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ON INTEGRABILITY CONDITIONS OF FUNCTIONS RELATED TO THE FORMAL TRIGONOMETRIC SERIES BELONGING TO ORLICZ SPACE

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Abstract. In this paper we have introduced a new class of numerical sequences named as Mean Rest Bounded Variation Sequence of second order. This class is used to show some integrability conditions of the functions $\sin x g(x)$ and $\sin x f(x)$ such that these functions belong to the Orlicz space, where $g(x)$ and $f(x)$ denote formal sine and cosine trigonometric series, respectively. This study may be taken as an continuation of some recent foregoing results proved by L. Leindler [**5**] and S. Tikhonov [**14**].

1. Introduction

Many authors have studied the integrability of the formal series

$$g(x) := \sum_{n=1}^{\infty} \lambda_n \sin nx \tag{1.1}$$

and

$$f(x) := \sum_{n=1}^{\infty} \lambda_n \cos nx \tag{1.2}$$

requiring certain conditions on the coefficients λ_n (see [**6**]–[**7**] and [**2**]–[**15**]).

As initial example, R. P. Boas in [**1**] proved the following result for (1.1).

Theorem 1.1. *If $\lambda_n \downarrow 0$, then for $0 \le \gamma \le 1$, $x^{-\gamma} g(x) \in L[0,\pi]$ if and only if $\sum_{n=1}^{\infty} n^{\gamma-1}\lambda_n$ converges.*

This result had previously been proved for $\gamma = 0$ by W.H. Young [**15**] and was later extended by P. Heywood [**4**] for $1 < \gamma < 2$.

Later the monotonicity condition on the coefficients λ_n was replaced to more general ones by S. M. Shah [**12**] and L. Leindler [**6**].

In 2004 S. Tikhonov [**14**] proved two theorems providing sufficient conditions of $g(x)$ and $f(x)$ belonging to Orlicz space. Before we state his theorems, we will recall some notions and notations.

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Leindler ([**6**]) introduced the following definition. A sequence $c := \{c_n\}$ of positive numbers tending to zero is of rest bounded variation, or briefly R_0^+BVS, if it possesses the property

$$\sum_{n=m}^{\infty} |c_n - c_{n+1}| \leq K(c)c_m \tag{1.3}$$

for all natural numbers m, where $K(c)$ is a constant depending only on c.

A sequence $\gamma := \{\gamma_n\}$ of positive terms will be called almost increasing (decreasing) if there exists constant $C := C(\gamma) \geq 1$ such that

$$C\gamma_n \geq \gamma_m \quad (\gamma_n \leq C\gamma_m)$$

holds for any $n \geq m$.

Here and further C, C_i denote positive constants that are not necessarily the same at each occurrence, and also we use the notion $u \ll w$ $(u \gg w)$ at inequalities if there exists a positive constant C such that $u \leq Cw$ $(u \geq Cw)$ holds.

We will denote (see [**9**]) by $\triangle(p,q)$, $(0 \leq q \leq p)$ the set of all nonnegative functions $\Phi(x)$ defined on $[0,1)$ such that $\Phi(0) = 0$ and $\Phi(x)/x^p$ is nonincreasing and $\Phi(x)/x^q$ is nondecreasing. It is clear that $\triangle(p,q) \subset \triangle(p,0)$, $0 < q \leq p$. As an example, $\triangle(p,0)$ contains the function $\Phi(x) = \log(1+x)$.

Here and in the sequel, a function $\gamma(x)$ is defined by the sequence γ in the following way: $\gamma\left(\frac{\pi}{n}\right) := \gamma_n$, $n \in \mathbb{N}$ and there exist positive constants C_1 and C_2 such that $C_1\gamma_{n+1} \leq \gamma(x) \leq C_2\gamma_n$ for $x \in \left(\frac{\pi}{n+1}, \frac{\pi}{n}\right)$.

