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UPPER SIGNED k-DOMINATION NUMBER OF DIRECTED GRAPHS

H. ARAM, S. M. SHEIKHOLESLAMI and L. VOLKMANN

Abstract. Let $k \geq 1$ be an integer, and let $D = (V, A)$ be a finite simple digraph in which $d_D^-(v) \geq k-1$ for all $v \in V$. A function $f\colon V \to \{-1, 1\}$ is called a signed k-dominating function (SkDF) if $f(N^-[v]) \geq k$ for each vertex $v \in V$. An SkDF f of a digraph D is minimal if there is no SkDF $g \neq f$ such that $g(v) \leq f(v)$ for each $v \in V$. The maximum values of $\sum_{v\in V} f(v)$, taken over all minimal signed k-dominating functions f, is called the *upper signed k-domination number* $\Gamma_{kS}(D)$. In this paper, we present a sharp upper bound for $\Gamma_{kS}(D)$.

1. Introduction

In this paper, D is a finite simple digraph with vertex set $V(D) = V$ and arc set $A(G) = A$. A digraph without directed cycles of length 2 is an *oriented graph*. The *order* $n(D) = n$ of a digraph D is the number of its vertices and the number of its arcs is the *size* $m(D) = m$. We write $d_D^+(v) = d^+(v)$ for the outdegree of a vertex v and $d_D^-(v) = d^-(v)$ for its indegree. The *minimum* and *maximum indegree* and *minimum* and *maximum outdegree* of D are denoted by $\delta^-(D) = \delta^-$, $\Delta^-(D) = \Delta^-$, $\delta^+(D) = \delta^+$ and $\Delta^+(D) = \Delta^+$, respectively. If uv is an arc of D, then we also write $u \to v$ and say that v is an *out-neighbor* of u and u is an *in-neighbor* of v. For every vertex $v \in V$, let $N_D^-(v) = N^-(v)$ be the set consisting of all vertices of D from which arcs go into v and let $N_D^-[v] = N^-[v] = N^-(v) \cup \{v\}$. If $X \subseteq V(D)$, then $D[X]$ is the subdigraph induced by X. If $X \subseteq V(D)$ and $v \in V(D)$, then $E(X, v)$ is the set of arcs from X to v and $d_X^-(v) = |E(X, v)|$. For a real-valued function $f\colon V(D) \to \mathbb{R}$ the weight of f is $w(f) = \sum_{v\in V} f(v)$, and for $S \subseteq V$, we define $f(S) = \sum_{v\in S} f(v)$, so $w(f) = f(V)$. Consult [**4**] for the notation and terminology which are not defined here.

Let $k \geq 1$ be an integer and let D be a digraph such that $\delta^-(D) \geq k-1$. A *signed k-dominating function* (abbreviated SkDF) of D is a function $f\colon V \to \{-1, 1\}$ such that $f[v] = f(N^-[v]) \geq k$ for every $v \in V$. An SkDF f of a digraph D is minimal if there is no SkDF $g \neq f$ such that $g(v) \leq f(v)$ for each $v \in V$. The maximum

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values of $\sum_{v\in V} f(v)$, taken over all minimal signed k-dominating functions f, is called the *upper signed k-domination number* $\Gamma_{kS}(D)$. For any SkDF f of D we define $P = \{v \in V \mid f(v) = 1\}$ and $M = \{v \in V \mid f(v) = -1\}$. The concept of the signed k-dominating function of digraphs D was introduced by Atapour et al. [**1**].

The concept of the upper signed k-domination number $\Gamma_{kS}(G)$ of undirected graphs G was introduced by Delić and Wang [**2**]. The special case $k = 1$ was defined and investigated in [**3**].

In this article, we present an upper bound on the upper signed k-domination number of digraphs. We make use of the following result.

Lemma 1. *An SkDF f of a digraph D is minimal if and only if for every $v \in V$ with $f(v) = 1$, there exists at least one vertex $u \in N^+[v]$ such that $f[u] = k$* or *$k+1$.*

Proof. Let f be a minimal signed k-dominating function of D. Suppose to the contrary that there exists a vertex $v \in V(D)$ such that $f(v) = 1$ and $f[u] \geq k+2$ for each $u \in N^+[v]$. Then the mapping $g\colon V(D) \to \{-1, 1\}$, defined by $g(v) = -1$ and $g(x) = f(x)$ for $x \in V(D) - \{v\}$, is clearly an SkDF of D such that $g \neq f$ and $g(u) \leq f(u)$ for each $u \in V(D)$, a contradiction.

