APPROXIMATION METHODS FOR SOLUTIONS OF SYSTEM OF SPLIT EQUILIBRIUM PROBLEMS

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ABSTRACT. In this paper, we introduce a new algorithm for finding a common fixed point of a finite family of continuous pseudocontractive mappings which is a unique solution of some variational inequality problem and whose image under some bounded linear operator is a common solution of some system of equilibrium problems in a real Hilbert space. Our result generalize and improve some well-known results.

1. Introduction and preliminaries

Let H be a real Hilbert space. A mapping T with domain $D(T) \subset H$ and range $R(T)$ in H is called *pseudocontractive* if for each $x, y \in D(T)$ we have

$$\langle Tx - Ty, x - y \rangle \leq ||x - y||^2. \tag{1.1}$$

T is called *strongly pseudocontractive* if there exists $k \in (0, 1)$ such that

$$\langle Tx - Ty, x - y \rangle \leq k||x - y||^2, \quad \forall x, y \in D(T),$$

and T is said to be k *strictly pseudocontractive* if there exists a constant $0 \leq k < 1$ such that

$$\langle Tx - Ty, x - y \rangle \leq ||x - y||^2 - k||(I - T)x - (I - T)y||^2,$$

for all $x, y \in D(T)$. The operator T is called *Lipschitian* if there exists $L \geq 0$ such that $||Tx - Ty|| \leq L||x - y||$, for all $x, y \in D(T)$. If $L = 1$, then T is called *nonexpansive*, and if $L \in [0, 1)$, then T is called a *contraction*. As a result of [11], it follows from inequality (1.1) that T is pseudocontractive if and only if the inequality

$$||Tx - Ty|| \leq ||(1 + t)(x - y) - t(Tx - Ty)||,$$

holds for each $x, y \in D(T)$ and for all $t > 0$. Apart from being an important generalization of nonexpansive, strongly pseudocontractive and k-strictly pseudocontractive mappings. Interest in pseudocontractive mappings stem mainly from their firm connection with the important class of nonlinear accretive operator, where a mapping A with domain $D(A)$ and range $R(A)$ in H is called accretive if the inequality

$$||Ax - Ay|| \leq ||x - y - s(Ax - Ay)||,$$

holds for every $x, y \in D(A)$ and for all $s > 0$. We observe that A is accretive if and only if $T := I - A$ is pseudocontractive, and thus a zero of A, $N(A) := \{x \in D(A) : Ax = 0\}$, is a fixed point of T, $F(T) := \{x \in D(T) : Tx = x\}$ and vice-versa.

It is now well known that if A is accretive then the solutions of the equation $Ax = 0$ correspond to the equilibrium points of some evolution systems. Consequently, considerable research efforts have been devoted to iterative methods for approximating fixed points of T when T is pseudocontractive (see, for example [5],[16] and the references contained in them).

Let H be a real Hilbert space and C be a closed convex subset of H. Let $g : C \times C \to \mathbb{R}$ be a bi-function. The classical equilibrium problem (EP for short) is defined as follows.

(EP) Find $p \in C$ such that $g(p, y) \geq 0, \forall y \in C$.

The symbol $EP(g)$ is used to denote the set of all solutions of the problem (EP), that is,

$$EP(g) = \{p \in C : g(p, y) \geq 0, \forall y \in C\}.$$

Let $\mathcal{G} = \{g_i\}$ be a family of bifunctions from $C \times C$ to $\mathbb{R}$. The system of equilibrium problem $\mathcal{G} = \{g_i\}$ is to determine common equilibrium points for $\mathcal{G} = \{g_i\}$ i.e., the set

$$EP(\mathcal{G}) = \{p \in C : g_i(p, y) \geq 0, \forall y \in C, \ i \in I\}.$$

It is known that the problem (EP) contain optimization problems, complementary problems, variational inequalities problems, saddle point problems, fixed point problems, bilevel problems, semiinfinite problems and others as special cases and have many applications in physics and economics; for detail, one can refer to ([1], [15], [21], [19]) and references therein. Recently, a lot of research efforts are devoted to finding a solution of split equilibrium or fixed point problems see, for instance, ([3], [10], [14]) and the references therein.
In last ten years or so, the problem (EP) has been generalized and improved to find a common element of the set of fixed points of a nonlinear operator and the

set of solutions of the problem (EP). More precisely, many authors have studied the following problem (FTEP) (see, for instance, [4], [20]):

(FTEP)   Find $p \in C$ such that $Tp = p$ and $g(p, y) \geq 0, \forall y \in C,$

where C is a closed convex subset of a Hilbert space H, $g : C \times C \to \mathbb{R}$ is a bi-function and $T : C \to C$ is a nonlinear operator.

