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ON M-PROJECTIVE CURVATURE TENSOR OF A GENERALIZED SASAKIAN SPACE FORM

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ABSTRACT. In the present paper, we have studied M-projectively flat generalized Sasakian space form, η-Einstein generalized Sasakian space form and irrotational M-projective curvature tensor on a Sasakian space form.

1. INTRODUCTION

A Riemannian manifold with constant sectional curvature C is known as real-space-form and its curvature tensor is given by

$$R(X,Y)Z = C\{g(Y,Z)X - g(X,Z)Y\}.$$

A Sasakian manifold (M, ϕ, ξ, η, g) is said to be a Sasakian space form [**3**], if all the ϕ-sectional curvatures $K(X \wedge \phi X)$ are equal to a constant C, where $K(X \wedge \phi X)$ denotes the sectional curvature of the section spanned by the unit vector field X, orthogonal to ξ and ϕX. In such a case, the Riemannian curvature tensor of M is given by

$$\begin{aligned} R(X,Y)Z = {} & \frac{C+3}{4}\{g(Y,Z)X - g(X,Z)Y\} \\ & + \frac{C-1}{4}\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ & + \frac{C-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ & + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}. \end{aligned} \tag{1.1}$$

As a natural generalization of these manifolds, P. Alegre, D. E. Blair and A. Carriazo [**3**], [**1**] introduced the notion of generalized Sasakian space form.

Sasakian space form and Generalized Sasakian space form have been studied by several authors, viz., [**3**], [**2**], [**6**], [**14**], [**10**].

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In 1971, G. P. Pokhariyal and R. S. Mishra [**13**] defined a tensor field W^* on a Riemannian manifold as

$$W^*(X,Y)Z = R(X,Y)Z - \frac{1}{4n}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \tag{1.2}$$

Such a tensor field W^* is known as M-projective curvature tensor.

The properties of the M-projective curvature tensor in Sasakian and Kaehler manifold were studied by R. H. Ojha [**11**] [**12**]. He showed that it bridges the gap between the conformal curvature tensor, coharmonic curvature tensor and concircular curvature tensor. S. K. Chaubey and R. H. Ojha [**8**] studied the properties of the M-projective curvature tensor in Riemannian and Kenmotsu manifold. S. K. Chaubey [**9**] also studied the properties of M-projective curvature tensor in LP-Sasakian manifold. C. S. Bagewadi, E. Girish Kumar and Venkatesha [**4**] studied irrotational D-conformal curvature tensor in Kenmotsu and trans-Sasakian manifolds. C.S. Bagewadi, Venkatesha and N.S. Basavarajappa [**5**] proved that if pseudo projective curvature tensor in a LP-Sasakian manifold is irrotational, then the manifold is Einstein. Motivated by these ideas, in the present paper, we made an attempt to study the properties of M-projective curvature tensor in generalized Sasakian space form. The present paper is organized as follows.

In Section 2, we review some preliminary results. In Section 3, we study M-projectively flat generalized Sasakian space form and obtain necessary and sufficient conditions for a generalized Sasakian space form to be M-projectively flat. And in Section 4, we study η-Einstein generalized Sasakian space form satisfying $W^*(\xi, X) \cdot R = 0$. Finally in Section 5, we prove that M-projective curvature tensor in an η-Einstein generalized Sasakian space form is irrotational if and only if $f_3 = \frac{3f_2}{(1-2n)}$.

2. Preliminaries

An odd-dimensional Riemannian manifold (M, g) is called an almost contact manifold if there exists a $(1,1)$ tensor field ϕ, a vector field ξ and a 1-form η on M, such that

$$\phi^2(X) = -X + \eta(X)\xi, \tag{2.1}$$

$$\eta(\phi X) = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$\phi\xi = 0, \qquad \eta(\xi) = 0, \qquad g(X,\xi) = \eta(X), \tag{2.4}$$

for any vector fields X, Y on M.