A locally integrable almost everywhere positive function $\gamma(x)\colon [0,\pi] \to [0,\infty)$ is said to be a weight function. Let $\Phi(t)$ be a nondecreasing continuous function defined on $[0,\infty)$ such that $\Phi(0) = 0$ and $\lim_{t\to\infty} \Phi(t) = +\infty$. For a weight $\gamma(x)$ the weighted Orlicz space $L(\Phi,\gamma)$ is defined by

$$L(\Phi,\gamma) = \left\{h : \int_0^{\pi} \gamma(x)\Phi(\varepsilon|h(x)|)dx < \infty \quad \text{for some} \quad \varepsilon > 0\right\}. \tag{1.4}$$

Tikhonov's results now can be read as follows.

Theorem 1.2. *Let $\Phi(x) \in \triangle(p,0)$, $0 \leq p$. If $\lambda_n \in R_0^+BVS$ and the sequence $\{\gamma_n\}$ is such that $\{\gamma_n n^{-1+\varepsilon}\}$ is almost decreasing for some $\varepsilon > 0$, then*

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n^2}\Phi(n\lambda_n) < \infty \quad \Rightarrow \quad \psi(x) \in L(\Phi,\gamma), \tag{1.5}$$

where a function $\psi(x)$ is either a sine or cosine series.

Theorem 1.3. *Let $\Phi(x) \in \triangle(p,q)$, $0 \leq q \leq p$. If $\lambda_n \in R_0^+BVS$ and the sequence $\{\gamma_n\}$ is such that $\{\gamma_n n^{-(1+q)+\varepsilon}\}$ is almost decreasing for some $\varepsilon > 0$, then*

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n^{2+q}}\Phi(n^2\lambda_n) < \infty \Rightarrow g(x) \in L(\Phi,\gamma). \tag{1.6}$$

A null-sequence c of nonnegative numbers possessing the property

$$\sum_{n=2m}^{\infty} |c_n - c_{n+1}| \leq \frac{K(c)}{m} \sum_{\nu=m}^{2m-1} c_\nu \tag{1.7}$$

is called a sequence of mean rest bounded variation, in symbols, $c \in MRBVS$.

In [**5**], L. Leindler extended Theorem 1.2 and Theorem 1.3, so that the sequence $\{\lambda_n\}$ belongs to the class $MRBVS$ instead of the class R_0^+BVS. His results are formulated as follows.

Theorem 1.4. *Theorems 1.2 and 1.3 can be improved when the condition* $\lambda_n \in R_0^+BVS$ *is replaced by the assumption* $\lambda_n \in MRBVS$*. Furthermore the conditions of* (1.5) *and* (1.6) *may be modified as follows:*

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n^2} \Phi\left(\sum_{\nu=n}^{2n-1} \lambda_\nu\right) < \infty \Rightarrow \psi(x) \in L(\Phi, \gamma), \tag{1.8}$$

and

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n^{2+q}} \Phi\left(n \sum_{\nu=n}^{2n-1} \lambda_\nu\right) < \infty \Rightarrow g(x) \in L(\Phi, \gamma), \tag{1.9}$$

respectively.

In 2009, B. Szal [**11**] introduced a new class of sequences as follows.

Definition 1.1. A sequence $\alpha := \{c_k\}$ of nonnegative numbers tending to zero is called Rest Bounded Second Variation of second order, or briefly, $\{c_k\} \in RBSVS$, if it has the property

$$\sum_{k=m}^{\infty} |c_k - c_{k+2}| \leq K(\alpha) c_m$$

for all natural numbers m, where $K(\alpha)$ is positive, depending only on the sequence $\{c_k\}$, and we assume that the sequence is bounded.

Motivated by the above definition, we introduce a new class of numerical sequences.

Definition 1.2. A null-sequence c of nonnegative numbers possessing the property

$$\sum_{n=2m}^{\infty} |\triangle^2 c_n + \triangle^2 c_{n+1}| \leq \frac{K(c)}{m} \sum_{\nu=m}^{2m-1} |c_\nu - c_{\nu+2}| \tag{1.10}$$

is said to be a sequence of Mean Rest Bounded Variation of second order, in symbols, $c \in MRBSVS$, where $\triangle^2 c_n = c_n - 2c_{n+1} + c_{n+2}$.