Let H be a real Hilbert space, a mapping $G : D(G) \subset H \to H$ is said to be *monotone* if for all $x, y \in D(G)$,

$$\langle Gx - Gy, x - y \rangle \geq 0,$$

where $D(G)$ denote the domain of G. For some $\eta \in (0,1)$, G is called $\eta -$ *strongly monotone* if for all $x, y \in D(G)$,

$$\langle Gx - Gy, x - y \rangle \geq \eta ||x - y||^2.$$

A map $G : H \to H$ is said to be strongly positive if there exists a constant $\eta > 0$ such that

$$\langle Gx, x \rangle \geq \eta ||x||^2, \ \forall x \in D(G).$$

For a strongly positive bonded linear operator G and any $x, y \in D(G)$, we have

$$\langle Gx - Gy, x - y \rangle \geq \eta ||x - y||^2.$$

This implies that G is $\eta-$strongly monotone. In this case, by simple calculation, the following relation also holds:

$$\langle Gx - Gy, x - y \rangle \leq \frac{(1 + ||G||^2)}{2}||x - y||^2 - \frac{1}{2}||(I - G)x - (I - G)y||^2.$$

This implies that $G/||G||$ is $1/2-$strictly pseudocontractive.

Let K be a nonempty, closed and convex subset of H and $G : K \to H$ be a nonlinear mapping. The variational inequality problem is to:

find $u \in K$ such that $\langle Gu, v - u \rangle \geq 0, \ \forall v \in K.$

The set of solution of variational inequality problem is denoted by $VI(K, G)$, which was introduced and studied by [17].

[13] introduced the viscosity approximation method for nonexpansive mappings. Let T be a nonexpansive mappings and f be contraction on H, starting with an arbitrary $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \ n \geq 0, \tag{1.2}$$

where $\{\alpha_n\}$ is a sequence in (0,1). He proved that under certain appropriate conditions on $\{\alpha_n\}$, the sequence $\{x_n\}$ generated by (1.2) strongly converges to the unique solution x^* in $F(T)$ of the variational inequality

$$\langle (I - f)x^*, x - x^* \rangle \geq 0, \ \text{for all } x \in F(T).$$

[27] proved, under some condition on the real sequence $\{\alpha_n\}$, that the sequence $\{x_n\}$ defined by $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = \alpha_n b + (I - \alpha_n A)Tx_n, \ n \geq 0, \tag{1.3}$$

converges strongly to $x^* \in F(T)$ which is the unique solution of the minimization problem

$$\min_{x \in F(T)} \frac{1}{2} \langle Ax, x \rangle - \langle x, b \rangle,$$

where A is a strongly positive bounded linear operator (i.e. $\exists \bar{\gamma} > 0$ such that $\langle Ax, x \rangle \geq \bar{\gamma} ||x||^2, \ \forall x \in H$).

Combining the iterative method (1.2) and (1.3), [12] studied the following general iterative method:

$$x_{n+1} = \alpha_n f(x_n) + (I - \alpha_n A) T x_n, \ n \geq 0, \tag{1.4}$$

they proved that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.4) converges strongly to $x^* \in F(T)$ which solves the variational inequality problem

$$\langle (\gamma f - A) x^*, x - x^* \rangle \leq 0 \ \forall x \in F(T),$$

which is the optimality condition for the minimization problem

$$\min_{x \in F(T)} \frac{1}{2} \langle Ax, x \rangle - h(x),$$

where h is a potential function for γf (i.e. $h'(x) = \gamma f(x)$ for $x \in H$).

