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STRENGTHENED CONVERSES OF THE JENSEN AND EDMUNDSON–LAH–RIBARIČ INEQUALITIES

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Abstract. In this paper, we give converses of the Jensen and Edmundson–Lah–Ribarič inequalities which are more accurate than the existing ones. These converses are given in a difference form and they rely on the recent refinement of the Jensen inequality obtained via linear interpolation of a convex function. As an application, we also derive improved converse relations for generalized means, for the Hölder and Hermite–Hadamard inequalities as well as for the inequalities of Giaccardi and Petrović.

1. Introduction

One of the most interesting inequalities in present mathematics is the Jensen inequality due to the fact that it implies a whole series of classical inequalities such as the arithmetic-geometric mean inequality, the Hölder and Minkowski inequalities, the Ky Fan inequality etc. Applications of this inequality in various fields of mathematics, especially in mathematical analysis and statistics, have certainly contributed to its importance. During decades, the Jensen inequality was extensively studied by numerous authors and was generalized in different directions. For a comprehensive inspection of the Jensen inequality including history, proofs and diverse applications, the reader is referred to [5], [11], [14], and [16].

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In this paper, we refer to a quite general form of the Jensen inequality for positive linear functionals. In order to present our results, we first introduce the appropriate setting. Let E be a nonempty set and L be a vector space of real-valued functions $f\colon E \to \mathbb{R}$ having following properties:

L1: $f, g \in L \Rightarrow af + bg \in L$ for all $a, b \in \mathbb{R}$;
L2: $1 \in L$, where 1 denotes the constant function.

In other words, L is a subspace of the vector space $\mathbb{R}^E$ over $\mathbb{R}$ containing 1.
We also consider positive linear functionals $A\colon L \to \mathbb{R}$, that is, we assume that:

A1: $A(af + bg) = aA(f) + bA(g)$ for $f, g \in L$ and $a, b \in \mathbb{R}$;
A2: $f \in L$, $f(t) \geq 0$ for every $t \in E$ implies that $A(f) \geq 0$ (that is A is positive).

In addition, if $A(1) = 1$, we say that A is a positive normalized linear functional.
In [7], one can find the following generalization of the Jensen inequality for convex functions involving positive normalized linear functionals.

Theorem 1.1. (see [7]) *Let L fulfills properties* (L1), (L2), *and let $\phi : I \to \mathbb{R}$ be a continous convex function. If A is a positive normalized linear functional, then for all $f \in L$ such that $\phi(f) \in L$, it follows that $A(f) \in I$ and*

$$\phi(A(f)) \leq A(\phi(f)). \tag{1.1}$$

Inequality (1.1) is sometimes called the Jessen inequality (see also [14, p.47]), but in this paper it will be referred to as the Jensen inequality, for the sake of simplicity.

Closely connected to the Jensen inequality is the Edmundson–Lah–Ribarič inequality. It has been proved in 1973. by Lah and Ribarič [9]. Since then, there have been many papers written on the subject of its generalizations, refinements and reverses. Now, we state a generalization of the Edmundson–Lah–Ribarič inequality for positive linear functionals, proved by Beesack and Pečarić [1] (see also [14, p.98]).

Theorem 1.2. (see [1]) *Let $\phi : [m, M] \to \mathbb{R}$ be a convex function, let L fulfills conditions* (L1), (L2), *and let A be any positive normalized linear functional on L. If $f \in L$ is such that $\phi(f) \in L$ (so that $m \leq f(t) \leq M$ for all $t \in E$), then*

$$A(\phi(f)) \leq \frac{M - A(f)}{M - m}\phi(m) + \frac{A(f) - m}{M - m}\phi(M). \tag{1.2}$$

Recently, Jakšić and Pečarić [6], obtained several converses of inequalities (1.1) and (1.2), provided that $\phi : I \to \mathbb{R}$ is a continuous convex function such that $[m, M] \subseteq \operatorname{Int} I$, where $\operatorname{Int} I$ stands for the interior of an interval I. They showed that if $\phi \circ f \in L$, then the following difference type converses of Jensen inequality

(1.1) hold:

$$\begin{aligned}0 &\le A(\phi(f)) - \phi(A(f)) \\ &\le (M - A(f))(A(f) - m) \sup_{t\in(m,M)} \Psi_\phi(t; m, M) \\ &\le (M - A(f))(A(f) - m)\frac{\phi'_-(M) - \phi'_+(m)}{M - m} \\ &\le \frac{1}{4}(M - m)(\phi'_-(M) - \phi'_+(m))\end{aligned} \tag{1.3}$$

and

$$\begin{aligned}0 \le A(\phi(f)) - \phi(A(f)) &\le \frac{1}{4}(M - m)^2\Psi_\phi(A(f); m, M) \\ &\le \frac{1}{4}(M - m)(\phi'_-(M) - \phi'_+(m)),\end{aligned} \tag{1.4}$$

where $\Psi_\phi(\cdot; m, M)\colon (m, M) \to \mathbb{R}$ is defined by

$$\Psi_\phi(t; m, M) = \frac{1}{M - m}\Big(\frac{\phi(M) - \phi(t)}{M - t} - \frac{\phi(t) - \phi(m)}{t - m}\Big), \tag{1.5}$$

and where $\phi'_-(M)$, $\phi'_+(m)$ are one-sided derivatives of function ϕ at the corresponding points.

In the same paper, the authors also proved the following difference type converses of the Edmundson–Lah–Ribarič inequality (1.2):

$$\begin{aligned}0 &\le \frac{A(f) - m}{M - m}\phi(M) + \frac{M - A(f)}{M - m}\phi(m) - A(\phi(f)) \\ &\le A[(M - f)(f - m)] \sup_{t\in(m,M)} \Psi_\phi(t; m, M) \\ &\le \frac{A[(M - f)(f - m)]}{M - m}(\phi'_-(M) - \phi'_+(m)) \\ &\le \frac{(M - A(f))(A(f) - m)}{M - m}(\phi'_-(M) - \phi'_+(m)) \\ &\le \frac{1}{4}(M - m)(\phi'_-(M) - \phi'_+(m)),\end{aligned} \tag{1.6}$$

$$\begin{aligned}0 &\le \frac{A(f) - m}{M - m}\phi(M) + \frac{M - A(f)}{M - m}\phi(m) - A(\phi(f)) \\ &\le A[(M - f)(f - m)] \sup_{t\in(m,M)} \Psi_\phi(t; m, M) \\ &\le (M - A(f))(A(f) - m) \sup_{t\in(m,M)} \Psi_\phi(t; m, M) \\ &\le \frac{(M - A(f))(A(f) - m)}{M - m}(\phi'_-(M) - \phi'_+(m)) \\ &\le \frac{1}{4}(M - m)(\phi'_-(M) - \phi'_+(m)),\end{aligned} \tag{1.7}$$

and

$$\begin{aligned}0 &\leq \frac{A(f)-m}{M-m}\phi(M)+\frac{M-A(f)}{M-m}\phi(m)-A(\phi(f))\\ &\leq \frac{1}{4}(M-m)^2A(\Psi_\phi(f;m,M))\\ &\leq \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m)),\end{aligned} \tag{1.8}$$

provided that $\Psi_\phi(f;m,M)\in L$ in (1.8).

