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LATTICE PLATONIC SOLIDS AND THEIR EHRHART POLYNOMIAL

E. J. IONASCU

Abstract. First, we calculate the Ehrhart polynomial associated with an arbitrary cube with integer coordinates for its vertices. Then, we use this result to derive relationships between the Ehrhart polynomials for regular lattice tetrahedra and those for regular lattice octahedra. These relations allow one to reduce the calculation of these polynomials to only one coefficient.

1. INTRODUCTION

In the 1960's, Eugène Ehrhart ([**14**], [**15**]) proved that given a d-dimensional compact simplicial complex in $\mathbb{R}^n$ $(1 \leq d \leq n)$, denoted here generically by $\mathcal{P}$, whose vertices are in the lattice $\mathbb{Z}^n$, there exists a polynomial $L(\mathcal{P}, t) \in Q[t]$ of degree d, associated with $\mathcal{P}$ and satisfying

$$L(\mathcal{P}, t) = \text{the cardinality of } \{t\mathcal{P}\} \cap \mathbb{Z}^n, \qquad t \in \mathbb{N}. \tag{1}$$

It is known that

$$L(\mathcal{P}, t) = \operatorname{Vol}(\mathcal{P})t^n + \frac{1}{2}\operatorname{Vol}(\partial\mathcal{P})t^{n-1} + \ldots + \chi(\mathcal{P}),$$

where $\operatorname{Vol}(\mathcal{P})$ is the usual volume of $\mathcal{P}$, $\operatorname{Vol}(\partial\mathcal{P})$ is the surface area of $\mathcal{P}$ normalized with respect to the sublattice on each face of $\mathcal{P}$ and $\chi(\mathcal{P})$ is the Euler characteristic of $\mathcal{P}$. In general, the other coefficients are less understandable, but significant progress has been done (see [**5**], [**27**] and [**28**]).

In [**13**], Eugène Ehrhart classified the regular convex polyhedra in $\mathbb{Z}^3$. It turns out that only cubes, regular tetrahedra and regular octahedra can be embedded in the usual integer lattice. We arrived at the same result in [**23**] using a construction of these polyhedra from equilateral triangles. This led us to the following simple description of all cubes in $\mathbb{Z}^3$. If we take an odd positive integer, say d, and a primitive solution of the Diophantine equation $a^2+b^2+c^2 = 3d^2$ ($\gcd(a, b, c) = 1$), the there are equilateral triangles in any plane having equation $ax + by + cz = f$, which can be parameterized in terms of two integers m and n (see [**18**], [**19**] and [**22**]). The side-lengths of such a triangle are equal to $d\sqrt{2(m^2 + mn + n^2)}$. In

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order to rise in space from such a triangle to form a regular tetrahedron, we needto satisfy the necessary and sufficient condition

$$m^2+mn+n^2=k^2 \text{ for some odd } k\in\mathbb{N}. \tag{2}$$

If (2) is satisfied, there are two possibilities. If k is a multiple of 3, then we can complete the triangle in both sides of the plane to a regular tetrahedron in $\mathbb{Z}^3$, and if k is not divisible by 3, then we can complete the triangle in exactly one side to form a regular tetrahedron in $\mathbb{Z}^3$ (see Figure 1). Every such regular tetrahedron can then be completed to a cube in $\mathbb{Z}^3$ with side-lengths equal to dk. Every regular octahedron in $\mathbb{Z}^3$ is the dual of the double of a cube in $\mathbb{Z}^3$. We will make these constructions very specific in the last section.

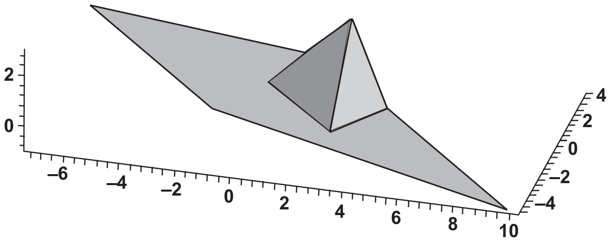

Figure 1. Expand an equilateral triangle.

It is natural to ask the question that we think Ehrhart himself asked: "What is the form that the polynomial in (1) takes for these regular lattice polyhedra?". The purpose of this paper is to answer this question for cubes (in a very simple way), and give some partial answers for regular tetrahedra and octahedra.