On the other hand, [28] introduced the following hybrid iterative method:

$$x_{n+1} = T x_n - \lambda_n \mu G T x_n, \ n \geq 0, \tag{1.5}$$

where G is a κ-Lipschitzian and η-strongly monotone operator with $\kappa > 0, \eta > 0$ and $0 < \mu < 2\eta/\kappa^2$. Under some appropriate conditions, he proved that the sequence $\{x_n\}$ generated by (1.5) converges strongly to the unique solution of the variational inequality problem

$$\langle G x^*, x - x^* \rangle \geq 0, \ \forall x \in F(T).$$

Recently, combining (1.4) and (1.5), [23] considered the following general iterative method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu G) T(x_n), \tag{1.6}$$

and proved that the sequence $\{x_n\}$ generated by (1.6) converges strongly to the unique solution $x^* \in F(T)$ of the variational inequality problem

$$\langle (\gamma f - \mu G) x^*, x - x^* \rangle \leq 0, \ \forall x \in F(T).$$

[22] studied method of approximation of common solutions of fixed points of continuous pseudo-contractive mappings and zero points of the sum of monotone mappings. [24] considered method of approximation of common solutions of fixed points of continuous pseudo-contractive mappings and solutions of a continuous monotone variational inequality.

[9] considered the following split equilibrium problem. Let H_1 and H_2 be two real Hilbert spaces. Let C be a closed convex subset of H_1 and K be a closed convex subset of H_2. Let $f : C \times C \to \mathbb{R}$ and $g : K \times K \to \mathbb{R}$ be two bifunctions, and

$A : H_1 \to H_2$ be a bounded linear operator. The split equilibrium problem (SEP, in short) is defined as follows:

(SEP) find $p \in C$ such that $f(p, y) \geq 0 \,\forall\, y \in C$ and $u := Ap$ satisfying $g(u, v) \geq 0$,

for all v in K. The author established weak convergence algorithms and strong convergence algorithms for SEP (see [9] for more details).

Motivated and inspired by the above results, in this paper, we introduce a new algorithm for finding an element in the set of common fixed points of finite family of continuous pseudocontractive mappings which is a unique solution of some variational inequality problems such that its image under a given bounded linear operator is a common solution of finite family of some equilibrium problems in a real Hilbert space. Our result generalize and improve some well-known results. In particular our result improve and extend the result in [25] and in [2] from family of nonexpansive maps to finite family of continuous pseudocontractive maps, and from equilibrium problem to the case of finite family of equilibrium problem.

Let K be a closed convex subset of a real Hilbert space H. The metric projection from H onto K is the mapping $P_K : H \to K$ for each $x \in H$, there exists a unique point $z = P_K(x)$ such that

$$||x - z|| = \inf_{y \in K} ||x - y||.$$

Lemma 1.1. *Let $x \in H$ and $z \in K$ be any point. Then we have*

   (i) *$z = P_K(x)$ if and only if the following relation holds*

$$\langle x - z, y - z \rangle \leq 0, \ \forall y \in K.$$

  (ii) *There holds the relation*

$$\langle P_K(x) - P_K(y), x - y \rangle \geq ||P_K(x) - P_K(y)||^2, \ \forall x, y \in H.$$

 (iii) *For $x \in H$ and $y \in K$*

$$||y - P_K(x)||^2 + ||x - P_K(x)||^2 \leq ||x - y||^2.$$

A Banach space E is said to satisfy Opial's condition if for each sequence $\{x_n\}$ in E which converges weakly to a point $x \in E$, we have

$$\liminf_{n \to \infty} ||x_n - x|| < \liminf_{n \to \infty} ||x_n - y||, \ \forall y \in E, y \neq x.$$

It is well known that every Hilbert space satisfies Opial's condition.
We shall make use of the following well known results.

Lemma 1.2. (see [1]) *Let K be a nonempty closed, convex subset of H and g be a bi-function of $K \times K$ into $\mathbb{R}$ satisfying the following conditions;*

  (A1) *$g(x, x) = 0$ for all $x \in K$;*

  (A2) *g is monotone, that is $g(x, y) + g(y, x) \leq 0$ for all $x, y \in K$;*

  (A3) *for each $x, y, z \in K$, $\limsup_{t \downarrow 0} g(tz + (1 - t)x, y) \leq g(x, y)$;*

  (A4) *for each $x \in K, y \mapsto g(x, y)$ is convex and lower semi-continuous.*

Let $r > 0$ and $x \in H$. Then there exists $z \in K$ such that $g(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0$ for all $y \in K$.