The main objective of this paper is to give improvements of the above converses. Our main results rely on the refinement of the Edmundson–Lah–Ribarič inequality obtained by Klaričić Bakula et.al. [8], and the recent refinement of the Jensen inequality via linear interpolation, established by Choi et.al. [4]. As an application, we shall also derive improved converse relations for generalized means, for the Hölder and Hermite–Hadamard inequalities as well as for the inequalities of Giaccardi and Petrović.

2. Main results

In order to derive improvements of the relations stated in the introduction, we will also need to equip our linear class L of functions with an additional property:

L3: $f,g\in L\Rightarrow \min\{f,g\}\in L\wedge\max\{f,g\}\in L$ (lattice property).

Hence, from now on, without further noticing, L stands for a vector space of real-valued functions $f\colon E\to\mathbb{R}$ satisfying properties (L1), (L2), and (L3), while A denotes a positive normalized linear functional acting on L.

In order to obtain the corresponding results, we first state the improved form of the Edmundson–Lah–Ribarič inequality obtained by Klaričić Bakula et.al. [8].

Theorem 2.1. (see [8]) *Let $\phi:[m,M]\to\mathbb{R}$ be a convex function and $f\in L$ be such that $\phi\circ f\in L$. Then, $A(f)\in[m,M]$ and*

$$A(\phi(f))\leq \frac{M-A(f)}{M-m}\phi(m)+\frac{A(f)-m}{M-m}\phi(M)-A(\tilde{f})\delta_\phi, \tag{2.1}$$

where

$$\tilde{f}=\frac{1}{2}-\frac{|f-\frac{m+M}{2}|}{M-m},\quad \delta_\phi=\phi(m)+\phi(M)-2\phi\Big(\frac{m+M}{2}\Big). \tag{2.2}$$

Our basic results will also include the function $\Psi_\phi(\cdot;m,M)$ defined by (1.5). It should be noticed here that the expression (1.5) is actually the second order divided difference of the function ϕ at points m,t, and M, for every $t\in(m,M)$.

First we give improved forms of converse Jensen relations (1.3) and (1.4).

Theorem 2.2. *Let $\phi : I \to \mathbb{R}$ be a continuous convex function and $[m, M] \subseteq \operatorname{Int} I$. If $f \in L$ is such that $f(E) \subseteq [m, M]$ and $\phi \circ f \in L$, then*

$$\begin{aligned}
0 &\le A(\phi(f)) - \phi(A(f)) \\
&\le (M - A(f))(A(f) - m) \sup_{t\in(m,M)} \Psi_\phi(t; m, M) - A(\tilde{f})\delta_\phi \\
&\le (M - A(f))(A(f) - m)\frac{\phi'_-(M) - \phi'_+(m)}{M - m} - A(\tilde{f})\delta_\phi \qquad (2.3)\\
&\le \frac{1}{4}(M - m)(\phi'_-(M) - \phi'_+(m)) - A(\tilde{f})\delta_\phi
\end{aligned}$$

and

$$\begin{aligned}
0 \le A(\phi(f)) - \phi(A(f)) &\le \frac{1}{4}(M - m)^2\Psi_\phi(A(f); m, M) - A(\tilde{f})\delta_\phi \\
&\le \frac{1}{4}(M - m)(\phi'_-(M) - \phi'_+(m)) - A(\tilde{f})\delta_\phi, \qquad (2.4)
\end{aligned}$$

where $\Psi_\phi, \tilde{f}$ and δ_ϕ are defined by (1.5) and (2.2). If ϕ is concave on I, then the inequality signs in (2.3) and (2.4) are reversed.

Proof. According to the property (L3), we have

$$\tilde{f} = \min\left\{\frac{M - f}{M - m}, \frac{f - m}{M - m}\right\} = \frac{1}{2} - \frac{|f - \frac{m+M}{2}|}{M - m} \in L.$$

Further, if $A(f) = m$ or $A(f) = M$ the inequalities hold trivially, so without loss of generality we can suppose that $A(f) \in (m, M)$. The first inequality in (2.3) is Jensen inequality (1.1). By Theorem 2.1, we have

$$\begin{aligned}
&A(\phi(f)) - \phi(A(f)) \le \frac{M - A(f)}{M - m}\phi(m) + \frac{A(f) - m}{M - m}\phi(M) - \phi(A(f)) - A(\tilde{f})\delta_\phi \\
&= \frac{(M - A(f))(A(f) - m)}{M - m}\left\{\frac{\phi(M) - \phi(A(f))}{M - A(f)} - \frac{\phi(A(f)) - \phi(m)}{A(f) - m}\right\} - A(\tilde{f})\delta_\phi \\
&= (M - A(f))(A(f) - m)\Psi_\phi(A(f); m, M) - A(\tilde{f})\delta_\phi \\
&\le (M - A(f))(A(f) - m) \sup_{t\in(m,M)} \Psi_\phi(t; m, M) - A(\tilde{f})\delta_\phi,
\end{aligned}$$

so the second inequality in (2.3) holds. Now, the third inequality sign in (2.3) follows from

$$\begin{aligned}
&\sup_{t\in(m,M)} \Psi_\phi(t; m, M) = \frac{1}{M - m} \sup_{t\in(m,M)} \left\{\frac{\phi(M) - \phi(t)}{M - t} - \frac{\phi(t) - \phi(m)}{t - m}\right\} \\
&\le \frac{1}{M - m}\left(\sup_{t\in(m,M)} \frac{\phi(M) - \phi(t)}{M - t} + \sup_{t\in(m,M)} \frac{-(\phi(t) - \phi(m))}{t - m}\right) \\
&= \frac{1}{M - m}\left(\sup_{t\in(m,M)} \frac{\phi(M) - \phi(t)}{M - t} - \inf_{t\in(m,M)} \frac{\phi(t) - \phi(m)}{t - m}\right) = \frac{\phi'_-(M) - \phi'_+(m)}{M - m}.
\end{aligned}$$

Finally, the last inequality in (2.3) holds due to the classical arithmetic-geometric mean inequality $\frac{(M-A(f))(A(f)-m)}{M-m} \le \frac{1}{4}(M - m)$, since $A(f) \in [m, M]$.

The proof of the series of inequalities in (2.4) is clear from the proof of (2.3). At last, if ϕ is concave, then $-\phi$ is convex, so applying (2.3) and (2.4) to $-\phi$ we obtain relations with a reversed signs of inequalities. □

Remark 2.3. It should be noticed here that the function ϕ (Theorem 2.2) is defined on interval I such that $[m, M] \subseteq \operatorname{Int} I$. This condition assures finiteness of the one-sided derivatives at points m and M, which implies that the expression $\Psi_\phi(t; m, M)$ is meaningful for all $t \in [m, M]$.