For completeness and due credit, we include Ehrhart's idea in [**16**] to characterize all cubes in $\mathbb{Z}^3$. This is based on a theorem of Olinde Rodrigues: *The set of 3-by-3 orthogonal matrices can be given by four real parameters a, b, c, d, not simultaneously zero, as follows*

$$\frac{\pm 1}{a^2+b^2+c^2+d^2}\begin{bmatrix} a^2+b^2-c^2-d^2 & 2(bc+da) & 2(bd-ca)\\ 2(bc-da) & a^2-b^2+c^2-d^2 & 2(cd+ba)\\ 2(bd+ca) & 2(cd-ba) & a^2-b^2-c^2+d^2\end{bmatrix}. \tag{3}$$

It is clear that every cube in $\mathbb{Z}^3$ can be translated in such way that a vertex becomes the origin and the three vectors defined by the three sides starting from the origin give an orthogonal basis for $\mathbb{R}^3$. Hence, we can construct a 3-by-3 orthogonal matrix from these vectors which has rational entries. Conversely, we can construct a cube in $\mathbb{Z}^3$ from such an orthogonal matrix which has rational entries. In what follows we will do this association so that the vectors (points) are determined by the rows. The construction here is to take four integers a, b, c and d in (3), simplify by whatever is possible and then get rid of the denominators to obtain the three vectors with integer coordinates that determine the cube. This construction is similar to the classical parametrization of the Heronian triangles.

Our approach to the classification allows us to start in terms of the side lengths. However, Ehrhart's construction is useful answering other questions about these objects. For instance, we can see that there are such cubes of any side length (other than the trivial ones, multiples of the unit cube) since every natural number can be written as a sum of four perfect squares. It turns out that there are only odd number side lengths for irreducible cubes, i.e., a cube which is not an integer multiple of a smaller cube in $\mathbb{Z}^3$.

Let us begin with some of the smallest irreducible cubes. We introduce them here by orthogonal matrices with rational entries and define up to the usual symmetries of the space (equivalent classes relative to the 48-order subgroup of all orthogonal matrices with entries 0 or ± 1, denoted by $\mathcal{S}_o$). As we mentioned before, this will make a difference, the cubes are essentially determined by the rows. Obviously, the Ehrhart polynomials are identical for all cubes in the same equivalence class (left or right).

We will denote the Ehrhart polynomial for an irreducible cube C_ℓ of side-length $\ell = 2k-1$, $k \in \mathbb{N}$, by $L(C_\ell, t)$. From the general theory we have

$$L(C_\ell, t) = \ell^3 t^3 + \lambda_1 t^2 + \lambda_2 t + 1, \qquad t \in \mathbb{N}, \tag{4}$$

where λ_1 is half the sum of the areas of the faces of the cube C_ℓ, each face being normalized by the area of a fundamental domain of the sublattice contained in that face. The coefficient λ_2 is in general a problem (see, for example [**6**]), but in this case it takes a simple form as we will show in Section 3.

For the unit cube $C_1 = I$ (the identity matrix), obviously, $L(C_1, t) = (t+1)^3$. There is only one cube (right or left equivalence classes modulo $\mathcal{S}_o$) for each $\ell = 2k-1$ for $k = 1, 2, 3, 4, 5$ and 6:

$$C_1 = I \qquad C_3 := \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix},$$

$$C_5 := \frac{1}{5}\begin{bmatrix} 4 & 3 & 0 \\ 3 & -4 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \qquad C_7 := \frac{1}{7}\begin{bmatrix} -2 & 6 & 3 \\ 3 & -2 & 6 \\ 6 & 3 & -2 \end{bmatrix},$$

$$C_9 := \frac{1}{9}\begin{bmatrix} 7 & 4 & -4 \\ 4 & 1 & 8 \\ -4 & 8 & 1 \end{bmatrix}, \qquad C_{11} := \frac{1}{11}\begin{bmatrix} 2 & 9 & 6 \\ 9 & 2 & -6 \\ 6 & -6 & 7 \end{bmatrix}.$$