Lemma 1.3. (see [7]) *Let K be a nonempty closed and convex subset of H and let g be a bi-function of $K \times K$ into $\mathbb{R}$ satisfying (A1)-(A4). For $r > 0$, define a mapping*

$$T_r^g x = \{z \in K : g(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \ \forall y \in K\}$$

for all $x \in H$. Then the following holds:

  (i) T_r^g *is single-valued;*
  (ii) $F(T_r^g) = EP(g)$ *for $r > 0$;*
  (iii) $EP(g)$ *is closed and convex;*
  (iv) T_r^g *is firmly nonexpansive, that is, for any $x, y \in H$,*

$$||T_r^g(x) - T_r^g(y)||^2 \leq \langle T_r^g(x) - T_r^g(y), x - y \rangle.$$

Lemma 1.4. (see,[9],[10]) *Let the mapping T_r^g be defined as in Lemma 1.3. Then, for $r, s > 0$ and $x, y \in H$,*

$$||T_r^g(x) - T_s^g(y)|| \leq ||x - y|| + \frac{|s - r|}{s}||T_s^g(y) - y||.$$

In particular, $||T_r^g(x) - T_r^g(y)|| \leq ||x - y||$ for any $r > 0$ and $x, y \in H$, that is, T_r^g is nonexpansive for any $r > 0$.

Lemma 1.5. ([4]) *Let H be a real Hilbert space. Then the following hold:*

  (a) $||x + y||^2 \leq ||y||^2 + 2\langle x, x + y \rangle$;
  (b) $||x - y||^2 = ||x||^2 + ||y||^2 - 2\langle x, y \rangle$ *for all $x, y \in H$;*
  (c) $||\alpha x + (1-\alpha)y||^2 = \alpha||x||^2 + (1-\alpha)||y||^2 - \alpha(1-\alpha)||x-y||^2$ *for all $x, y \in H$ and $\alpha \in [0, 1]$.*

Lemma 1.6. ([29]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to H$ be a continuous pseudocontractive mapping. Then, for $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$\langle y - z, Tz \rangle - \frac{1}{r}\langle y - z, (1 + r)z - x \rangle \leq 0, \ \forall y \in C.$$

Lemma 1.7. ([29]), *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a continuous pseudocontractive mapping. Then, for $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r x := \{z \in C : \langle y - z, Tz \rangle - \frac{1}{r}\langle y - z, (1 + r)z - x \rangle \leq 0, \ \forall y \in C\}$$

for all $x \in H$. Then, the following holds:

  (1) T_r *is single valued;*
  (2) T_r *is firmly nonexpansive type mapping, that is for all $x, y \in H$,*

$$||T_r x - T_r y||^2 \leq \langle T_r x - T_r y, x - y \rangle;$$

  (3) $F(T_r) = F(T)$;
  (4) $F(T)$ *is closed and convex.*

Lemma 1.8. ([15]) *Let C be a nonempty closed convex subset of H and $\{r_n\} \subset (0,1)$ be a sequence converging to $r > 0$, for a bi-function $F : C \times C \to \mathbb{R}$, satisfying condition (A1) - (A4), define $T_{r_n}^F$ and T_r^F for $n \in \mathbb{N}$ as in Lemma 1.3. Then for every $x \in H$, we have $\lim_{n \to \infty} ||T_{r_n}^F x - T_r^F x|| = 0$.*

Lemma 1.9. ([8]) *Let C be a nonempty closed and convex subset of a Hilbert space, and T be a nonexpansive mapping from C into itself. Then $I - T$ is demiclosed at zero, i.e., $x_n \rightharpoonup x$, $x_n - Tx_n \to 0$ as $n \to \infty$ implies $x = Tx$.*

Lemma 1.10. ([18]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf \beta_n \le \limsup \beta_n < 1$. Suppose that $x_{n+1} = \beta_n y_n + (1 - \beta_n)x_n$ for all integer $n \ge 1$ and $\limsup_{n \to \infty}(||y_{n+1} - y_n|| - ||x_{n+1} - x_n||) \le 0$. Then, $\lim_{n \to \infty} ||y_n - x_n|| = 0$.*

Lemma 1.11. ([26]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1 - b_n)a_n + c_n, \ n \ge 0,$$

where $\{b_n\}$ is a sequences in $(0,1)$ and $\{c_n\}$ is a sequence satisfying the following conditions:

(i) $\sum_{n=0}^{\infty} b_n = \infty$,
(ii) *either* $\limsup_{n \to \infty} c_n/b_n \le 0$ *or* $\sum_{n=0}^{\infty} |c_n| < \infty$.