Conversely, let f be a signed k-dominating function of D such that for every $v \in V$ with $f(v) = 1$, there exists at least one vertex $u \in N^+[v]$ such that $f[u] = k$ or $k+1$. Suppose to the contrary that f is not minimal. Then there is an SkDF g of D such that $g \neq f$ and $g(u) \leq f(u)$ for each $u \in V(D)$. Since $g \neq f$, there is a vertex $v \in V$ such that $g(v) < f(v)$. Then $g(v) = -1$ and $f(v) = 1$ because $f(v), g(v) \in \{-1, 1\}$. Since g is an SkDF of D, $g[u] \geq k$ for each $u \in N^+[v]$. It follows that $f[u] = g[u] + 2 \geq k+2$ for each $u \in N^+[v]$ which is a contradiction. This completes the proof. $\square$

2. An upper bound

Theorem 2. *Let k be a positive integer and let D be a digraph of order n with minimum indegree $\delta^- \geq k-1$ and maximum indegree Δ^-. Then*

$$\Gamma_{kS}(D) \leq \begin{cases} \dfrac{\Delta^-(\delta^- + k + 4) - \delta^- + k + 2}{\Delta^-(\delta^- + k + 4) + \delta^- - k - 2} n & \text{if} \quad \delta^- - k \equiv 0 \pmod 2 \\[2ex] \dfrac{\Delta^-(\delta^- + k + 5) - \delta^- + k + 1}{\Delta^-(\delta^- + k + 5) + \delta^- - k - 1} n. & \text{if} \quad \delta^- - k \equiv 1 \pmod 2). \end{cases}$$

Proof. If $\delta^- = k-1$ or k, then the result is clearly true. Let $\delta^- \geq k+1$ and let f be a minimal SkDF such that $\Gamma_{ks}(D) = f(V(D))$. Suppose that $P = \{v \in V(D) \mid f(v) = 1\}$, $M = \{v \in V(D) \mid f(v) = -1\}$, $p = |P|$ and $q = |M|$. Then $\Gamma_{ks}(D) = f(V) = |P| - |M| = p - q = n - 2q$.

Since f is an SkDF,

$$(d^-(v) - d^-_M(v)) + 1 - d^-_M(v) \geq k$$

for each $v \in P$. It follows that $d^-_M(v) \leq \frac{\Delta^- - k + 1}{2}$ when $v \in P$. Similarly, $d^-_M(v) \leq \frac{\Delta^- - k - 1}{2}$ when $v \in M$. Define $A_i = \{v \in P \mid d^-_M(v) = i\}$, $a_i = |A_i|$ for each

$0 \le i \le \lfloor \frac{\Delta^- + 1 - k}{2} \rfloor$ and $B_i = \{v \in M \mid d_M^-(v) = i\}$, $b_i = |B_i|$ for each $0 \le i \le \lfloor \frac{\Delta^- - 1 - k}{2} \rfloor$. Then the sets $A_0, A_1, \ldots, A_{\lfloor (\Delta^- - k+1)/2 \rfloor}$ form a partition of P and $B_0, B_1, \ldots, B_{\lfloor (\Delta^- - k-1)/2 \rfloor}$ form a partition of M.

Since f is a minimal SkDF, it follows from Lemma 1 that for every $v \in P$, there is at least one vertex $u_v \in N^+[v]$ such that $f[u_v] \in \{k, k+1\}$. For each $v \in A_0$, since v has no in-neighbor in M,

$$f[v] = d^-(v) + 1 \ge \delta^- + 1 \ge k + 2.$$

Therefore $u_v \notin A_0$ for each $v \in P$.