For $k = 7$ we have $C_{13} := \frac{1}{13}\begin{bmatrix} -3 & 12 & 4 \\ 4 & -3 & 12 \\ 12 & 4 & -3 \end{bmatrix}$, and an extra orthogonal matrix:

$$\hat{C}_{13} := \frac{1}{13}\begin{bmatrix} 5 & 12 & 0 \\ 12 & -5 & 0 \\ 0 & 0 & 13 \end{bmatrix}.$$

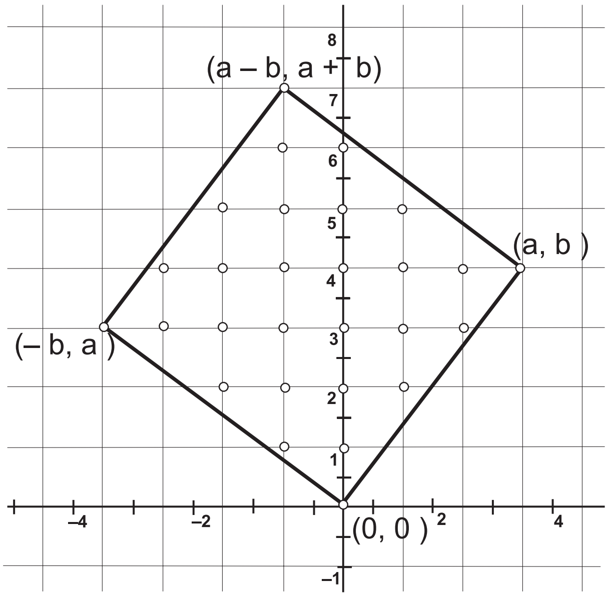

Figure 2. One face of the cube.

One peculiar thing about the Ehrhart polynomials associated with these cubes so far is that there is an unexpected factor in their factorization:

$$L(C_3,t) = (3t+1)(9t^2+1), \qquad L(C_5,t) = (5t+1)(25t^2+2t+1),$$
$$L(C_7,t) = (7t+1)(49t^2-4t+1), \qquad L(C_9,t) = (9t+1)(81t^2-6t+1),$$
$$L(C_{11},t) = (11t+1)(121t^2-8t+1), \quad L(C_{13},t) = (13t+1)(169t^2-10t+1),$$

and

$$L(\hat{C}_{13},t) = (13t+1)(169t^2+2t+1).$$

This suggests that

$$L(C_\ell,t) = (\ell t+1)(\ell^2t^2+\alpha t+1), \quad t \in \mathbb{N}, \text{ and some } \alpha \in \mathbb{Z}. \tag{5}$$

We can easily prove that this is indeed the case for cubes of a special form like C_5 and $\hat{C}_{13}$ above. Let us consider a primitive Pythagorean triple (a,b,c) with $a^2+b^2=c^2$. In the xy-plane, we construct the square with vertices $O(0,0,0)$, $A(a,b,0)$, $B(a-b,a+b,0)$, and $C(-b,a,0)$ (Figure 2). We then translate this face along the vector $c\overrightarrow{k}$ to form a cube of side-lengths equal to c. Let us denote this cube by $C_{a,b,c}$. It is easy to argue that (a,b,c) is primitive because we have no lattice points on the sides of $OABC$, other than its vertices. The coefficient λ_1 in (4) is equal to $\frac{1}{2}(c^2+c^2+4c)$ because two of the faces have to be normalized by 1 and four of the faces have to be normalized by $c(\frac{c}{c}) = c$. By Pick's theorem, applied to $OABC$, we have

$$\begin{aligned} c^2 &= \frac{\#\{\text{points on the sides}\}}{2} + \#\{\text{interior points of OABC}\} - 1 \\ &= \#\{\text{interior points of OABC}\} + 1. \end{aligned}$$

Hence the number of lattice points in the interior of $OABC$ is $c^2 - 1$. Therefore the number of lattice points in $C_{a,b,c}$ is $(c+1)(c^2+3) = c^3 + c^2 + 3c + 3$. The polynomial

$$L(C_{a,b,c}, t) = c^3t^3 + (c^2 + 2c)t^2 + (c+2)t + 1 = (ct+1)(c^2t^2 + 2t + 1)$$

satisfies exactly the condition $L(C_{a,b,c}, 1) = (c+1)(c^2+3)$. So we have shown that (5) is true for infinitely many cubes C_ℓ.