If in addition, ξ is a Killing vector field, then M is said to be a K-contact manifold. It is well known that a contact metric manifold is a K-contact manifold if and only if

$$(\nabla_X \xi) = -\phi(X) \tag{2.5}$$

for any vector field X on M.

On the other hand, the almost contact metric structure on M is said to be normal if $[\phi,\phi](X,Y) = -2d\eta(X,Y)\xi$ for any X, Y, where $[\phi,\phi]$ denotes the Nijenhuis tensor of ϕ given by

$$[\phi,\phi](X,Y) = \phi^2[X,Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y].$$

A normal contact metric manifold is called a Sasakian manifold. It can be proved that Sasakian manifold is K-contact, and that an almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi)(Y) = g(X,Y)\xi - \eta(Y)X. \tag{2.6}$$

Given an almost contact metric manifold (M, ϕ, ξ, η, g), we say that M is an generalized Sasakian space form if there exists three functions f_1, f_2 and f_3 on M such that

$$\begin{aligned} R(X,Y)Z = {} & f_1\{g(Y,Z)X - g(X,Z)Y\} \\ & + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ & + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ & + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} \end{aligned} \tag{2.7}$$

for any vector fields X, Y, Z on M, where R denotes the curvature tensor of M. This kind of manifold appears as a natural generalization of the well-known Sasakian space form $M(C)$, which can be obtained as particular cases of generalized Sasakian space form by taking $f_1 = \frac{C+3}{4}$ and $f_2 = f_3 = \frac{C-1}{4}$.

Further in a $(2n+1)$-dimensional generalized Sasakian space form, we have [**1**]

$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n-1)f_3)\eta(X)\xi, \tag{2.8}$$

$$S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y), \tag{2.9}$$

$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3, \tag{2.10}$$

$$R(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y], \tag{2.11}$$

$$R(\xi,X)Y = (f_1 - f_3)[g(X,Y)\xi - \eta(Y)X], \tag{2.12}$$

$$\eta(R(X,Y)Z) = (f_1 - f_3)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{2.13}$$

$$S(X,\xi) = 2n(f_1 - f_3)\eta(X). \tag{2.14}$$

3. M-projectively flat generalized Sasakian space form

For a $(2n+1)$-dimensional $(n > 1)$ M-projectively flat generalized Sasakian space form, from (1.2), we have

$$R(X,Y)Z = \frac{1}{4n}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]. \tag{3.1}$$

In view of (2.8) and (2.9), the equation (3.1) takes the form

$$\begin{aligned} R(X,Y)Z = &\frac{1}{4n}[2(2nf_1+3f_2-f_3)\{g(Y,Z)X-g(X,Z)Y\} \\ &-(3f_2+(2n-1)f_3)\{\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y \\ &+g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi\}]. \end{aligned} \tag{3.2}$$

Using (2.7), the equation (3.2) reduces to

$$\begin{aligned} &f_1\{g(Y,Z)X-g(X,Z)Y\} \\ &+f_2\{g(X,\phi Z)\phi Y-g(Y,\phi Z)\phi X+2g(X,\phi Y)\phi Z\} \\ &+\ f_3\{\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X\}+g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi\} \\ =&\frac{1}{4n}[2(2nf_1+3f_2-f_3)\{g(Y,Z)X-g(X,Z)Y\} \\ &-(3f_2+(2n-1)f_3)\{\eta(Y)\eta(Z)X \\ &-\eta(X)\eta(Z)Y+g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi\}]. \end{aligned} \tag{3.3}$$

Replacing Z by ϕZ in (3.3), we obtain

$$\begin{aligned} &f_1\{g(Y,\phi Z)X-g(X,\phi Z)Y\} \\ &+f_2\{g(X,\phi^2 Z)\phi Y-g(Y,\phi^2 Z)\phi X+2g(X,\phi Y)\phi^2 Z\} \\ &+f_3\{g(X,\phi Z)\eta(Y)\xi-g(Y,\phi Z)\eta(X)\xi\} \\ =&\frac{1}{4n}[2(2nf_1+3f_2-f_3)\{g(Y,\phi Z)X-g(X,\phi Z)Y\} \\ &-(3f_2+(2n-1)f_3)\{g(Y,\phi Z)\eta(X)\xi-g(X,\phi Z)\eta(Y)\xi\}]. \end{aligned} \tag{3.4}$$