Then, $\lim_{n \to \infty} a_n = 0$.

Lemma 1.12. (Lemma 2.5 of [2])*Let $\lambda \in (0,1), \mu > 0$, and $F : C \to H$ be an $\kappa-$Lipschitzian and $\eta-$strongly monotone operator. In association with a nonexpansive mapping $T : C \to C$, define a mapping $T^\lambda : C \to H$ by $T^\lambda x = Tx - \lambda\mu FT(x)$, for all $x \in C$. Then T^λ is a contraction provided $\mu < \frac{2\eta}{\kappa^2}$, that is*

$$||T^\lambda x - T^\lambda y|| \le (1 - \lambda\nu)||x - y||, \quad \forall x, y \in C,$$

where $\nu = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$.

2. Main results

Lemma 2.1. *Let C be a nonempty closed convex subset of H. For $x \in H$, let the mapping T_r be the same as in Lemma 1.7. Then for $r, s > 0$ and $F(T_r) \ne \emptyset$, for any $x, y \in H$, and $p \in F(T_r)$*

(i) $||T_r(x) - T_s(y)|| \le ||y - x|| + \frac{|s-r|}{s}||T_s(y) - y||$;
(ii) $||T_r(x) - x||^2 \le ||x - p||^2 - ||T_r(x) - p||^2$.

In particular, $||T_r(x) - T_r(y)|| \le ||x - y||$ for any $r > 0$ and $x, y \in H$, that is, T_r is nonexpansive for any $r > 0$.

Proof. For $r, s > 0$ and $x, y \in H$, by (1) of Lemma 1.7, let $z_1 = T_r(x)$ and $z_2 = T_s(y)$. By the definition of T_r, we have

$$\langle u - z_1, Tz_1 \rangle - \frac{1}{r}\langle u - z_1, (1 + r)z_1 - x \rangle \le 0, \ \forall u \in C \qquad (2.1)$$

and

$$\langle u - z_2, T z_2 \rangle - \frac{1}{s}\langle u - z_2, (1 + s)z_2 - y \rangle \le 0, \ \forall u \in C \tag{2.2}$$

Putting $u := z_2$ in (2.1) and $u := z_1$ in (2.2), we have

$$\langle z_2 - z_1, T z_1 \rangle - \frac{1}{r}\langle z_2 - z_1, (1 + r)z_1 - x \rangle \le 0, \tag{2.3}$$

and

$$\langle z_1 - z_2, T z_2 \rangle - \frac{1}{s}\langle z_1 - z_2, (1 + s)z_2 - y \rangle \le 0, \tag{2.4}$$

Adding (2.3) and (2.4), we have

$$\langle z_2 - z_1, T z_1 - T z_2 \rangle - \langle z_2 - z_1, \frac{(1 + r)z_1 - x}{r} - \frac{(1 + s)z_2 - y}{s} \rangle \le 0,$$

$$\Leftrightarrow \langle z_2 - z_1, (I - T)z_2 - (I - T)z_1 \rangle - \langle z_2 - z_1, \frac{z_1 - x}{r} - \frac{z_2 - y}{s} \rangle \le 0.$$

Since T is pseudocontractive, we have

$$\langle z_2 - z_1, \frac{z_1 - x}{r} - \frac{z_2 - y}{s} \rangle \ge 0,$$

and hence

$$\langle z_2 - z_1, z_1 - x - \frac{r}{s}(z_2 - y) \rangle \ge 0,$$

which implies

$$\langle z_2 - z_1, z_2 - z_1 \rangle \le \langle z_2 - z_1, z_2 - x - \frac{r}{s}(z_2 - y) \rangle,$$

therefore

$$||z_2 - z_1||^2 \le ||z_2 - z_1|| ||z_2 - x - \frac{r}{s}(z_2 - y)||$$

then

$$\begin{aligned}
||z_2 - z_1|| &\le ||z_2 - x - \frac{r}{s}(z_2 - y)|| \\
&= ||(z_2 - y) + (y - x) - \frac{r}{s}(z_2 - y)|| \\
&= ||(y - x) + (1 - \frac{r}{s})(z_2 - y)|| \\
&\le ||y - x|| + \frac{|s - r|}{s}||z_2 - y||.
\end{aligned}$$

