pv''] - \frac{u}{v} [v(pu'')' - u(Pv'')'] \right\} \\ (13) \quad & = (p - P)u''^2 + (q - Q)u^2 - v'(Pv'')' \left(\frac{u'}{v'} - \frac{u}{v} \right)^2 \\ & \quad + P \left(u'' - \frac{u'v''}{v'} \right)^2. \end{aligned}$$

The following comparison theorem of the Leighton type concerning the more general fourth-order linear differential equations

$$(14) \quad (p(t)u'')'' - (r(t)u')' + q(t)u = 0$$

and

$$(15) \quad (P(t)v'')'' - (R(t)v')' + Q(t)v = 0$$

can be obtained as a special case of the results in [7].

Theorem C. *Suppose that there exists a nontrivial solution of (14) which satisfies (12) and*

$$(16) \quad \int_a^b [(p - P)u^2 + (r - R)u'^2 + (q - Q)u''^2] dt \geq 0.$$

If v satisfies (15) with $P(t) \geq 0$ in (a, b) ,

$$(17) \quad v'[R(t)v' - (P(t)v'')'] \geq 0 \quad \text{and} \quad R(t)v' - (P(t)v'')' \neq 0 \quad \text{in} \quad (a, b)$$

then at least one of v and v' has a zero in $[a, b]$.

The purpose of this paper is to generalize the identity (13) to the case of half-linear differential equations of the fourth order and use it in proving comparison theorems of the Sturm-Picone and Leighton type.

For related results concerning the linear case see also [6] and [12].

2. MAIN RESULTS

Consider the operators

$$(18) \quad l_\alpha[x] \equiv (p(t)\varphi(x''))'' - (r(t)\varphi(x'))' + q(t)\varphi(x)$$

and

$$(19) \quad L_\alpha[y] \equiv (P(t)\varphi(y''))'' - (R(t)\varphi(y'))' + Q(t)\varphi(y)$$

where p, r, q, P, R and Q are continuous functions defined on $[a, b] \subset I$ and $\varphi[u] := |u|^\alpha \operatorname{sgn} u, \alpha > 0$, as before.

Let $D_{l_\alpha}(I)$ (resp. $D_{L_\alpha}(I)$) denote the set of all continuous functions x (resp. y) defined on I such that x (resp. y) is two times continuously differentiable on I and also $(r\varphi(x'))'$ and $(p\varphi(x''))''$ (resp. $(R\varphi(y'))'$ and $(P\varphi(y''))''$) exist and are continuous on I .

Denote by Φ_α the form defined for $u, v \in \mathbb{R}$ and $\alpha > 0$ by

$$(20) \quad \Phi_\alpha(u, v) := u\varphi(u) + \alpha v\varphi(v) - (\alpha + 1)u\varphi(v).$$

It follows from the Young inequality that $\Phi_\alpha(u, v) \geq 0$ for all $u, v \in \mathbb{R}$ and the equality holds if and only if $u = v$.

The following lemma can be verified by a direct computation.

Lemma. *If $x \in D_{l_\alpha}(I)$ and $y \in D_{L_\alpha}(I)$ on an interval I and if none of y and y' vanish in I , then*

$$(21) \quad \begin{aligned} & \frac{d}{dt} \left\{ \frac{x'}{\varphi(y')} [\varphi(y')p\varphi(x'') - \varphi(x')P\varphi(y'')] \right. \\ & \quad - \frac{x}{\varphi(y)} [\varphi(y)(p\varphi(x''))' - \varphi(x)(P\varphi(y''))'] \\ & \quad \left. - \frac{x}{\varphi(y)} [\varphi(y)r\varphi(x') - \varphi(x)R\varphi(y')] \right\} \\ & = \frac{x}{\varphi(y)} \{ \varphi(x)L_\alpha[y] - \varphi(y)l_\alpha[x] \} \\ & \quad + (q - Q)|x|^{\alpha+1} + (r - R)|x'|^{\alpha+1} + (p - P)|x''|^{\alpha+1} \\ & \quad + P\Phi_\alpha\left(x'', \frac{x'y''}{y'}\right) + y'[R\varphi(y') - (P\varphi(y''))']\Phi_\alpha\left(\frac{x'}{y'}, \frac{x}{y}\right). \end{aligned}$$