Proposition 1.1. *Given a primitive Pythagorean triple, $a^2+b^2 = c^2$, the cubes in the class of* $C_{a,b,c} := \frac{1}{c}\begin{bmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & c \end{bmatrix}$ *have the same Ehrhart polynomial given by*

$$L(C_c, t) = (ct+1)(c^2t^2 + 2t + 1), \ t \in \mathbb{N}.$$

This proposition follows easily from the general theory since the polytope is a product of a square and a segment. The general formula is proved in Section 3. Section 2 is basically dealing with the second coefficient in (4). In Section 4, we look at the Ehrhart polynomial for regular tetrahedra and regular octahedra with lattice vertices.

2. The coefficient λ_1

Let us prove the following well known lemma (see the acknowledgement note) which we include here for completeness.

Lemma 2.1. *For $n \in \mathbb{N}$, $n \geq 2$, let a_1, a_2, $\ldots$, a_n be n integers such that* $\gcd(a_1, a_2, \ldots, a_n) = 1$. *Then the determinant of the lattice $\mathcal{L}$ of points $(x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n$ in the hyperplane $a_1x_1 + \ldots + a_nx_n = 0$ is given by $\sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}$.*

Proof. We define $p = a_1^2 + a_2^2 + \ldots + a_n^2$ and consider the sublattice L of points $(x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n$ such that $a_1x_1 + \ldots + a_nx_n = 0$ (mod p). Since $\gcd(a_1, a_2, \ldots, a_n) = 1$, the index of L in $\mathbb{Z}^n$ is p and hence the determinant of L is p. On the other hand, a basis for L can be obtained by appending a basis for the lattice $\mathcal{L}$ by the vector $(a_1, a_2, \ldots, a_n)$ whose length is $\sqrt{p}$ and which is perpendicular to all other basis vectors. Therefore, the determinant of the lattice $\mathcal{L}$ is $\frac{p}{\sqrt{p}} = \sqrt{p}$. □

Let us now assume that we have an arbitrary cube in $\mathbb{Z}^3$,

$$C_\ell = \frac{1}{\ell}\left[\begin{array}{ccc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_2 & c_3 \end{array}\right], \tag{6}$$

with a_i, b_i and c_i integers such that $a_ia_j + b_ib_j + c_ic_j = \delta_{i,j}\ell^2$ for all i, j in $\{1,2,3\}$. We define $d_i := \gcd(a_i, b_i, c_i)$. It is clear that d_i are divisors of ℓ. Let us also introduce the numbers $d_i' = \frac{\ell}{d_i}$, $i = 1,2,3$. Then, we have the following expression for the first coefficient in (4).

Theorem 2.2. *The coefficient* λ_1 *is given by*

$$\lambda_1 = \ell(d_1 + d_2 + d_3) \textit{where } d_i := \gcd(a_i, b_i, c_i),\ i \in \{1,2,3\}. \tag{7}$$

Proof. We use Lemma 2.1 for each of the faces of the cube. Opposite faces will have the same contribution. Say we take a face containing the points (a_1, b_1, c_1) and (a_2, b_2, c_2). The irreducible normal vector to this face is clearly $\frac{1}{d_3}(a_3, b_3, c_3)$. The area of a fundamental domain here is given by $\sqrt{\frac{1}{d_3^2}(a_3^2 + b_3^2 + c_3^2)} = D_3$. By the general theory $\lambda_1 = \frac{1}{2}(2\frac{\ell^2}{d_1'} + 2\frac{\ell^2}{d_2'} + 2\frac{\ell^2}{d_3'}) = \ell(d_1 + d_2 + d_3)$. ☐

Naturally, at this point, the question is whether or not it is possible to have all of the d_i's bigger than one. It turns out that this is possible and as before, in our line of similar investigations, the first ℓ is $\ell = 1105 = 5(13)(17)$

$$C_{1105} = \frac{1}{1105}\left[\begin{array}{rrr} -65 & 156 & 1092 \\ 420 & 1015 & -120 \\ 1020 & -408 & 119 \end{array}\right]. \tag{8}$$