The aim of this paper is to extend Tikhonov's results and Leindler's result, so that the sequence $\{\lambda_n\}$ belongs to the class $MRBSVS$ instead of the classes R_0^+BVS and $MRBVS$. To achieve this aim, we need some helpful statements given in next section.

2. Auxiliary Lemmas

We shall use the following lemmas for the proof of the main results.

Lemma 2.1 ([**9**]). *Let* $\Phi \in \triangle(p,q)$, $0 \le q \le p$, *and* $t_j \ge 0$, $j = 1, 2, \ldots, n$, $n \in \mathbb{N}$. *Then*

(1) $\theta^p \Phi(t) \le \Phi(\theta t) \le \theta^q \Phi(t)$, $0 \le \theta \le 1$, $\quad t \ge 0$,

(2) $\Phi\left(\sum_{j=1}^n t_j\right) \le \left(\sum_{j=1}^n \Phi^{1/p*}(t_j)\right)^{p*}$, $\quad p* := \max(1,p)$.

Lemma 2.2 ([**5**]). *Let* $\Phi \in \triangle(p,q)$, $0 \le q \le p$. *If* $\rho_n > 0$, $a_n \ge 0$ *and if*

$$\sum_{\nu=2^m}^{2^{m+1}-1} a_\nu \ll \sum_{\nu=1}^{2^m-1} a_\nu \tag{2.1}$$

holds for all $m \in \mathbb{N}$, *then*

$$\sum_{k=1}^{\infty} \rho_k \Phi\left(\sum_{\nu=1}^{k} a_\nu\right) \ll \sum_{k=1}^{\infty} \Phi\left(\sum_{\nu=k}^{2k-1} a_\nu\right) \rho_k \left(\frac{1}{k\rho_k}\sum_{\nu=k}^{\infty} \rho_\nu\right)^{p*},$$

where $p* := \max(1,p)$.

Lemma 2.3. *The following representations of* $g(x)$ *and* $f(x)$

$$2\sin x g(x) = -\sum_{k=1}^{\infty} (\lambda_k - \lambda_{k+2}) \cos(k+1)x$$

and

$$2\sin x f(x) = \sum_{k=1}^{\infty} (\lambda_k - \lambda_{k+2}) \sin(k+1)x,$$

where we have assumed that $\lambda_1 = \lambda_2 = 0$, *hold.*

Proof. We start from obvious equality

$$\sum_{k=1}^{\infty} \lambda_k \cos kx = \frac{1}{2}\sum_{k=1}^{\infty} (\lambda_k + \lambda_{k+1}) \cos kx + \frac{1}{2}\sum_{k=1}^{\infty} (\lambda_k - \lambda_{k+1}) \cos kx,$$

or

$$\frac{1}{2}\sum_{k=1}^{\infty} \lambda_k \cos kx = \frac{1}{2}\sum_{k=1}^{\infty} (\lambda_k + \lambda_{k+1}) \cos kx - \frac{1}{2}\cos x \sum_{k=2}^{\infty} \lambda_k \cos kx$$
$$- \frac{1}{2}\sin x \sum_{k=2}^{\infty} \lambda_k \sin kx.$$

Thus we have

$$\frac{1+\cos x}{2}\sum_{k=2}^{\infty} \lambda_k \cos kx$$
$$= \frac{1}{2}\sum_{k=1}^{\infty} (\lambda_k + \lambda_{k+1}) \cos kx - \frac{1}{2}\sin x \sum_{k=2}^{\infty} \lambda_k \sin kx - \frac{1}{2}\lambda_1 \cos x$$

or since $\lambda_1 = 0$, we obtain

$$\begin{aligned}\sum_{k=2}^{\infty} \lambda_k \cos kx \\ = \frac{1}{2\cos^2 \frac{x}{2}} \left\{ \sum_{k=1}^{\infty} (\lambda_k + \lambda_{k+1}) \cos kx - \sin x \sum_{k=2}^{\infty} \lambda_k \sin kx \right\}.\end{aligned} \tag{2.2}$$