Theorem 1 (Leighton-type comparison). *If there exists a nontrivial $u \in D_{l_\alpha}([a, b])$ such that*

$$(22) \quad \int_a^b ul_\alpha[u]dt \leq 0,$$

$$(23) \quad u(a) = u'(a) = u(b) = u'(b) = 0$$

and

$$(24) \quad V_\alpha[u] \equiv \int_a^b [(p - P)|u''|^{\alpha+1} + (r - R)|u'|^{\alpha+1} + (q - Q)|u|^{\alpha+1}] dt \geq 0,$$

then for any $v \in D_{L_\alpha}([a, b])$ satisfying

$$(25) \quad vL_\alpha[v] \geq 0 \quad \text{in } (a, b), \quad P(t) \geq 0,$$

$$(26) \quad \begin{aligned} v' [R(t)\varphi(v') - (P(t)\varphi(v''))'] &\geq 0, \\ R(t)\varphi(v') - (P(t)\varphi(v''))' &\neq 0 \quad \text{in } (a, b), \end{aligned}$$

v or v' has a zero in $[a, b]$.

Proof. Suppose to the contrary that there exists a function $v \in D_{L_\alpha}([a, b])$ satisfying the inequality (25) in (a, b) such that $v(t) \neq 0$ and $v'(t) \neq 0$ in $[a, b]$. Integrating the identity (21) where $x = u$ and $y = v$ on $[a, b]$, we obtain

$$(27) \quad 0 \geq V_\alpha[u] + \int_a^b v' [R(t)\varphi(v') - (P(t)\varphi(v''))'] \Phi_\alpha \left(\frac{u'}{v'}, \frac{u}{v} \right) dt \geq 0.$$

Thus, we get

$$\int_a^b v' [R(t)\varphi(v') - (P(t)\varphi(v''))'] \Phi_\alpha \left(\frac{u'}{v'}, \frac{u}{v} \right) dt = 0.$$

The assumption (26) implies that $\Phi_\alpha(u'/v', u/v) \equiv 0$ in (a, b) which means that $u = cv$ on $[a, b]$ for some nonzero constant c . Since $u(a) = u(b) = 0$ and $v(t) \neq 0$ on $[a, b]$, this leads to a contradiction. The proof is complete. \square

Corollary (Sturm-Picone comparison). *If*

$$(28) \quad p(t) \geq P(t) > 0, \quad r(t) \geq R(t) \quad \text{and} \quad q(t) \geq Q(t)$$

on $[a, b]$ and there exists a nontrivial solution u of

$$(29) \quad (p(t)\varphi(u''))'' - (r(t)\varphi(u'))' + q(t)\varphi(u) = 0, \quad a < t < b,$$

satisfying (23), then for any solution v of the majorant equation

$$(30) \quad (P(t)\varphi(v''))'' - (R(t)\varphi(v'))' + Q(t)\varphi(v) = 0, \quad a < t < b,$$

satisfying (26) in (a, b) , v or v' must have a zero in $[a, b]$.

3. DISCONJUGACY CRITERION

Consider Eq. (29) in an interval I . Two points $a, b \in I$ are called *conjugate with respect to* (29) if there exists a nontrivial solution $u \in D_{L_\alpha}([a, b])$ satisfying (23). Eq. (29) is called *disconjugate on* I if no two points of I are conjugate with respect to (29).

The following disconjugacy criterion for Eq. (29) is an immediate consequence of Theorem 1.

Theorem 2. *Eq. (29) is disconjugate on I if there exist a half-linear differential operator L_α defined by (19) and a function $v \in D_{L_\alpha}(I)$ satisfying*

$$(31) \quad p(t) \geq P(t) \geq 0, \quad r(t) \geq R(t) \quad \text{and} \quad q(t) \geq Q(t) \quad \text{in} \quad I,$$

$$(32) \quad vL_\alpha[v] \geq 0 \quad \text{in} \quad I, \quad v(t) \neq 0 \quad \text{in} \quad I,$$

and

$$(33) \quad v' [R(t)\varphi(v') - (P(t)\varphi(v''))'] > 0 \quad \text{in} \quad I.$$

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