Let $T = \{u \mid u \in N^+(A_0) \text{ and } f[u] = k \text{ or } k+1\}$. If $0 \le i \le \lfloor \frac{\delta^- - k - 1}{2} \rfloor$ and $v \in A_i$, then we have $f[v] = d^-(v) + 1 - 2i \ge k+2$. Similarly, if $0 \le i \le \lfloor \frac{\delta^- - k - 3}{2} \rfloor$ and $v \in B_i$, then we have $f[v] = d^-(v) - 1 - 2i \ge k + 2$. This implies that

$$T \subseteq \left(\bigcup_{\lfloor (\delta^- - k+1)/2 \rfloor}^{\lfloor (\Delta^- - k+1)/2 \rfloor} A_i \right) \cup \left(\bigcup_{\lfloor (\delta^- - k-1)/2 \rfloor}^{\lfloor (\Delta^- - k-1)/2 \rfloor} B_i \right).$$

If $\lfloor \frac{\delta^- - k+1}{2} \rfloor \le i \le \lfloor \frac{\Delta^- - k+1}{2} \rfloor$ and $v \in T \cap A_i$, then $d^-(v) + 1 - 2i = f[v] = k$ or $k+1$ which implies that $d^-(v) = 2i + k$ or $2i + k - 1$. Hence each $v \in T \cap A_i$ has at most $i+k$ in-neighbors in A_0 and so $T \cap A_i$, has at most $(i+k)|T \cap A_i|$ in-neighbors in A_0. Similarly, if $\lfloor \frac{\delta^- - k-1}{2} \rfloor \le i \le \lfloor \frac{\Delta^- - k-1}{2} \rfloor$, then $T \cap B_i$ has at most $(i+k+2)|T \cap B_i|$ in-neighbors in A_0.

Since f is a minimal SkDF of D and $f[v] = d^-(v) + 1 \ge \delta^- + 1 \ge k+2$ for every $v \in A_0$, we deduce that $N^+(v) \ne \emptyset$ for every $v \in A_0$. Note that

$$A_0 \subseteq \left(\bigcup_{\lfloor (\delta^- - k+1)/2 \rfloor}^{\lfloor (\Delta^- - k+1)/2 \rfloor} N^-(T \cap A_i) \right) \cup \left(\bigcup_{\lfloor (\delta^- - k-1)/2 \rfloor}^{\lfloor (\Delta^- - k-1)/2 \rfloor} N^-(T \cap B_i) \right).$$

Thus

$$\begin{aligned} a_0 &\le \left| \bigcup_{\lfloor (\delta^- - k+1)/2 \rfloor}^{\lfloor (\Delta^- - k+1)/2 \rfloor} N^-(T \cap A_i) \right| + \left| \bigcup_{\lfloor (\delta^- - k-1)/2 \rfloor}^{\lfloor (\Delta^- - k-1)/2 \rfloor} N^-(T \cap B_i) \right| \\ &= \sum_{\lfloor (\delta^- - k+1)/2 \rfloor}^{\lfloor (\Delta^- - k+1)/2 \rfloor} |N^-(T \cap A_i)| + \sum_{\lfloor (\delta^- - k-1)/2 \rfloor}^{\lfloor (\Delta^- - k-1)/2 \rfloor} |N^-(T \cap B_i)| \qquad (1) \\ &\le \sum_{\lfloor (\delta^- - k+1)/2 \rfloor}^{\lfloor (\Delta^- - k+1)/2 \rfloor} (i+k) a_i + \sum_{\lfloor (\delta^- - k-1)/2 \rfloor}^{\lfloor (\Delta^- - k-1)/2 \rfloor} (i+k+2) b_i. \end{aligned}$$

Obviously,

$$n = \sum_{i=0}^{\lfloor (\Delta^- - k+1)/2 \rfloor} a_i + \sum_{i=0}^{\lfloor (\Delta^- - k-1)/2 \rfloor} b_i. \qquad (2)$$

Since the number $e(M, V(D))$ of arcs cannot be more than $q\Delta^-$, we have

$$\sum_{i=1}^{\lfloor(\Delta^- - k+1)/2\rfloor} ia_i + \sum_{i=1}^{\lfloor(\Delta^- - k-1)/2\rfloor} ib_i \le q\Delta^-. \tag{3}$$