Next, we give improved forms of converses (1.6), (1.7), and (1.8), related to the Edmundson–Lah–Ribarič inequality.

Theorem 2.4. *Let $\phi : I \to \mathbb{R}$ be a continuous convex function and $[m, M] \subseteq \operatorname{Int} I$. If $f \in L$ is such that $f(E) \subseteq [m, M]$ and $\phi \circ f \in L$, then*

$$\begin{aligned}
0 &\le \frac{A(f)-m}{M-m}\phi(M) + \frac{M-A(f)}{M-m}\phi(m) - A(\phi(f)) - A(\tilde{f})\delta_\phi \\
&\le A[(M-f)(f-m)] \sup_{t\in(m,M)} \Psi_\phi(t; m, M) - A(\tilde{f})\delta_\phi \\
&\le \frac{A[(M-f)(f-m)]}{M-m}(\phi'_-(M) - \phi'_+(m)) - A(\tilde{f})\delta_\phi \\
&\le \frac{(M-A(f))(A(f)-m)}{M-m}(\phi'_-(M) - \phi'_+(m)) - A(\tilde{f})\delta_\phi \\
&\le \frac{1}{4}(M-m)(\phi'_-(M) - \phi'_+(m)) - A(\tilde{f})\delta_\phi,
\end{aligned} \tag{2.5}$$

$$\begin{aligned}
0 &\le \frac{A(f)-m}{M-m}\phi(M) + \frac{M-A(f)}{M-m}\phi(m) - A(\phi(f)) - A(\tilde{f})\delta_\phi \\
&\le A[(M-f)(f-m)] \sup_{t\in(m,M)} \Psi_\phi(t; m, M) - A(\tilde{f})\delta_\phi \\
&\le (M-A(f))(A(f)-m) \sup_{t\in(m,M)} \Psi_\phi(t; m, M) - A(\tilde{f})\delta_\phi \\
&\le \frac{(M-A(f))(A(f)-m)}{M-m}(\phi'_-(M) - \phi'_+(m)) - A(\tilde{f})\delta_\phi \\
&\le \frac{1}{4}(M-m)(\phi'_-(M) - \phi'_+(m)) - A(\tilde{f})\delta_\phi,
\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}
0 &\le \frac{A(f)-m}{M-m}\phi(M) + \frac{M-A(f)}{M-m}\phi(m) - A(\phi(f)) - A(\tilde{f})\delta_\phi \\
&\le \frac{1}{4}(M-m)^2 A(\Psi_\phi(f; m, M)) - A(\tilde{f})\delta_\phi \\
&\le \frac{1}{4}(M-m)(\phi'_-(M) - \phi'_+(m)) - A(\tilde{f})\delta_\phi,
\end{aligned} \tag{2.7}$$

where $\Psi_\phi, \tilde{f}$ and δ_ϕ are defined by (1.5) and (2.2), and $\Psi_\phi(f; m, M) \in L$ in (2.7). If ϕ is concave on I, then the inequality signs in (2.5), (2.6) and (2.7) are reversed.

Proof. It has been shown in the proof of Theorem 2.2 that $\tilde{f} \in L$. Now, since $f(E) \subseteq [m, M]$, following the lines as in the proof of Theorem 2.2, we obtain scalar relations

$$\begin{aligned}
&\frac{f(t)-m}{M-m}\phi(M) + \frac{M-f(t)}{M-m}\phi(m) - \phi(f(t)) \\
&\leq (M-f(t))(f(t)-m) \sup_{t\in(m,M)} \Psi_\phi(t; m, M) \\
&\leq \frac{(M-f(t))(f(t)-m)}{M-m}(\phi'_-(M) - \phi'_+(m)) \\
&\leq \frac{1}{4}(M-m)(\phi'_-(M) - \phi'_+(m))
\end{aligned}$$

and

$$\begin{aligned}
&\frac{f(t)-m}{M-m}\phi(M) + \frac{M-f(t)}{M-m}\phi(m) - \phi(f(t)) \\
&\leq \frac{1}{4}(M-m)^2 \Psi_\phi(f; m, M) \\
&\leq \frac{1}{4}(M-m)(\phi'_-(M) - \phi'_+(m)).
\end{aligned}$$

In addition, applying a positive normalized linear functional A to the above series of inequalities, and taking into account the inequality (2.1), we obtain the first three inequality signs in (2.5) and the series of inequalities in (2.7). To prove the fourth inequality in (2.5), we need to notice that the quadratic function $g(t) = (M-t)(t-m)$ is concave, so by Jensen inequality (1.1) we have $A(g(f)) - A(\tilde{f})\delta_\phi \leq g(A(f)) - A(\tilde{f})\delta_\phi$. Finally, the last inequality in (2.5) follows from the arithmetic-geometric mean inequality.

To conclude the proof, it suffices to justify the third inequality sign in (2.6). Clearly, it follows again by virtue of concavity of the function $g(t) = (M-t)(t-m)$ and Jensen inequality (1.1). □

Remark 2.5. Obviously, our Theorems 2.2 and 2.4 are improvements of relations (1.3), (1.4), (1.6), (1.7) and (1.8) presented in the introduction, since under the required assumptions we have

$$A(\tilde{f})\delta_\phi = A\Big(\frac{1}{2} - \frac{|f - \frac{m+M}{2}|}{M-m}\Big)\Big(\phi(m) + \phi(M) - 2\phi\Big(\frac{m+M}{2}\Big)\Big) \geq 0.$$

Remark 2.6. It should be noticed here that the series of inequalities in (2.5) and (2.6) differ only in the third line. This occurs as a result of an order of estimating expressions $A[(M-f)(f-m)]$ and $\Psi_\phi(t; m, M)$. Therefore, without further noticing, inequalities related to (2.6) will be omitted in the sequel.

Improved converse relations obtained in Theorems 2.2 and 2.4 rely on the refinement of the Edmundson–Lah–Ribarič inequality given by (2.1). Recently, Pečarić and Perić [13], established even more accurate version of the Edmundson–Lah–Ribarič inequality. The corresponding result is derived by virtue of the refinement of the Jensen inequality via linear interpolation obtained by Choi et.al. [4].

Theorem 2.7. (see [13]) *Let $\phi : [m, M] \to \mathbb{R}$ be a convex function and $f \in L$ be such that $\phi \circ f \in L$. Then, $A(f) \in [m, M]$ and*

$$A(\phi(f)) \leq \frac{M - A(f)}{M - m}\phi(m) + \frac{A(f) - m}{M - m}\phi(M) - R_{\phi,A}(m, M; f), \tag{2.8}$$

where

$$R_{\phi,A}(m, M; f) = \sum_{n=0}^{N-1}\sum_{k=1}^{2^n} \Delta_\phi(m, M, n, k) A\Big(r_n \chi_{(\frac{k-1}{2^n}, \frac{k}{2^n})}\Big(\frac{f - m}{M - m}\Big)\Big), \tag{2.9}$$

$r_0(t) = \min\{t, 1 - t\}$, $r_n(t) = \min\{2r_{n-1}(t), 1 - 2r_{n-1}(t)\}$, $0 \leq t \leq 1$,

$$\begin{aligned}\Delta_\phi(m, M, n, k) =& \phi\Big(\frac{(2^n - k + 1)m + (k - 1)M}{2^n}\Big) + \phi\Big(\frac{(2^n - k)m + kM}{2^n}\Big) \\ &- 2\phi\Big(\frac{(2^{n+1} - 2k + 1)m + (2k - 1)M}{2^{n+1}}\Big),\end{aligned}$$

and where χ stands for a characteristic function of the corresponding interval.