Corollary 2.3. *For a matrix* C_ℓ *as in* (6) *such that the* C_ℓ^{-1} *in the same equivalence class modulo* $\mathcal{S}_o$, *we have*

$$d_1 + d_2 + d_3 = \gcd(a_1, a_2, a_3) + \gcd(b_1, b_2, b_3) + \gcd(c_1, c_2, c_3). \tag{9}$$

Proof. The Ehrhart polynomial must be the same for the corresponding cubes in the same equivalence class. ☐

We believe that this corollary applies to all $\ell < 1105$, and of course to a lot of other cases, but we do not have a proof of this fact. A counterexample to the hypothesis of this corollary is given by the matrix given in (8). In this case, $d_1 + d_2 + d_3 = 35$ and $\gcd(a_1, a_2, a_3) + \gcd(b_1, b_2, b_3) + \gcd(c_1, c_2, c_3) = 7$.

3. THE COEFFICIENT λ_2

The main idea in calculating the coefficient a_2 is to take advantage of the fact that every cube defined by (6) can be used to form a fundamental domain (wandering set) W for the space, under integer translations along the vectors $\overrightarrow{\alpha} = (a_1, b_1, c_1)$, $\overrightarrow{\beta} = (a_2, b_2, c_2)$ and $\overrightarrow{\gamma} = (a_3, b_3, c_3)$, i.e.,

$$\mathbb{R}^3 = \overset{\circ}{\bigcup_{i,j,k\in\mathbb{Z}}} (W + i\overrightarrow{\alpha} + j\overrightarrow{\beta} + k\overrightarrow{\gamma}),$$

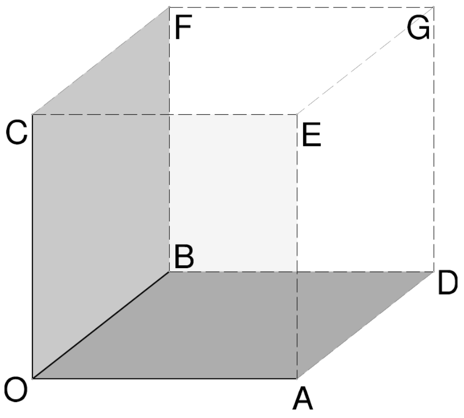

Figure 3. Wandering set determined by the cube.

where $\overset{\circ}{\bigcup}$ means a union of mutually disjoint sets.

The fundamental domain W that we will consider here, associated with a generic cube as in Figure 3, is the set of all points formed by the interior points of the cube to which we add the points of the faces OAEC, OADB and OBFC except the (closed) edges AD, DB, BF, FC, CE, and EA. It is easy to see that such a set is indeed a wandering set. We were informed that this notion is known as well as the *half-open fundamental parallelepiped* for the cone formed by $\overrightarrow{\alpha}$, $\overrightarrow{\beta}$ and $\overrightarrow{\gamma}$. In our setting we think of $\overrightarrow{\alpha}$, $\overrightarrow{\beta}$ and $\overrightarrow{\gamma}$ as the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$.

We will need to use the following well known result fact.

Theorem 3.1 (Ehrhart-Macdonald Reciprocity Law)**.** *Given a compact simplicial lattice complex P (as before) of dimension n, then*

$$L(\overset{\circ}{P},t)=(-1)^nL(P,-t),\ t\in\mathbb{N},$$

where $\overset{\circ}{P}$ denotes the interior of P, as usual.

With the notation from the previous section, we have the following result.

Theorem 3.2. *The coefficient λ_2 is equal to $d_1+d_2+d_3$.*

Proof. Let us denote by k the number of lattice points in W. For $n\in\mathbb{N}$, the number of lattice points in

$$\overset{\circ}{\bigcup_{i,j,k\in\{1,2,\ldots,n\}}}(W+i\overrightarrow{\alpha}+j\overrightarrow{\beta}+k\overrightarrow{\gamma}),$$

is equal to n^3k. On the other hand, this number is equal to $L(C_\ell,n)+K$, where K is the number of lattice points on three big faces of nC. It is easy to see that K is $O(n^2)$, and so $k=\lim_{n\to\infty}\frac{1}{n^3}(L(C_\ell,n)+O(n^2))=\ell^3$.