Putting $X=\xi$ in (3.4), we get

$$\begin{aligned} &4nf_1g(Y,\phi Z)\xi-4nf_3g(Y,\phi Z)\xi \\ =&[4nf_1+3f_2-(1+2n)f_3]g(Y,\phi Z)\xi. \end{aligned} \tag{3.5}$$

Simplifying (3.5), we get

$$[(1-2n)f_3-3f_2]g(Y,\phi Z)\xi=0. \tag{3.6}$$

Since $g(Y,\phi Z)\neq 0$, it follows from (3.6) that

$$f_3=\frac{3f_2}{(1-2n)}. \tag{3.7}$$

Conversely, suppose that

$$f_3=\frac{3f_2}{(1-2n)}$$

holds. Then in view of (2.7) and (2.9), we can write the equation (1.2) as

$$\begin{aligned} \grave{W}^*(X,Y,Z,W) = &f_2\{g(X,\phi Z)g(\phi Y,W)-g(Y,\phi Z)g(\phi X,W) \\ &+2g(X,\phi Y)g(\phi Z,W)\}+f_3\{\eta(X)\eta(Z)g(Y,W) \\ &-\eta(Y)\eta(Z)g(X,W)+g(X,Z)\eta(Y)\eta(W) \\ &-g(Y,Z)\eta(X)\eta(W)+g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\}, \end{aligned} \tag{3.8}$$

where $\grave{W}^*(X,Y,Z,W) = g(W^*(X,Y)Z,W)$.

Replacing X by ϕX and Y by ϕY in (3.8), we get

$$
\begin{aligned}
\grave{W}^*(\phi X,\phi Y,Z,W) = f_2\{&g(\phi X,\phi Z)g(\phi^2 Y,W) - g(\phi Y,\phi Z)g(\phi^2 X,W)\\
&+ 2g(\phi X,\phi^2 Y)g(\phi Z,W)\} + f_3\{g(\phi Y,Z)g(\phi X,W)\\
&- g(\phi X,Z)g(\phi Y,W)\}.
\end{aligned} \tag{3.9}
$$

Putting $Y = W = e_i$ where $\{e_i\}$, is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over i $(1 \leq i \leq 2n+1)$, we get

$$
\begin{aligned}
\sum_{i=1}^{2n+1} \grave{W}^*(\phi X,\phi e_i,Z,e_i) = f_2\{&-g(\phi X,\phi Z)g(\phi e_i,\phi e_i)\\
&+ g(\phi^2 Z,\phi^2 X) + 2g(\phi^2 X,\phi^2 Z)\}\\
&- f_3 g(\phi Z,\phi X).
\end{aligned} \tag{3.10}
$$

Putting $X = Z = e_i$, where e_i, is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over i $(1 \leq i \leq 2n+1)$, we get after simplification that $f_2 = 0$. But then $f_3 = 0$ by (3.7).

Therefore,

$$R(X,Y)Z = f_1[g(Y,Z)X - g(X,Z)Y]. \tag{3.11}$$

The above equation gives

$$S(X,Y) = 2nf_1 g(X,Y). \tag{3.12}$$

Hence in view of (1.2), we have $W^*(X,Y)Z = 0$. This leads us to state the following.

Theorem 3.1. *A $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian space form is M-projectively flat if and only if $f_3 = \frac{3f_2}{1-2n}$.*

But in [**14**], the author proved that if a $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian space form is Ricci semisymmetric, then $f_3 = \frac{3f_2}{1-2n}$. Hence we conclude the following.