Remark 2.8. Any summation having $\sum_{n=0}^{N-1}$ is assumed to be zero for $N = 0$, therefore inequality (2.8) may be regarded as a generalization of inequality (2.1). The functions r_n, $n \in \mathbb{N}$, are non-negative and it has been shown in [4] that they can be rewritten in an explicit form

$$r_n(t) = \begin{cases} 2^n t - k + 1, & \frac{k-1}{2^n} \leq t \leq \frac{2k-1}{2^{n+1}}, \\ k - 2^n t, & \frac{2k-1}{2^{n+1}} < t \leq \frac{k}{2^n}, \end{cases} \tag{2.10}$$

for $k = 1, 2, \dots, 2^n$. In addition, if $N \geq 1$, then $R_{\phi,A}(m, M; f)$ can be rewritten in the following way:

$$\begin{aligned}R_{\phi,A}(m, M; f) =& \Delta_\phi(m, M, 0, 1) A\Big(r_0 \chi_{(0,1)}\Big(\frac{f - m}{M - m}\Big)\Big) \\ &+ \sum_{n=1}^{N-1}\sum_{k=1}^{2^n} \Delta_\phi(m, M, n, k) A\Big(r_n \chi_{(\frac{k-1}{2^n}, \frac{k}{2^n})}\Big(\frac{f - m}{M - m}\Big)\Big).\end{aligned}$$

Now, since $\chi_{(0,1)}\Big(\frac{f-m}{M-m}\Big) = 1$, $\Delta_\phi(m, M, 0, 1) = \phi(m) + \phi(M) - 2\phi\Big(\frac{m + M}{2}\Big)$, and

$$r_0\Big(\frac{f - m}{M - m}\Big) = \min\Big\{\frac{f - m}{M - m}, 1 - \frac{f - m}{M - m}\Big\} = \frac{1}{2} - \frac{|f - \frac{m+M}{2}|}{M - m},$$

it follows that the inequality (2.8) provides sharper estimate for the Edmundson–Lah–Ribarič inequality than inequality (2.1).

According to the previous remark we can give strengthened Theorems 2.2 and 2.4. More precisely, following the lines of the proofs of Theorems 2.2 and 2.4 with a term $R_{\phi,A}(m, M; f)$ instead of $A(\tilde{f})\delta_\phi$, and taking into account relation (2.8), we give now sharper forms for converses of the Jensen and Edmundson–Lah–Ribarič inequalities than those established in Theorems 2.2 and 2.4.

Theorem 2.9. *Let $\phi : I \to \mathbb{R}$ be a continuous convex function and $[m, M] \subseteq \operatorname{Int} I$. If $f \in L$ is such that $f(E) \subseteq [m, M]$ and $\phi \circ f \in L$, then*

$$\begin{aligned} 0 &\le A(\phi(f)) - \phi(A(f)) \\ &\le (M - A(f))(A(f) - m) \sup_{t\in(m,M)} \Psi_\phi(t; m, M) - R_{\phi,A}(m, M; f) \\ &\le (M - A(f))(A(f) - m) \frac{\phi'_-(M) - \phi'_+(m)}{M - m} - R_{\phi,A}(m, M; f) \\ &\le \frac{1}{4}(M - m)(\phi'_-(M) - \phi'_+(m)) - R_{\phi,A}(m, M; f) \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} 0 \le A(\phi(f)) - \phi(A(f)) &\le \frac{1}{4}(M - m)^2 \Psi_\phi(A(f); m, M) - R_{\phi,A}(m, M; f) \\ &\le \frac{1}{4}(M - m)(\phi'_-(M) - \phi'_+(m)) - R_{\phi,A}(m, M; f), \end{aligned} \tag{2.12}$$

where Ψ_ϕ and $R_{\phi,A}$ are defined by (1.5) and (2.9). If ϕ is concave on I, then the inequality signs in (2.11) and (2.12) are reversed.

Theorem 2.10. *Let $\phi : I \to \mathbb{R}$ be a continuous convex function and $[m, M] \subseteq \operatorname{Int} I$. If $f \in L$ is such that $f(E) \subseteq [m, M]$ and $\phi \circ f \in L$, then*

$$\begin{aligned} 0 &\le \frac{A(f) - m}{M - m}\phi(M) + \frac{M - A(f)}{M - m}\phi(m) - A(\phi(f)) - R_{\phi,A}(m, M; f) \\ &\le A[(M - f)(f - m)] \sup_{t\in(m,M)} \Psi_\phi(t; m, M) - R_{\phi,A}(m, M; f) \\ &\le \frac{A[(M - f)(f - m)]}{M - m}(\phi'_-(M) - \phi'_+(m)) - R_{\phi,A}(m, M; f) \\ &\le \frac{(M - A(f))(A(f) - m)}{M - m}(\phi'_-(M) - \phi'_+(m)) - R_{\phi,A}(m, M; f) \\ &\le \frac{1}{4}(M - m)(\phi'_-(M) - \phi'_+(m)) - R_{\phi,A}(m, M; f) \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} 0 &\le \frac{A(f) - m}{M - m}\phi(M) + \frac{M - A(f)}{M - m}\phi(m) - A(\phi(f)) - R_{\phi,A}(m, M; f) \\ &\le \frac{1}{4}(M - m)^2 A(\Psi_\phi(f; m, M)) - R_{\phi,A}(m, M; f) \\ &\le \frac{1}{4}(M - m)(\phi'_-(M) - \phi'_+(m)) - R_{\phi,A}(m, M; f) \end{aligned} \tag{2.14}$$

where Ψ_ϕ and $R_{\phi,A}$ are defined by (1.5) and (2.9), and $\Psi_\phi(f; m, M) \in L$ in (2.14). If ϕ is concave on I, then the inequality signs in (2.13) and (2.14) are reversed.

Remark 2.11. Our main results presented in this section cover the classical discrete and integral case. Namely, common examples of positive linear functionals are $A(f) = \int_E f d\mu$ or $A(f) = \sum_{k\in E} p_k f_k$, where μ is positive measure on E in the first case, and in the other, $E = \mathbb{N}$ is a countable set with the discrete measure $\mu(k) = p_k \ge 0$, $0 < \sum_{k\in E} p_k < \infty$, $f(k) = f_k$, defined on it.