Similarly as above, we obtain

$$\sum_{k=1}^{\infty} \lambda_k \sin kx = \frac{1}{2} \sum_{k=1}^{\infty} (\lambda_k + \lambda_{k+1}) \sin kx + \frac{1}{2} \sum_{k=1}^{\infty} (\lambda_k - \lambda_{k+1}) \sin kx,$$

or

$$\begin{aligned}\frac{1}{2} \sum_{k=1}^{\infty} \lambda_k \sin kx = \frac{1}{2} \sum_{k=1}^{\infty} (\lambda_k + \lambda_{k+1}) \sin kx \\ - \frac{1}{2} \cos x \sum_{k=2}^{\infty} \lambda_k \sin kx + \frac{1}{2} \sin x \sum_{k=2}^{\infty} \lambda_k \cos kx.\end{aligned} \tag{2.3}$$

Inserting (2.2) into (2.3), we have ($\lambda_1 = 0$)

$$\begin{aligned}\frac{1}{2} \sum_{k=1}^{\infty} \lambda_k \sin kx &= \frac{1}{2} \sum_{k=1}^{\infty} (\lambda_k + \lambda_{k+1}) \sin kx - \frac{1}{2} \cos x \sum_{k=2}^{\infty} \lambda_k \sin kx \\ &\quad + \frac{\sin \frac{x}{2}}{2\cos \frac{x}{2}} \sum_{k=1}^{\infty} (\lambda_k + \lambda_{k+1}) \cos kx - \frac{\sin \frac{x}{2} \sin x}{2\cos \frac{x}{2}} \sum_{k=2}^{\infty} \lambda_k \sin kx \\ &= \frac{1}{2} \sum_{k=1}^{\infty} (\lambda_k + \lambda_{k+1}) \sin kx + \frac{\sin \frac{x}{2}}{2\cos \frac{x}{2}} \sum_{k=1}^{\infty} (\lambda_k + \lambda_{k+1}) \cos kx \\ &\quad - \left(\frac{\cos x}{2} + \frac{\sin \frac{x}{2} \sin x}{2\cos \frac{x}{2}} \right) \sum_{k=2}^{\infty} \lambda_k \sin kx\end{aligned}$$

or

$$\sum_{k=1}^{\infty} \lambda_k \sin kx = \frac{1}{2\cos \frac{x}{2}} \sum_{k=1}^{\infty} (\lambda_k + \lambda_{k+1}) \sin \left(k + \frac{1}{2} \right) x$$

Applying the summation by parts to the above equality and taking into account that $\lambda_1 = \lambda_2 = 0$, we obtain

$$\sum_{k=1}^{\infty} \lambda_k \sin kx = \frac{1}{2\cos \frac{x}{2}} \sum_{k=1}^{\infty} (\lambda_k - \lambda_{k+2}) \sum_{i=0}^{k} \sin \left(i + \frac{1}{2} \right) x,$$

or finally, noting that

$$\sum_{i=0}^{k} 2 \sin \left(i + \frac{1}{2} \right) x \sin \frac{x}{2} = 1 - \cos(k+1)x,$$

we get

$$\sum_{k=1}^{\infty}\lambda_k \sin kx = -\frac{1}{2\sin x}\sum_{k=1}^{\infty}(\lambda_k-\lambda_{k+2})\cos(k+1)x,$$

which clearly proves the first part of this lemma.

For the proof of the second part of this lemma, it is enough to put $n=1$ into the equality (3.10), see [**11**, page 167]. □

Lemma 2.4. *If* $\lambda := \{\lambda_n\} \in MRBSVS$ *and* $D_n := \frac{1}{n}\sum_{k=n}^{2n-1}|\lambda_k-\lambda_{k+2}|$, *then*

$$D_k \ll D_\ell$$

holds for all $k \geq 2\ell$.