Hence, according to Theorem 3.1, the number of lattice points in the interior of C_ℓ is $\ell^3-\lambda_1+\lambda_2-1$. So the number of lattice points on the boundary of C_ℓ

is $2\lambda_1 + 2$. Let us denote by σ the number of lattice points on the interior of the sides $\overline{OA}$, $\overline{OB}$ and $\overline{OC}$. Then we have

$$2\lambda_1 + 2 = 2[k - (\ell^3 - \lambda_1 + \lambda_2 - 1)] + 2\sigma + 6 \ \Rightarrow\ \lambda_2 = \sigma + 3.$$

Since $\sigma = (d_1 - 1) + (d_2 - 1) + (d_3 - 1)$, the claim follows. $\square$

Putting these facts together we obtain the following theorem.

Theorem 3.3. *Given a cube C_ℓ constructed from a matrix as in* (6)*, its Ehrhart polynomial is given by*

$$L(C_\ell, t) = (\ell t + 1)[\ell^2 t^2 + (d_1 + d_2 + d_3 - \ell)t + 1], \ t \in \mathbb{N}. \tag{10}$$

There are some natural questions at this point. One of them is: "What is the maximum number of lattice points that can be contained in a lattice cube of side lengths ℓ?" We have the following corollary to the above theorem.

Corollary 3.4. *Given a cube C_ℓ constructed from a matrix as in* (6)*, the maximum of lattice points inside or on the boundary of this cube cannot be more than $(\ell + 1)^3$. This value is attained for the cube ℓC_1.*

Proof. Since d_i is a divisor of ℓ, we have $d_i \le \ell$, so the corollary follows from (10). $\square$

What is the maximum of lattice points contained in an irreducible cubes of sides ℓ? This is a more complicated problem which depends heavily on ℓ and relates to the number of irreducible cubes (their Ehrhart polynomials, in fact) of sides ℓ.

4. Regular tetrahedra and regular octahedra

We remind the reader that a cube in space (Figure 4) is determined by an orthogonal matrix as in (6) by taking its vertices O (the origin), A, B, C, D, E, F and G whose position vectors are $\overrightarrow{OA} = \overrightarrow{\alpha} = (a_1, b_1, c_1)$, $\overrightarrow{OB} = \overrightarrow{\beta} = (a_2, b_2, c_2)$, $\overrightarrow{OC} = \overrightarrow{\gamma} = (a_3, b_3, c_3)$, $\overrightarrow{OD} = \overrightarrow{\alpha} + \overrightarrow{\beta}$, $\overrightarrow{OF} = \overrightarrow{\beta} + \overrightarrow{\gamma}$, $\overrightarrow{OE} = \overrightarrow{\gamma} + \overrightarrow{\alpha}$ and $\overrightarrow{OG} = \overrightarrow{\alpha} + \overrightarrow{\beta} + \overrightarrow{\gamma}$.

In [**23**], we rediscovered Ehrhart's characterization ([**13**]) of all regular polyhedra which can be imbedded in $\mathbb{Z}^3$. Only the cubes, the tetrahedra and octahedra exist in $\mathbb{Z}^3$ and there are infinitely many in each class. We have constructed all of these equilateral triangles. In general, once a tetrahedron is constructed, this can be always completed to a cube. Vice versa, for a cube given by (6), there are two regular tetrahedra as shown in Figure 4, which are in the same equivalence class, modulo the orthogonal matrices with entries ± 1, denoted earlier by $\mathcal{S}_0$. Regular octahedra can be obtained by doubling the coordinates of the cube C_ℓ and then taking the centers of each face. This procedure is exhaustive. An octahedron in the same class can be obtained by simply taking the vertices whose position vectors are $\pm\overrightarrow{\alpha}$, $\pm\overrightarrow{\beta}$ and $\pm\overrightarrow{\gamma}$. We will use the notations T_ℓ and O_ℓ for the tetrahedra and octahedra constructed this way from C_ℓ. Since we are interested in irreducible