Moreover, let $X\colon \Omega \to [m, M]$ be a random variable on a probability space (Ω, p) with finite expectation $\mathbb{E}[X]$. Then, setting $A = \mathbb{E}$ and $f = X$, our Theorems 2.2, 2.4, 2.9 and 2.10 yield probabilistic versions of converses for the Jensen and Edmundson–Lah–Ribarič inequalities, provided that $\mathbb{E}[\phi(X)] < \infty$.

3. Applications

In this section, we give applications of our main results to generalized means, to the Hölder and Hermite–Hadamard inequalities and to inequalities of Giaccardi and Petrović. In such a way, we will obtain more precise converses for these inequalities. In some cases, we will also obtain the refinements of some aforementioned inequalities.

3.1. **Generalized means.** Let $\psi\colon I \to \mathbb{R}$ be continuous and strictly monotonic function, and let $f \in L$ be such that $\psi(f) \in L$. A generalized mean with respect to the functional A and the function ψ, for $f \in L$, is defined by

$$M_\psi(f, A) = \psi^{-1} A(\psi(f)).$$

First, we state two already known results referring to a comparison of these generalized means, and then, we give the corresponding converses based on our main results.

Theorem 3.1. (see [14]) *Let $\psi, \chi\colon I \to \mathbb{R}$ be continuous and strictly monotonic functions, and let $f \in L$ be such that $\psi(f), \chi(f) \in L$. Then,*

$$M_\psi(f, A) \leq M_\chi(f, A),$$

provided either χ is increasing and $\phi = \chi \circ \psi^{-1}$ is convex, or χ is decreasing and $\phi = \chi \circ \psi^{-1}$ is concave.

Theorem 3.2. (see [14]) *Let ψ and χ be as in Theorem 3.1, but with $I = [m, M]$. If $f \in L$ is such that $f(E) \subseteq [m, M]$, then*

$$(\psi(M) - \psi(m))A(\chi(f)) - (\chi(M) - \chi(m))A(\psi(f)) \leq \psi(M)\chi(m) - \chi(M)\psi(m),$$

provided that $\phi = \chi \circ \psi^{-1}$ is convex. The inequality is reversed if ϕ is concave.

By virtue of Theorems 2.9 and 2.10, we give now converse relations that correspond to Theorems 3.1 and 3.2.

Theorem 3.3. *Let $\psi, \chi\colon I \to \mathbb{R}$ be continuous and strictly monotonic functions such that the function $\phi = \chi \circ \psi^{-1}$ is convex on I. Let $[m, M] \subseteq \operatorname{Int} I$, and let*

$f \in L$ be such that $\psi(f), \chi(f) \in L$ and $f(E) \subseteq [m, M]$. Then,

$$\begin{aligned}
0 &\le \chi(M_\chi(f,A)) - \chi(M_\psi(f,A)) \\
&\le (M_\psi - A(\psi(f)))(A(\psi(f)) - m_\psi) \sup_{t\in(m_\psi,M_\psi)} \Psi_\phi(\psi(t); m_\psi, M_\psi) \\
&\quad - R_{\phi,A}(m_\psi, M_\psi, \psi(f)) \\
&\le (M_\psi - A(\psi(f)))(A(\psi(f)) - m_\psi) \frac{\phi'_-(M_\psi) - \phi'_+(m_\psi)}{M_\psi - m_\psi} \\
&\quad - R_{\phi,A}(m_\psi, M_\psi, \psi(f)) \\
&\le \frac{1}{4}(M_\psi - m_\psi)(\phi'_-(M_\psi) - \phi'_+(m_\psi)) - R_{\phi,A}(m_\psi, M_\psi, \psi(f))
\end{aligned}$$

and

$$\begin{aligned}
0 &\le \chi(M_\chi(f,A)) - \chi(M_\psi(f,A)) \\
&\le \frac{1}{4}(M_\psi - m_\psi)^2 \Psi_\phi(A(\psi(f)); m_\psi, M_\psi) - R_{\phi,A}(m_\psi, M_\psi, \psi(f)) \\
&\le \frac{1}{4}(M_\psi - m_\psi)(\phi'_-(M_\psi) - \phi'_+(m_\psi)) - R_{\phi,A}(m_\psi, M_\psi, \psi(f)),
\end{aligned}$$

where Ψ_ϕ and $R_{\phi,A}$ are defined by (1.5) and (2.9), and $[m_\psi, M_\psi] = \psi([m, M])$. If ϕ is concave, then the signs of inequalities are reversed.

Proof. Since $f(E) \subseteq [m, M]$, it follows that $m_\psi \le \psi(f(t)) \le M_\psi$ for every $t \in E$ (if ψ is increasing, then $m_\psi = \psi(m)$ and $M_\psi = \psi(M)$; if ψ is decreasing, then $m_\psi = \psi(M)$ and $M_\psi = \psi(m)$). Therefore, the conditions as in Theorem 2.9 are satisfied, so we obtain required inequalities by putting $m = m_\psi$, $M = M_\psi$ and replacing f with $\psi \circ f$ in (2.11) and (2.12) respectively. □

Theorem 3.4. *Suppose that the assumptions as in Theorem 3.3 are fulfilled. Then,*

$$\begin{aligned}
0 &\le \frac{A(\psi(f)) - m_\psi}{M_\psi - m_\psi}\chi(M_\psi) + \frac{M_\psi - A(\psi(f))}{M_\psi - m_\psi}\chi(m_\psi) - \chi(M_\chi(f,A)) \\
&\quad - R_{\phi,A}(m_\psi, M_\psi, \psi(f)) \\
&\le A[(M_\psi - \psi(f))(\psi(f) - m_\psi)] \sup_{t\in(m_\psi,M_\psi)} \Psi_\phi(\psi(t); m_\psi, M_\psi) \\
&\quad - R_{\phi,A}(m_\psi, M_\psi, \psi(f)) \\
&\le \frac{A[(M_\psi - \psi(f))(\psi(f) - m_\psi)]}{M_\psi - m_\psi}(\phi'_-(M_\psi) - \phi'_+(m_\psi)) \\
&\quad - R_{\phi,A}(m_\psi, M_\psi, \psi(f)) \\
&\le \frac{(M_\psi - A(\psi(f)))(A(\psi(f)) - m_\psi)}{M_\psi - m_\psi}(\phi'_-(M_\psi) - \phi'_+(m_\psi)) \\
&\quad - R_{\phi,A}(m_\psi, M_\psi, \psi(f)) \\
&\le \frac{1}{4}(M_\psi - m_\psi)(\phi'_-(M_\psi) - \phi'_+(m_\psi)) - R_{\phi,A}(m_\psi, M_\psi, \psi(f))
\end{aligned}$$

and

$$\begin{aligned}0 &\leq \frac{A(\psi(f)) - m_\psi}{M_\psi - m_\psi}\chi(M_\psi) + \frac{M_\psi - A(\psi(f))}{M_\psi - m_\psi}\chi(m_\psi) - \chi(M_\chi(f, A))\\ &\quad - R_{\phi,A}(m_\psi, M_\psi, \psi(f))\\ &\leq \frac{1}{4}(M_\psi - m_\psi)^2 A(\Psi_\phi(\psi(f); m_\psi, M_\psi)) - R_{\phi,A}(m_\psi, M_\psi, \psi(f))\\ &\leq \frac{1}{4}(M_\psi - m_\psi)(\phi'_-(M_\psi) - \phi'_+(m_\psi)) - R_{\phi,A}(m_\psi, M_\psi, \psi(f)),\end{aligned}$$

where Ψ_ϕ *and* $R_{\phi,A}$ *are defined by* (1.5) *and* (2.9)*, and* $[m_\psi, M_\psi] = \psi([m, M])$*. If* ϕ *is concave, then the signs of inequalities are reversed.*