Case 1. $\delta^- - k \equiv 0 \pmod 2$.
Then $\lfloor(\delta^- - k + 1)/2\rfloor = (\delta^- - k)/2$ and $\lfloor(\delta^- - k - 1)/2\rfloor = (\delta^- - k - 2)/2$. Note that $i + k + 1 \le i(\delta^- + k + 2)/(\delta^- - k)$ when $i \ge \frac{\delta^- - k}{2}$ and $i + k + 3 \le i(\delta^- + k + 4)/(\delta^- - k - 2)$ when $i \ge \frac{\delta^- - k - 2}{2}$. By (1), (2) and (3), we get

$$\begin{aligned}
n &\le \sum_{i=0}^{\lfloor(\Delta^- - k+1)/2\rfloor} a_i + \sum_{i=0}^{\lfloor(\Delta^- - k-1)/2\rfloor} b_i \\
&= \sum_{i=0}^{\lfloor(\delta^- - k-2)/2\rfloor} a_i + \sum_{i=(\delta^- - k)/2}^{\lfloor(\Delta^- - k+1)/2\rfloor} a_i + \sum_{i=0}^{\lfloor(\delta^- - k-4)/2\rfloor} b_i + \sum_{i=(\delta^- - k-2)/2}^{\lfloor(\Delta^- - k-1)/2\rfloor} b_i \\
&\le \sum_{i=1}^{\lfloor(\delta^- - k-2)/2\rfloor} a_i + \sum_{i=(\delta^- - k)/2}^{\lfloor(\Delta^- - k+1)/2\rfloor} (i+k+1)a_i + \sum_{i=0}^{\lfloor(\delta^- - k-4)/2\rfloor} b_i \\
&\quad + \sum_{i=(\delta^- - k-2)/2}^{\lfloor(\Delta^- - k-1)/2\rfloor} (i+k+3)b_i \\
&\le b_0 + \frac{\delta^- + k + 2}{\delta^- - k} \sum_{i=1}^{\lfloor(\Delta^- - k+1)/2\rfloor} ia_i + \frac{\delta^- + k + 4}{\delta^- - k - 2} \sum_{i=1}^{\lfloor(\Delta^- - k-1)/2\rfloor} ib_i \\
&\le b_0 + \frac{\delta^- + k + 4}{\delta^- - k - 2}\left(\sum_{i=1}^{\lfloor(\Delta^- - k+1)/2\rfloor} ia_i + \sum_{i=1}^{\lfloor(\Delta^- - k-1)/2\rfloor} ib_i \right) \\
&\le q + \frac{\delta^- + k + 4}{\delta^- - k - 2} q\Delta^-.
\end{aligned}$$

By solving the above inequality for q, we obtain that

$$q \ge \frac{n(\delta^- - k - 2)}{\Delta^-(\delta^- + k + 4) + \delta^- - k - 2}.$$

Hence,

$$\Gamma_{ks}(D) = n - 2q \le \frac{\Delta^-(\delta^- + k + 4) - \delta^- + k + 2}{\Delta^-(\delta^- + k + 4) + \delta^- - k - 2} n.$$

Case 2. $\delta^- - k \equiv 1 \pmod 2$.
Then $\lfloor(\delta^- - k + 1)/2\rfloor = (\delta^- - k + 1)/2$ and $\lfloor(\delta^- - k - 1)/2\rfloor = (\delta^- - k - 1)/2$.

Note that $i+k+1 \le i(\delta^- + k+3)/(\delta^- - k+1)$ when $i \ge \frac{\delta^- - k+1}{2}$ and $i+k+3 \le i(\delta^- + k+5)/(\delta^- - k-1)$ when $i \ge \frac{\delta^- - k-1}{2}$. By (1), (2) and (3), we get