Corollary 3.1. *If a $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian space form is Ricci semisymmetric, then it is M-projectively flat.*

4. An η-Einstein generalized Sasakian space form satisfying $W^*(\xi,X)R = 0$

In view of (2.4), (2.8), (2.9) and (2.12), (1.2) becomes

$$W^*(\xi,X)Y = \frac{1}{4n}[(1-2n)f_3 - 3f_2]\{g(X,Y)\xi - \eta(Y)X\}. \tag{4.1}$$

Now we have

$$
\begin{aligned}
(W^*(\xi,X)R)(Y,Z)U = {}& W^*(\xi,X)R(Y,Z)U - R(W^*(\xi,X)Y,Z)U\\
&- R(Y,W^*(\xi,X)Z)U - R(Y,Z)W^*(\xi,X)U.
\end{aligned} \tag{4.2}
$$

But as we assume $W^*(\xi,X)R=0$, (4.2) takes the form

$$\begin{aligned} W^*(\xi,X)R(Y,Z)U - R(W^*(\xi,X)Y,Z)U \\ - R(Y,W^*(\xi,X)Z)U - R(Y,Z)W^*(\xi,X)U = 0. \end{aligned} \tag{4.3}$$

Using (2.4), (2.11), (2.12), (2.13) and (4.1) in (4.3), we get

$$\begin{aligned} &\frac{1}{4n}[(1-2n)f_3 - 3f_2][\grave{R}(X,Y,Z,U)\xi + \eta(Y)R(X,Z)U \\ &+ \eta(Z)R(Y,X)U + \eta(U)R(Y,Z)X - (f_1-f_3)\{g(Z,U)\eta(Y)X \\ &- g(Y,U)\eta(Z)X + g(X,Y)g(Z,U)\xi - g(X,Y)\eta(U)Z \\ &- g(X,Z)g(Y,U)\xi + g(X,Z)\eta(U)Y + g(X,U)\eta(Z)Y \\ &- g(X,U)\eta(Y)Z\}] = 0, \end{aligned} \tag{4.4}$$

where

$$\grave{R}(X,Y,Z,U) = g(X,R(Y,Z)U). \tag{4.5}$$

Taking inner product of (4.4) with respect to the Riemannian metric g and then using (2.4) and (2.13), we have

$$\begin{aligned} &\frac{1}{4n}[(1-2n)f_3 - 3f_2][\grave{R}(X,Y,Z,U) - (f_1-f_3)\{g(X,Y)g(Z,U) \\ &- g(X,Z)g(Y,U)\}] = 0. \end{aligned} \tag{4.6}$$

Then

$$f_3 = \frac{3f_2}{(1-2n)}$$

or

$$\grave{R}(X,Y,Z,U) = (f_1-f_3)\{g(X,Y)g(Z,U) - g(X,Z)g(Y,U)\}. \tag{4.7}$$

Using (2.4) and (4.5) in (4.7), we get

$$R(Y,Z)U \quad = \quad (f_1-f_3)\{g(Z,U)Y - g(Y,U)Z\}. \tag{4.8}$$

Contracting (4.8) with respect to the vector field Y, we find

$$S(Z,U) = 2n(f_1-f_3)g(Z,U). \tag{4.9}$$

Therefore,

$$QZ = 2n(2n+1)(f_1-f_3)Z. \tag{4.10}$$

Hence,

$$r = 2n(2n+1)(f_1-f_3) \qquad \text{and so} \qquad f_3 = \frac{3f_2}{(1-2n)}. \tag{4.11}$$

Thus, we state following theorem.

Theorem 4.1. *A* $(2n+1)$*-dimensional* $(n>1)$ η*-Einstein generalized Sasakian space form satisfies the condition* $W^*(\xi,X)R=0$ *if and only if* $f_3 = \frac{3f_2}{(1-2n)}$.

In the light of Theorems 3.1 and 4.1, we state next collorary.