Proof. It follows from Theorem 2.10 by utilizing the same substitutions as in the previous theorem. □

Remark 3.5. Power means

$$M^{[r]}(f, A) = \begin{cases} (A(f^r))^{1/r} & , r \neq 0 \\ \exp(A(\log f)) & , r = 0 \end{cases}$$

are the special case of generalized means, so by choosing $\chi(t) = t^s$, $\psi(t) = t^r$ in Theorems 3.3 and 3.4, which are either increasing or decreasing functions, one obtains the corresponding results for power means. Here, they are omitted. For more details about power means, the reader is referred to [2] and [14].

3.2. **The Hölder inequality.**

Generalizations of the Hölder inequality and its converse for positive linear functionals acting on the vector space of positive real-valued functions are stated in the following theorems:

Theorem 3.6. (see [14]) *Let* $p > 1$ *and* $q = p/(p-1)$*. If* $w, f, g \in L$ *are such that* $w, f, g \geq 0$ *and* $wf^p, wg^q, wfg \in L$*, then*

$$A(wfg) \leq A^{1/p}(wf^p)A^{1/q}(wg^q).$$

If $0 < p < 1$ *and* $A(wg^q) > 0$ *(or* $p < 0$ *and* $A(wf^p) > 0$*), then the inequality sign is reversed.*

Theorem 3.7. (see [14]) *Let* $p > 1$*,* $q = p/(p-1)$*, and* $w, f, g \in L$ *be such that* $w, f, g \geq 0$ *and* $wf^p, wg^q, wfg \in L$*. If* $fg^{-q/p}(E) \subseteq [m, M]$*, then*

$$(M - m)A(wf^p) + (mM^p - Mm^p)A(wg^q) \leq (M^p - m^p)A(wfg).$$

If $p < 0$*, then the inequality also holds provided that either* $A(wf^p) > 0$ *or* $A(wg^q) > 0$*. If* $0 < p < 1$*, then the reversed inequality holds provided that either* $A(wf^p) > 0$ *or* $A(wg^q) > 0$*.*

Our next application are converses in connection with the Hölder inequality and they give estimates for the difference between the right-hand side and the left-hand side of the inequalities from Theorems 3.6 and 3.7.

Theorem 3.8. *Let $p > 1$, $q = p/(p-1)$, $w, f, g \in L$ be such that $w, f, g \geq 0$ and $wf^p, wg^q, wfg \in L$. If $A(wg^q) > 0$ and $fg^{-q/p}(E) \subseteq [m, M]$, then*

$$\begin{aligned}
0 &\leq A(wf^p)A^{p/q}(wg^q) - A^p(wfg) \\
&\leq (MA(wg^q) - A(wfg))(A(wfg) - mA(wg^q)) \sup_{t\in(m,M)} \Psi_\phi(t; m, M)A^{p-2}(wg^q) \\
&\quad - \tilde{R}_{\phi,A}(m, M; fg^{-\frac{p}{q}})A^{p-1}(wg^q) \\
&\leq (MA(wg^q) - A(wfg))(A(wfg) - mA(wg^q))p\frac{M^{p-1} - m^{p-1}}{M - m}A^{p-2}(wg^q) \\
&\quad - \tilde{R}_{\phi,A}(m, M; fg^{-\frac{p}{q}})A^{p-1}(wg^q) \\
&\leq \frac{p}{4}(M - m)(M^{p-1} - m^{p-1})A^p(wg^q) - \tilde{R}_{\phi,A}(m, M; fg^{-\frac{p}{q}})A^{p-1}(wg^q)
\end{aligned}$$

and

$$\begin{aligned}
0 &\leq A(wf^p)A^{p/q}(wg^q) - A^p(wfg) \\
&\leq \frac{1}{4}(M - m)^2\Psi_\phi(\frac{A(wfg)}{A(wg^q)}; m, M)A^p(wg^q) - \tilde{R}_{\phi,A}(m, M; fg^{-\frac{p}{q}})A^{p-1}(wg^q) \\
&\leq \frac{p}{4}(M - m)(M^{p-1} - m^{p-1})A^p(wg^q) - \tilde{R}_{\phi,A}(m, M; fg^{-\frac{p}{q}})A^{p-1}(wg^q),
\end{aligned}$$

where $\phi(t) = t^p$, Ψ_ϕ is defined by (1.5), and

$$\begin{aligned}
&\tilde{R}_{\phi,A}(m, M; fg^{-\frac{p}{q}}) \\
&\quad = \sum_{n=0}^{N-1}\sum_{k=1}^{2^n} \Delta_\phi(m, M, n, k)A\Big(wg^q r_n \chi_{(\frac{k-1}{2^n}, \frac{k}{2^n})}\Big(\frac{fg^{-\frac{q}{p}} - m}{M - m}\Big)\Big),
\end{aligned}$$

where $\Delta_\phi(m, M, n, k)$ is defined in Theorem 2.7. If $A(wfg) > 0$, then the inequalities also hold for $p < 0$, while for $0 < p < 1$ the inequalities are reversed.

Proof. Clearly, the function $\phi(t) = t^p$ is convex (concave) for $p > 1$ and $p < 0$ (for $0 < p < 1$). We define $B(f) = \frac{A(wf)}{A(w)}$ for $w \in L$ such that $w \geq 0$ and $A(w) > 0$. Since $B(1) = 1$, the conditions of Theorem 2.9 are fulfilled. Now the required relations follow from (2.11) and (2.12) by replacing A with B, w with wg^q and f with $fg^{-q/p}$. □

In the same way, by virtue of Theorem 2.10, we get:

Theorem 3.9. *Suppose that the assumptions as in Theorem 3.8 are fulfilled. If $p > 1$ or $p < 0$, then*

$$\begin{aligned}
0 &\leq \frac{A(wfg)-mA(wg^q)}{M-m}M^p + \frac{MA(wg^q)-A(wfg)}{M-m}m^p - A(wf^p) \\
&\quad - \tilde{R}_{\phi,A}(m,M;fg^{-\frac{p}{q}}) \\
&\leq A(wg^q[(M-fg^{-q/p})(fg^{-q/p}-m)]) \sup_{t\in(m,M)} \Psi_\phi(t;m,M) \\
&\quad - \tilde{R}_{\phi,A}(m,M;fg^{-\frac{p}{q}}) \\
&\leq \frac{A(wg^q[(M-fg^{-q/p})(fg^{-q/p}-m)])}{M-m} p(M^{p-1}-m^{p-1}) \\
&\quad - \tilde{R}_{\phi,A}(m,M;fg^{-\frac{p}{q}}) \\
&\leq \frac{(MA(wg^q)-A(wfg))(A(wfg)-mA(wg^q))}{(M-m)A(wg^q)} p(M^{p-1}-m^{p-1}) \\
&\quad - \tilde{R}_{\phi,A}(m,M;fg^{-\frac{p}{q}}) \\
&\leq \frac{p}{4}(M-m)(M^{p-1}-m^{p-1})A(wg^q) - \tilde{R}_{\phi,A}(m,M;fg^{-\frac{p}{q}})
\end{aligned}$$

and

$$\begin{aligned}
0 &\leq \frac{A(wfg)-mA(wg^q)}{M-m}M^p + \frac{MA(wg^q)-A(wfg)}{M-m}m^p - A(wf^p) \\
&\quad - \tilde{R}_{\phi,A}(m,M;fg^{-\frac{p}{q}}) \\
&\leq \frac{1}{4}(M-m)^2 A(wg^q\Psi_\phi(fg^{-q/p};m,M)) - \tilde{R}_{\phi,A}(m,M;fg^{-\frac{p}{q}}) \\
&\leq \frac{p}{4}(M-m)(M^{p-1}-m^{p-1})A(wg^q) - \tilde{R}_{\phi,A}(m,M;fg^{-\frac{p}{q}}).
\end{aligned}$$

If $0<p<1$*, the inequalities are reversed.*

Remark 3.10. It should be noticed here that Theorems 3.1, 3.2, 3.6 and 3.7 hold regardless to the lattice property (L3).

3.3. **Hermite–Hadamard's inequality.** The Hermite–Hadamard inequality states that if $f\colon [a,b]\to\mathbb{R}$ is a convex function, then

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \leq \frac{f(a)+f(b)}{2}. \tag{3.1}$$

If f is concave, the inequalities in (3.1) are reversed. For more details about the Hermite–Hadamard inequality, the reader is referred to [3], [10], [12], [14], and references therein.

It is interesting that Theorems 2.9 and 2.10 can be utilized in obtaining converses for both inequalities in (3.1).

Theorem 3.11. *If $f\colon I \to \mathbb{R}$ is continuous convex function and $[a,b] \subseteq \operatorname{Int} I$, then*

$$\begin{aligned}
0 &\leq \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - f\Big(\frac{a+b}{2}\Big)\\
&\leq \frac{1}{4}(b-a)^2 \sup_{t\in(a,b)} \Psi_f(t;a,b) - R_f(a,b)\\
&\leq \frac{1}{4}(b-a)(f'_-(b)-f'_+(a)) - R_f(a,b)
\end{aligned}\tag{3.2}$$

and

$$\begin{aligned}
0 &\leq \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - f\Big(\frac{a+b}{2}\Big)\\
&\leq \frac{1}{4}(b-a)^2 \Psi_f\Big(\frac{a+b}{2};a,b\Big) - R_f(a,b)\\
&\leq \frac{1}{4}(b-a)(f'_-(b)-f'_+(a)) - R_f(a,b),
\end{aligned}\tag{3.3}$$

where

$$R_f(a,b) = \sum_{n=0}^{N-1} 2^{-n-2} \sum_{k=1}^{2^n} \Delta_f(a,b,n,k) \tag{3.4}$$

and $\Delta_f(a,b,n,k)$ is defined in Theorem 2.7. If f is concave, the inequalities are reversed.

Proof. Inequalities (3.2) and (3.3) follow from (2.11) and (2.12), respectively, by putting $A(f) = \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t$, $f(t) = t$ and replacing ϕ with f. The expression for $R_f(a,b)$ is calculated from (2.9) by making the same substitutions and utilizing the explicit form (2.10) of functions r_n. □

Theorem 3.12. *If $f\colon I \to \mathbb{R}$ is continuous convex function and $[a,b] \subseteq \operatorname{Int} I$, then*

$$\begin{aligned}
0 &\leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - R_f(a,b)\\
&\leq \frac{1}{6}(b-a)^2 \sup_{t\in(a,b)} \Psi_f(t;a,b) - R_f(a,b)\\
&\leq \frac{1}{6}(b-a)(f'_-(b)-f'_+(a)) - \frac{1}{4}R_f(a,b),
\end{aligned}\tag{3.5}$$

where $R_f(a,b)$ is defined by (3.4). If f is concave, the inequalities in (3.5) are reversed.

Proof. Inequalities (3.5) are obtained from (2.13) by making the same substitutions as in the proof of the previous theorem. □

Remark 3.13. It should be noticed here that the first inequality sign in (3.5) provides the refinement of the right inequality in (3.1).

Remark 3.14. If $f\colon I\to\mathbb{R}$ is continuous convex function such that $[a,b]\subseteq \operatorname{Int} I$, then, combining the relations from the previous two theorems, we have

$$\frac{f(a)+f(b)}{2}-\frac{1}{6}(b-a)^2\sup_{t\in(a,b)}\Psi_f(t;a,b)-R_f(a,b)\le\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t$$
$$\le f\Big(\frac{a+b}{2}\Big)+\frac{1}{4}(b-a)^2\sup_{t\in(a,b)}\Psi_f(t;a,b)-R_f(a,b),$$

where $R_f(a,b)$ is defined by (3.4).

3.4. **Inequalities of Giaccardi and Petrović.** Let $\mathbf{p}$ and $\mathbf{x}$ be r-tuples of non-negative real numbers such that

$$(x_i-x_0)\Big(\sum_{j=1}^r p_jx_j-x_i\Big)\ge 0, i=1,...,r;\ \sum_{k=1}^r p_kx_k\ne x_0;\ x_0,\sum_{i=1}^r p_ix_i\in[a,b]. \tag{3.6}$$

The Giaccardi inequality (see [17]) asserts that if $f\colon[a,b]\to\mathbb{R}$ is convex function, then

$$\sum_{i=1}^r p_if(x_i)\le Af\Big(\sum_{i=1}^r p_ix_i\Big)+B\Big(\sum_{i=1}^r p_i-1\Big)f(x_0),$$

where

$$A=\frac{\sum_{i=1}^r p_i(x_i-x_0)}{\sum_{i=1}^r p_ix_i-x_0},\ B=\frac{\sum_{i=1}^r p_ix_i}{\sum_{i=1}^r p_ix_i-x_0}. \tag{3.7}$$

The succeeding result is the refinement and converse of the Giaccardi inequality obtained directly from Theorem 2.10.