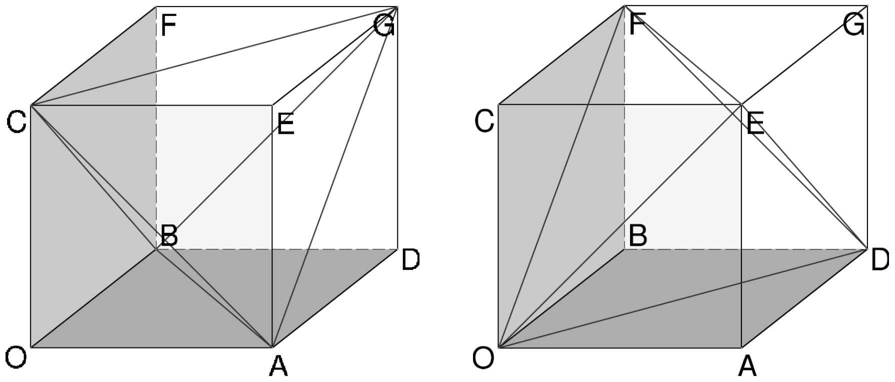

Figure 4. A cube determines two tetrahedra.

T_ℓ and O_ℓ, we may assume that ℓ is odd. The T_ℓ and O_ℓ have side-lengths equal to $\ell\sqrt{2}$. From the general Ehrhart theory (see [**3**]), we have

$$L(T_\ell,t)=\frac{\ell^3}{3}t^3+\mu_1t^2+\mu_2t+1,\quad L(O_\ell,t)=\frac{4\ell^3}{3}t^3+\nu_1t^2+\nu_2t+1. \tag{11}$$

Let us first look at some of the examples of the smallest side-lengths.

$$T_1:=\begin{bmatrix}1&1&0\\0&1&1\\1&0&1\end{bmatrix},\qquad O_1:=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\text{ with}$$

$$L(T_1,t)=\frac{t^3}{3}+t^2+\frac{5t}{3}+1\ \text{ and }\ L(O_1,t)=\frac{4}{3}t^3+2t^2+\frac{8t}{3}+1.$$

For the next side-lengths,

$$T_3:=\begin{bmatrix}1&1&4\\1&4&1\\4&1&1\end{bmatrix},\ O_3:=\begin{bmatrix}-1&2&2\\2&-1&2\\2&2&-1\end{bmatrix}\text{ with}$$

$$L(T_3,t)=9t^3+\frac{9}{2}t^2+\frac{13t}{2}+1\ \text{ and }\ L(O_3,t)=36t^3+9t^2-t+1.$$

These polynomials were computed with the help of a computer.

4.1. The coefficients μ_1 and ν_1

From the general theory we know that these coefficients can be computed in terms of the areas of faces and normalized by the area of the fundamental domains of the sub-lattice of $\mathbb{Z}^3$ corresponding to that face. Since the number of faces for O_ℓ is twice as big as T_ℓ and basically the parallel faces are in the same equivalence class, we have $\nu_1=2\mu_1$.

Let us introduce the divisors

$$
\begin{aligned}
D_1 &= \gcd(a_1+a_2+a_3, b_1+b_2+b_3, c_1+c_2+c_3), \\
D_2 &= \gcd(a_1+a_2-a_3, b_1+b_2-b_3, c_1+c_2-c_3), \\
D_3 &= \gcd(a_1-a_2+a_3, b_1-b_2+b_3, c_1-c_2+c_3) \text{ and} \\
D_4 &= \gcd(-a_1+a_2+a_3, -b_1+b_2+b_3, -c_1+c_2+c_3).
\end{aligned} \tag{12}
$$

Let us observe that the vectors $\overrightarrow{\alpha}+\overrightarrow{\beta}+\overrightarrow{\gamma}$, $\overrightarrow{\alpha}+\overrightarrow{\beta}-\overrightarrow{\gamma}$, $\overrightarrow{\alpha}-\overrightarrow{\beta}+\overrightarrow{\gamma}$, $-\overrightarrow{\alpha}+\overrightarrow{\beta}+\overrightarrow{\gamma}$ are vectors normal to the faces of the T_ℓ. By Lemma 2.1, we see that the area of each fundamental domain corresponding to a face of T_ℓ is given by one of the numbers $\frac{\ell\sqrt{3}}{D_i}$.