Proof. For $m \geq 2\ell$, we note that

$$\begin{aligned}\frac{1}{\ell}\sum_{k=\ell}^{2\ell-1}|\lambda_k-\lambda_{k+2}| &\gg \sum_{k=2\ell}^{\infty}|\triangle^2\lambda_k+\triangle^2\lambda_{k+1}| \\ &\geq \sum_{k=m}^{\infty}|\triangle^2\lambda_k+\triangle^2\lambda_{k+1}| \\ &\geq \sum_{k=m}^{\infty}\big||\lambda_k-\lambda_{k+2}|-|\lambda_{k+1}-\lambda_{k+3}|\big| \geq |\lambda_m-\lambda_{m+2}|.\end{aligned}$$

Summing up the both sides of the last inequality, when m goes from k to $2k-1$, we obtain

$$\frac{k}{\ell}\sum_{k=\ell}^{2\ell-1}|\lambda_k-\lambda_{k+2}| \gg \sum_{m=k}^{2k-1}|\lambda_m-\lambda_{m+2}|,$$

whence the required inequality follows immediately. □

3. Main Results

Our first theorem deals with integrability of both functions $\sin x g(x)$ and $\sin x f(x)$ simultaneously.

Theorem 3.1. *Let* $\Phi(x) \in \triangle(p,0)$, $0 \leq p$. *If* $\lambda_n \in MRBSVS$ *and the sequence* $\{\gamma_n\}$ *is such that* $\{\gamma_n n^{-1+\varepsilon}\}$ *is almost decreasing for some* $\varepsilon > 0$, *then*

$$\sum_{n=1}^{\infty}\frac{\gamma_n}{n^2}\Phi\left(\sum_{\nu=n}^{2n-1}|\lambda_\nu-\lambda_{\nu+2}|\right) < \infty \quad \Rightarrow \quad \sin x\psi(x) \in L(\Phi,\gamma), \tag{3.1}$$

where a function $\psi(x)$ *is either a sine or cosine series.*

Proof. For the proof we use the idea which Tikhonov and Leindler used for their results. For this, let $x \in \left(\frac{\pi}{n+1},\frac{\pi}{n}\right]$. Based on Lemma 2.3 and applying the summation by parts, we obtain

$$\begin{aligned}2|\sin x f(x)| \le & \sum_{k=1}^{n}|\lambda_k-\lambda_{k+2}|+\left|\sum_{k=n}^{\infty}(\lambda_k-\lambda_{k+2})\sin(k+1)x\right| \\ \le & \sum_{k=1}^{n}|\lambda_k-\lambda_{k+2}|+\sum_{k=n}^{\infty}|\triangle^2\lambda_k+\triangle^2\lambda_{k+1}|\left|\widetilde{D}_k^*(x)\right| \\ & +|\lambda_n-\lambda_{n+2}|\left|\widetilde{D}_n^*(x)\right|\end{aligned}$$

where $\widetilde{D}_k^*(x)$ are defined by

$$\widetilde{D}_k^*(x):=\sum_{i=0}^{k}\sin(i+1)x=\frac{\cos\frac{x}{2}-\cos\left(k+\frac{3}{2}\right)x}{2\sin\frac{x}{2}},\quad k\in\mathbb{N}.$$

Taking into account that $|\widetilde{D}_k^*(x)|=O\left(\frac{1}{x}\right)$ and $\{\lambda_n\}\in MRBSVS$, we have

$$\begin{aligned}2|\sin x f(x)| &\le \sum_{k=1}^{n}|\lambda_k-\lambda_{k+2}|+n\sum_{k=n}^{\infty}|\triangle^2\lambda_k+\triangle^2\lambda_{k+1}|+n|\lambda_n-\lambda_{n+2}| \\ &\ll \sum_{k=1}^{n}|\lambda_k-\lambda_{k+2}|+\sum_{k=\frac{n}{2}}^{n-1}|\lambda_k-\lambda_{k+2}|+n|\lambda_n-\lambda_{n+2}| \\ &\ll \sum_{k=1}^{n}|\lambda_k-\lambda_{k+2}|+n|\lambda_n-\lambda_{n+2}|.\end{aligned}$$