Theorem 3.15. *Let* $\mathbf{p}$ *and* $\mathbf{x}$ *be* r*-tuples of non-negative real numbers such that* (3.6) *holds. If* $f\colon I\to\mathbb{R}$ *is continuous convex function and* $[a,b]\subseteq\operatorname{Int} I$*, then*

$$\begin{aligned}0&\le Af\Big(\sum_{i=1}^r p_ix_i\Big)+B\Big(\sum_{i=1}^r p_i-1\Big)f(x_0)-\sum_{i=1}^r p_if(x_i)-R_f(m,M;\mathbf{x})\\
&\le\sum_{j=1}^r p_j\Big(\sum_{i=1}^r p_ix_i-x_j\Big)(x_j-x_0)\sup_{t\in(m,M)}\Psi_f\Big(t;x_0,\sum_{i=1}^r p_ix_i\Big)-R_f(m,M;\mathbf{x})\\
&\le\frac{\sum_{j=1}^r p_j(\sum_{i=1}^r p_ix_i-x_j)(x_j-x_0)}{M-m}(f'_-(M)-f'_+(m))-R_f(m,M;\mathbf{x})\\
&\le\Big(M-\frac{\sum_{i=1}^r p_ix_i}{\sum_{i=1}^r p_i}\Big)\Big(\frac{\sum_{i=1}^r p_ix_i}{\sum_{i=1}^r p_i}-m\Big)\frac{f'_-(M)-f'_+(m)}{M-m}\sum_{i=1}^r p_i-R_f(m,M;\mathbf{x})\\
&\le\frac{1}{4}(M-m)(f'_-(M)-f'_+(m))\sum_{i=1}^r p_i-R_f(m,M;\mathbf{x})\end{aligned}$$

and

$$\begin{aligned}0 &\leq Af\Big(\sum_{i=1}^{r} p_i x_i\Big) + B\Big(\sum_{i=1}^{r} p_i - 1\Big) f(x_0) - \sum_{i=1}^{r} p_i f(x_i) - R_f(m, M; \mathbf{x}) \\ &\leq \frac{1}{4}(M-m)^2 \sum_{i=1}^{r} p_i \Psi_f\Big(x_i; x_0, \sum_{i=1}^{r} p_i x_i\Big) - R_f(m, M; \mathbf{x}) \\ &\leq \frac{1}{4}(M-m)(f'_-(M) - f'_+(m)) \sum_{i=1}^{r} p_i - R_f(m, M; \mathbf{x}),\end{aligned}$$

where $m = \min\{x_0, \sum_{i=1}^{r} p_i x_i\}$, $M = \max\{x_0, \sum_{i=1}^{r} p_i x_i\}$,

$$R_f(m, M; \mathbf{x}) = \sum_{i=1}^{r} \sum_{n=0}^{N-1} \sum_{k=1}^{2^n} p_i \Delta_f(m, M, n, k) r_n \chi_{(\frac{k-1}{2^n}, \frac{k}{2^n})}\Big(\frac{x_i - m}{M - m}\Big),$$

$\Delta_f(m, M, n, k)$ *is defined in Theorem 2.7 and* A, B *are defined by (3.7). If* f *is concave, the inequalities are reversed.*

Proof. It follows directly from Theorem 2.10 for $A(\mathbf{x}) = \dfrac{\sum_{i=1}^{r} p_i x_i}{\sum_{i=1}^{r} p_i}$ and $\phi = f$. □

A special case of the Giaccardi inequality is the Petrović inequality (see [15]) which asserts that if $f\colon [0, a] \to \mathbb{R}$ is convex function, then

$$\sum_{i=1}^{r} f(x_i) \leq f\Big(\sum_{i=1}^{r} x_i\Big) + (r-1) f(0),$$

where $x_i, i = 1, ..., r$, are non-negative real numbers such that $x_1, ..., x_r, \sum_{i=1}^{r} x_i \in [0, a]$. Our last result, which gives the lower and upper bound for the difference between the right-hand side and the left-hand side of the Petrović inequality, can be obtained from Theorem 2.10, but can also be obtained as a special case of Theorem 3.15 for $p_1 = ... = p_r = 1$ and $x_0 = 0$.

Corollary 3.16. *Let* $f\colon I \to \mathbb{R}$ *be a continuous convex function and* $[0, a] \subseteq \operatorname{Int} I$. *If* $x_1, ..., x_r \in [0, a]$ *are real numbers such that* $\sum_{i=1}^{r} x_i \in (0, a]$, *then*

$$
\begin{aligned}
0 &\leq f\Big(\sum_{i=1}^{r} x_i\Big) + (r-1)f(0) - \sum_{i=1}^{r} f(x_i) - R_f(\mathbf{x}) \\
&\leq \sum_{j=1}^{r} x_j \Big(\sum_{i=1}^{r} x_i - x_j\Big) \sup_{t\in(0,\sum_{i=1}^{r} x_i)} \Psi_f\Big(t; 0, \sum_{i=1}^{r} x_i\Big) - R_f(\mathbf{x}) \\
&\leq \frac{\sum_{j=1}^{r} x_j (\sum_{i=1}^{r} x_i - x_j)}{\sum_{i=1}^{r} x_i} \Big(f'_-\Big(\sum_{i=1}^{r} x_i\Big) - f'_+(0)\Big) - R_f(\mathbf{x}) \\
&\leq \frac{r-1}{r}\Big(\sum_{i=1}^{r} x_i\Big)\Big(f'_-\Big(\sum_{i=1}^{r} x_i\Big) - f'_+(0)\Big) - R_f(\mathbf{x}) \\
&\leq \frac{r}{4}\Big(\sum_{i=1}^{r} x_i\Big)\Big(f'_-\Big(\sum_{i=1}^{r} x_i\Big) - f'_+(0)\Big) - R_f(\mathbf{x})
\end{aligned}
$$

and

$$
\begin{aligned}
0 &\leq f\Big(\sum_{i=1}^{r} x_i\Big) + (r-1)f(0) - \sum_{i=1}^{r} f(x_i) - R_f(\mathbf{x}) \\
&\leq \frac{1}{4}\Big(\sum_{i=1}^{r} x_i\Big)^2 \sum_{i=1}^{r} \Psi_f\Big(x_i; 0, \sum_{i=1}^{r} x_i\Big) - R_f(\mathbf{x}) \\
&\leq \frac{r}{4}\Big(\sum_{i=1}^{r} x_i\Big)\Big(f'_-\Big(\sum_{i=1}^{r} x_i\Big) - f'_+(0)\Big) - R_f(\mathbf{x}),
\end{aligned}
$$

where

$$
R_f(\mathbf{x}) = \sum_{i=1}^{r} \sum_{n=0}^{N-1} \sum_{k=1}^{2^n} \Delta_f(0, \sum_{i=1}^{r} x_i, n, k) r_n \chi_{(\frac{k-1}{2^n}, \frac{k}{2^n})}\Big(\frac{x_i}{\sum_{i=1}^{r} x_i}\Big),
$$

and $\Delta_f(0, \sum_{i=1}^{r} x_i, n, k)$ *is defined in Theorem 2.7. If* f *is concave, the inequalities are reversed.*

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