Corollary 4.1. *A $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian space form satisfies the condition $W^*(\xi, X)R = 0$ if and only if it is M-projectively flat.*

5. The irrotational M-projective curvature tensor

Definition 5.1. The rotation (curl) of M-projective curvature tensor W^* on a Riemannian manifold is given by [**1**]

$$\begin{aligned}\mathrm{Rot}W^* = (\nabla_U W^*)(X,Y)Z + (\nabla_X W^*)(U,Y)Z \\ + (\nabla_Y W^*)(X,U)Z - (\nabla_Z W^*)(X,Y)U.\end{aligned} \tag{5.1}$$

By virtue of second Bianchi identity, we have

$$(\nabla_U W^*)(X,Y)Z + (\nabla_X W^*)(U,Y)Z + (\nabla_Y W^*)(X,U)Z = 0.$$

Therefore, (5.1) becomes

$$\mathrm{Rot}W^* = -(\nabla_Z W^*)(X,Y)U. \tag{5.2}$$

If the M-projective curvature tensor is irrotational, then $\mathrm{curl}W^* = 0$, and so by (5.2) we get

$$(\nabla_Z W^*)(X,Y)U = 0.$$

Thus,

$$\begin{aligned}(\nabla_Z W^*)(X,Y)U = W^*(\nabla_Z X, Y)U + W^*(X, \nabla_Z Y)U \\ + W^*(X,Y)\nabla_Z U.\end{aligned} \tag{5.3}$$

Replacing $U = \xi$ in (5.3), we have

$$\begin{aligned}(\nabla_Z W^*)(X,Y)\xi = W^*(\nabla_Z X, Y)\xi + W^*(X, \nabla_Z Y)\xi \\ + W^*(X,Y)\nabla_Z \xi.\end{aligned} \tag{5.4}$$

Now, substituting $Z = \xi$ in (1.2) and then using (2.4), (2.8), (2.11) and (2.14), we obtain

$$(\nabla_Z W^*)(X,Y)\xi \quad = \quad k[\eta(Y)X - \eta(X)Y], \tag{5.5}$$

where

$$k = \frac{1}{4n}[(1-2n)f_3 - 3f_2]. \tag{5.6}$$

Using (5.5) in (5.4), we obtain

$$W^*(X,Y)\phi Z \quad = \quad k[g(Z,\phi X)Y - g(Z,\phi Y)X]. \tag{5.7}$$

Replacing Z by ϕZ in (5.7) and simplifying by using (2.1) and (2.3), we get

$$W^*(X,Y)Z = k[g(Z,Y)X - g(Z,X)Y]. \tag{5.8}$$

Also equations (1.2) and (5.8) give

$$\begin{aligned}k[g(Z,Y)X - g(Z,X)Y] = R(X,Y)Z - \frac{1}{4n}[S(Y,Z)X - S(X,Z)Y \\ + g(Y,Z)QX - g(X,Z)QY].\end{aligned} \tag{5.9}$$

Contracting the above equation with respect to the vector X and then using (5.6), we find

$$S(Y,Z) = 2n(f_1 - f_3)g(Y,Z), \tag{5.10}$$

which gives

$$r = 2n(2n+1)(f_1 - f_3). \tag{5.11}$$

In consequence of (1.2), (5.6), (5.8), (5.10) and (5.11) we can find

$$R(X,Y)Z = -(f_1 - f_3)[g(Y,Z)X - g(X,Z)Y]. \tag{5.12}$$

Therefore, we can state the following theorem.

Theorem 5.1. *The M-projective curvature tensor in an η-Einstein generalized Sasakian space form is irrotational if and only if $f_3 = \frac{3f_2}{(1-2n)}$.*

Theorem 4.1 together with Theorem 5.1 lead to the following corollaries.

Corollary 5.1. *A $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian space form satisfies the condition $W^*(\xi, X)R = 0$ if and only if the M-projective curvature tensor is irrotational.*

Corollary 5.2. *A $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian space form is irrotational if and only if it is M-projectively flat.*

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