The following estimates can be obtained by the same technique. We get

$$\begin{aligned}&2|\sin x g(x)| \\ &\le \sum_{k=1}^{n}|\lambda_k-\lambda_{k+2}|+\left|\sum_{k=n}^{\infty}(\lambda_k-\lambda_{k+2})\cos(k+1)x\right| \\ &\le \sum_{k=1}^{n}|\lambda_k-\lambda_{k+2}|+\sum_{k=n}^{\infty}|\triangle^2\lambda_k+\triangle^2\lambda_{k+1}|\left|D_k^*(x)\right|+|\lambda_n-\lambda_{n+2}|\left|D_n^*(x)\right| \\ &\le \sum_{k=1}^{n}|\lambda_k-\lambda_{k+2}|+n\sum_{k=n}^{\infty}|\triangle^2\lambda_k+\triangle^2\lambda_{k+1}|+n|\lambda_n-\lambda_{n+2}| \\ &\ll \sum_{k=1}^{n}|\lambda_k-\lambda_{k+2}|+\sum_{k=\frac{n}{2}}^{n-1}|\lambda_k-\lambda_{k+2}|+n|\lambda_n-\lambda_{n+2}| \\ &\ll \sum_{k=1}^{n}|\lambda_k-\lambda_{k+2}|+n|\lambda_n-\lambda_{n+2}|,\end{aligned}$$

where $D_k^*(x)$ are defined by

$$D_k^*(x):=\sum_{i=0}^{k}\cos(i+1)x=\frac{\sin\left(k+\frac{3}{2}\right)x-\sin\frac{x}{2}}{2\sin\frac{x}{2}},\qquad k\in\mathbb{N}.$$

Thus

$$|\sin x\psi(x)| \ll \sum_{k=1}^{n} |\lambda_k - \lambda_{k+2}| + n|\lambda_n - \lambda_{n+2}|,$$

where a function $\psi(x)$ is either $f(x)$ or $g(x)$.

Moreover, since $\{\lambda_n\} \in MRBSVS$,

$$n|\lambda_n - \lambda_{n+2}| \leq n\sum_{k=n}^{\infty} |\triangle^2\lambda_k + \triangle^2\lambda_{k+1}| \ll \sum_{k=1}^{n} |\lambda_k - \lambda_{k+2}|,$$

and hence

$$|\sin x\psi(x)| \ll \sum_{k=1}^{n} |\lambda_k - \lambda_{k+2}|. \tag{3.2}$$

According to Lemma 2.4, the condition (2.1) with $|\lambda_\nu - \lambda_{\nu+2}|$ in place of a_ν is satisfied, and thus we are ready to apply Lemma 2.2. Therefore, by (3.2), we obtain

$$\begin{aligned}\int_0^\pi \gamma(x)\Phi(|\sin x\psi(x)|)\mathrm{d}x &\ll \sum_{n=1}^{\infty} \Phi\left(\sum_{k=1}^{n} |\lambda_k - \lambda_{k+2}|\right) \int_{\pi/(n+1)}^{\pi/n} \gamma(x)\mathrm{d}x \\ &\ll \sum_{n=1}^{\infty} \frac{\gamma_n}{n^2}\Phi\left(\sum_{k=1}^{n} |\lambda_k - \lambda_{k+2}|\right) \\ &\ll \sum_{n=1}^{\infty} \Phi\left(\sum_{k=n}^{2n-1} |\lambda_k - \lambda_{k+2}|\right) \frac{\gamma_n}{n^2}\left(\frac{n}{\gamma_n}\sum_{\nu=n}^{\infty}\frac{\gamma_\nu}{\nu^2}\right)^{p*},\end{aligned}$$

where $p* := \max(1, p)$.