Proposition 4.1. *The coefficients* μ_1 *and* ν_1 *in* (11) *are given by*

$$
\mu_1 = \frac{\nu_1}{2} = \frac{\ell(D_1+D_2+D_3+D_4)}{4}. \tag{13}
$$

This explains the coefficients of t^2 in the next examples which were obtained by brute force counting using Maple:

$$
T_5 := \begin{bmatrix} 7 & -1 & 0 \\ 4 & 3 & 5 \\ 3 & -4 & 5 \end{bmatrix}, \quad O_5 := \begin{bmatrix} 4 & 3 & 0 \\ 3 & -4 & 0 \\ 0 & 0 & 5 \end{bmatrix} \quad \text{with}
$$

$$
L(T_5,t) = \frac{125}{3}t^3 + 5t^2 + \frac{1}{3}t + 1 \quad \text{and} \quad L(O_5,t) = \frac{500}{3}t^3 + 10t^2 + \frac{16}{3}t + 1.
$$

4.2. The coefficients μ_2 and ν_2

Let us observe that the cube in Figure 4 can be decomposed into four triangular pyramids OABC, DABG, FCGB and EGCA, which can be translated and some reflected into the origin to form half of O_ℓ and the regular tetrahedron T_ℓ. We remind the reader of a notation we used in the proof of Theorem 3.2 where we denoted by σ the number of lattice points on the interior of the edges $\overline{OA}$, $\overline{OB}$ and $\overline{OC}$. We showed that $\sigma = d_1+d_2+d_3-3$.

Let us balance the number M of the lattice interior points of C_ℓ using the above decomposition. According to Theorem 3.3, and Theorem 3.1 we have

$$
\begin{aligned}
M = -L(C_\ell,-1) &= (\ell-1)(\ell^2-d_1-d_2-d_3+\ell+1) \\
&= \ell^3-(d_1+d_2+d_3)(\ell-1)-1.
\end{aligned}
$$

Part of the lattice points counted in M are in the regular tetrahedron which are counted by $L(T_\ell,1)=\frac{\ell^3}{3}+\mu_1+\mu_2+1$, from which we need to subtract the number of lattice points on the interior of its sides, which we will denote by τ and subtract 4 for its vertices. The rest of the points counted in M is in the interior of the four pyramids. If we multiply this number by two and add the number of lattice points in the interior of the cube faces of the cube less τ, we get the number of interior points of O_ℓ minus $2\sigma+1$. The number of lattice interior points of the cube faces is equal to $2\lambda_1 + 2 - 4\sigma - 8$. In other words, we have

$$\begin{aligned}&2(M - \frac{\ell^3}{3} - \mu_1 - \mu_2 - 1 + \tau + 4) + 2\lambda_1 + 2 - 4\sigma - 8 - \tau\\ &= \frac{4\ell^3}{3} - \nu_1 + \nu_2 - 1 - (2\sigma + 1).\end{aligned}$$

Taking into account that $\nu_1 = 2\mu_1$ and $\lambda_1 = (d_1 + d_2 + d_3)\ell = (\sigma + 3)\ell$, this can be simplified to

$$\nu_2 + 2\mu_2 = 6 + \tau. \tag{14}$$

We close this section concluding what we have shown.

Theorem 4.1. *For a regular tetrahedron T_ℓ and a regular octahedron O_ℓ constructed as before from an orthogonal matrix with rational coefficients as in* (6)*, the coefficients μ_2 and ν_2 in* (11) *satisfy*

$$\nu_2 + 2\mu_2 = (d_1 + d_2 + d_3 + d_4 + d_5 + d_6), \tag{15}$$

where d_1, d_2, d_3 are defined as before and $d_4 = \gcd(a_1 - a_2, b_1 - b_2, c_1 - c_2)$, $d_5 = \gcd(a_1 - a_3, b_1 - b_3, c_1 - c_3)$ *and* $d_6 = \gcd(a_3 - a_2, b_3 - b_2, c_3 - c_2)$.

We have tried to find another relation that will help us find the two coefficients but it seems there is not an easy way to avoid, what are called in [**3**], the building blocks of the lattice-point enumeration, the Dedekind sums. These numbers require a little more computational power and we are wonder if a shortcut doesn't really exist. One would expect that the answer to our questions for such regular objects is encoded in the coordinates of their vertices in a relatively simple way. We leave this problem to the interest of a reader.

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E. J. Ionascu, Department of Mathematics Columbus State University 4225 University Avenue Columbus, GA 31907, U.S.A, *e-mail*: `ionascu@columbusstate.edu, math@ejionascu.ro`