$$
\begin{aligned}
n &\le \sum_{i=0}^{\lfloor(\Delta^- - k+1)/2\rfloor} a_i + \sum_{i=0}^{\lfloor(\Delta^- - k-1)/2\rfloor} b_i \\
&= \sum_{i=0}^{(\delta^- - k-1)/2} a_i + \sum_{i=(\delta^- - k+1)/2}^{\lfloor(\Delta^- - k+1)/2\rfloor} a_i + \sum_{i=0}^{\lfloor(\delta^- - k-3)/2\rfloor} b_i + \sum_{i=(\delta^- - k-1)/2}^{\lfloor(\Delta^- - k-1)/2\rfloor} b_i \\
&\le \sum_{i=1}^{(\delta^- - k-1)/2} a_i + \sum_{i=(\delta^- - k+1)/2}^{\lfloor(\Delta^- - k+1)/2\rfloor} (i+k+1)a_i + \sum_{i=0}^{(\delta^- - k-3)/2} b_i \\
&\quad + \sum_{i=(\delta^- - k-2)/2}^{\lfloor(\Delta^- - k-1)/2\rfloor} (i+k+3)b_i \qquad (4) \\
&\le b_0 + \frac{\delta^- + k+3}{\delta^- - k+1} \sum_{i=1}^{\lfloor(\Delta^- - k+1)/2\rfloor} ia_i + \frac{\delta^- + k+5}{\delta^- - k-1} \sum_{i=1}^{\lfloor(\Delta^- - k-1)/2\rfloor} ib_i \\
&< b_0 + \frac{\delta^- + k+5}{\delta^- - k-1} \left(\sum_{i=1}^{\lfloor(\Delta^- - k+1)/2\rfloor} ia_i + \sum_{i=1}^{\lfloor(\Delta^- - k-1)/2\rfloor} ib_i \right) \\
&\le q + \frac{\delta^- + k+5}{\delta^- - k-1} q\Delta^-.
\end{aligned}
$$

By solving the inequality (4) for q, we obtain

$$q \ge \frac{n(\delta^- - k-1)}{\Delta^-(\delta^- + k+5) + \delta^- - k-1}.$$

Thus

$$\Gamma_{ks}(D) = n - 2q \le \frac{\Delta^-(\delta^- + k+5) - \delta^- + k+1}{\Delta^-(\delta^- + k+5) + \delta^- - k-1} n.$$

This completes the proof. □

The *associated digraph* $D(G)$ of a graph G is the digraph obtained when each edge e of G is replaced by two oppositely oriented arcs with the same ends as e. We denote the associated digraph $D(K_n)$ of the complete graph K_n of order n by K_n^* and the associated digraph $D(C_n)$ of the cycle C_n of order n by C_n^*.

Let $V(K_6^*) = \{v_1, \ldots, v_6\}$ and $V(C_{46}^*) = \{u_1, \ldots, u_{46}\}$. Assume that D is obtained from $K_6^* + C_{46}^*$ by adding arcs which go from v_i to u_j for $1 \le i \le 3$ and $1 \le j \le 46$. Then $\delta^-(D) = 5$. Let $k = 1$ and define $f\colon V(D) \to \{-1, 1\}$ by $f(v_1) = f(v_2) = -1$ and $f(x) = 1$ for otherwise. Obviously f is a minimal signed dominating function of D with $\omega(f) = 48$. This example shows that the bound in Theorem 2 is sharp for $k = 1$.

Corollary 3. *Let D be an r-inregular digraph of order n. For any positive integer $k \leq r-1$,*

$$\Gamma_{kS}(D) \leq \begin{cases} \dfrac{r^2 + r(k+3) + k + 2}{r^2 + r(k+5) - k - 2} n & \text{if} \quad \delta^- - k \equiv 0 \pmod 2 \\[2ex] \dfrac{r^2 + r(k+4) + k + 1}{r^2 + r(k+6) - k - 1} n. & \text{if} \quad \delta^- - k \equiv 1 \pmod 2. \end{cases}$$

Corollary 4. *Let D be a nearly r-inregular digraph of order n. For any positive integer $k \leq r-1$,*

$$\Gamma_{kS}(D) \leq \begin{cases} \dfrac{r^2 + r(k+2) + k + 3}{r^2 + r(k+4) - k - 3} n & \text{if} \quad \delta^- - k \equiv 0 \pmod 2 \\[2ex] \dfrac{r^2 + r(k+3) + k + 2}{r^2 + r(k+5) - k - 2} n. & \text{if} \quad \delta^- - k \equiv 1 \pmod 2. \end{cases}$$

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H. Aram, Department of Mathematics, Azarbaijan University of Tarbiat Moallem, Tabriz, I. R. Iran

S. M. Sheikholeslami, Department of Mathematics, Azarbaijan University of Tarbiat Moallem, Tabriz, I. R. Iran, *e-mail*: `s.m.sheikholeslami@azaruniv.edu`

L. Volkmann, Lehrstuhl II für Mathematik, RWTH-Aachen University, 52056 Aachen, Germany, *e-mail*: `volkm@math2.rwth-aachen.de`