Finally, by the assumption on $\{\gamma_n\}$, we get

$$\frac{n}{\gamma_n}\sum_{\nu=n}^{\infty}\frac{\gamma_\nu}{\nu^2} \ll 1$$

which along with the above inequality immediately imply (3.1). The proof is completed. □

Theorem 3.2. *Let $\Phi(x) \in \triangle(p, q)$, $0 \leq q \leq p$. If $\lambda_n \in MRBSVS$ and the sequence $\{\gamma_n\}$ is such that $\{\gamma_n n^{-(1+q)+\varepsilon}\}$ is almost decreasing for some $\varepsilon > 0$, then*

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n^{2+q}}\Phi\left(\sum_{k=n}^{2n-1} k|\lambda_k - \lambda_{k+2}|\right) < \infty \quad \Rightarrow \quad \sin xf(x) \in L(\Phi, \gamma). \tag{3.3}$$

Proof. Let $x \in \left(\frac{\pi}{n+1}, \frac{\pi}{n}\right]$. Then

$$\begin{aligned}
2|\sin x f(x)| \le & \sum_{k=1}^{n}(k+1)x|\lambda_k - \lambda_{k+2}| + \left|\sum_{k=n+1}^{\infty}(\lambda_k - \lambda_{k+2})\sin(k+1)x\right| \\
& \ll x\sum_{k=1}^{n} k|\lambda_k - \lambda_{k+2}| \\
& \quad + \sum_{k=n}^{\infty} |\triangle^2\lambda_k + \triangle^2\lambda_{k+1}|\left|\widetilde{D}_k^*(x)\right| + |\lambda_n - \lambda_{n+2}|\left|\widetilde{D}_n^*(x)\right| \\
& \ll n^{-1}\sum_{k=1}^{n} k|\lambda_k - \lambda_{k+2}| + \sum_{k=\frac{n}{2}}^{n-1}|\lambda_k - \lambda_{k+2}| + n|\lambda_n - \lambda_{n+2}| \\
& \ll n^{-1}\sum_{k=1}^{n} k|\lambda_k - \lambda_{k+2}|.
\end{aligned} \tag{3.4}$$

According to Lemmas 2.1, 2.2, 2.4, and the estimate (3.4), we have

$$\begin{aligned}
& \int_0^{\pi} \gamma(x)\Phi(|\sin x f(x)|)\mathrm{d}x \\
& \ll \sum_{n=1}^{\infty} \Phi\left(n^{-1}\sum_{k=1}^{n} k|\lambda_k - \lambda_{k+2}|\right)\int_{\pi/(n+1)}^{\pi/n}\gamma(x)\mathrm{d}x \\
& \ll \sum_{n=1}^{\infty} \frac{\gamma_n}{n^{2+q}}\Phi\left(\sum_{k=1}^{n} k|\lambda_k - \lambda_{k+2}|\right) \\
& \ll \sum_{n=1}^{\infty} \Phi\left(\sum_{k=n}^{2n-1} k|\lambda_k - \lambda_{k+2}|\right)\frac{\gamma_n}{n^{2+q}}\left(\frac{n^{1+q}}{\gamma_n}\sum_{\nu=n}^{\infty}\frac{\gamma_\nu}{\nu^{2+q}}\right)^{p*},
\end{aligned} \tag{3.5}$$

where $p* := \max(1, p)$.

By the assumption on $\{\gamma_n\}$, we get

$$\frac{n^{1+q}}{\gamma_n}\sum_{\nu=n}^{\infty}\frac{\gamma_\nu}{\nu^{2+q}} \ll 1,$$

and hence (3.5) takes this form

$$\int_0^{\pi} \gamma(x)\Phi(|\sin x f(x)|)dx \ll \sum_{n=1}^{\infty}\frac{\gamma_n}{n^{2+q}}\Phi\left(\sum_{k=n}^{2n-1} k|\lambda_k - \lambda_{k+2}|\right),$$

which proves (3.3). With this the proof of theorem is finished. □

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