Thus $||T_r(x) - T_s(y)|| \le ||y - x|| + \frac{|s-r|}{s}||T_s(y) - y||$, and this complete the proof. We show that (ii) is satisfied. By (2) of Lemma 1.7 and (b) of Lemma 1.5; since $p = T_r(p)$

$$\begin{aligned}
||T_r(x) - p||^2 &= ||T_r(x) - T_r(p)||^2 \le \langle T_r(x) - T_r(p), x - p \rangle \\
&= \frac{1}{2}\Big(||T_r(x) - T_r(p)||^2 + ||x - p||^2 - ||T_r(x) - x||^2 \Big),
\end{aligned}$$

which show that $||T_r(x) - x||^2 \le ||x - p||^2 - ||T_r(x) - p||^2$, this complete the proof of (ii). $\qquad \square$

Let C be nonempty closed convex subsets of Hilbert space. Let $T_i : C \rightarrow C$, $i = 1, 2, \cdots, N$ be a finite family of continuous pseudocontractive mapping. For the rest of this paper, let $T_{[n]r_n}$ be a mapping defined as follows: for $x \in H$, $r_n \in (0, \infty)$

$$T_{[n]r_n}x := \{z \in C : \langle y - z, T_{[n]}z \rangle - \frac{1}{r_n}\langle y - z, (1 + r_n)z - x \rangle \leq 0, \ \forall y \in C\}$$

where $T_{[n]} := T_{nmodN}$, which satisfies Lemma 1.7.

Theorem 2.2. *Let H_1 and H_2 be two real Hilbert spaces. Let $C \subset H_1$, $K \subset H_2$ be two nonempty closed convex sets. Let $T_i : C \rightarrow C$, for $i = 1, 2, \cdots, N$ be continuous pseudocontractive mappings such that $F := \cap_{i=1}^N F(T_i) \neq \emptyset$, $F = F(T_N T_{N-1} T_{N-2} \cdots T_2 T_1) \neq \emptyset$ and $\mathcal{G} = \{g_k : K \times K \rightarrow \mathbb{R}, k = 1, 2, 3, \cdots, M\}$ be finite family of bi-function satisfying the conditions (A1)-(A4). Let $G : H_1 \rightarrow H_1$ be an η-strongly monotone and κ-Lipschitzian with $0 < \mu < \frac{2\eta}{\kappa^2}$, and let $f : H_1 \rightarrow H_1$ be a contraction with $\alpha \in (0, 1)$. Assume that $0 < \gamma < \frac{\tau}{\alpha}$, where $\tau := 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Let $A : H_1 \rightarrow H_2$ be a bounded linear operator with its adjoint B. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 \in H_1 \\ y_n = P_C(x_n + \lambda B(\mathfrak{J}_n^M - I)Ax_n) \\ z_n = \beta y_n + (1 - \beta)T_{[n]r_n}y_n \\ x_{n+1} = \alpha_n \gamma f(x_n) + \delta_n x_n + ((1 - \delta_n)I - \alpha_n \mu G)z_n, \ \forall n \in \mathbb{N}, \end{cases} \tag{2.5}$$

where $\mathfrak{J}_n^M = T_{s_{M,n}}^{g_M} T_{s_{M-1,n}}^{g_{M-1}} \cdots T_{s_{2,n}}^{g_2} T_{s_{1,n}}^{g_1}$, and $\mathfrak{J}_n^0 = I$ for all $n \in \mathbb{N}$, $\beta \in (0, 1), 0 < \liminf_{n \to \infty} \delta_n \leq \limsup_{n \to \infty} \delta_n < 1, \{r_n\} \in (0, +\infty)$ with $\liminf_{n \to +\infty} r_n > 0$, and $\lambda \in (0, \frac{1}{\|B\|^2})$ and $\{s_{k,n}\}_{k=1}^M \subset (0, +\infty)$ with $\liminf_{n \to +\infty} s_{k,n} > 0$, for every $k \in \{1, 2, 3, \cdots, M\}$ and

(C1) $\lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,

(C2) $\lim_{n \to \infty} |r_n - r_{n+1}| = 0$ and $\lim_{n \to \infty} |s_{k,n} - s_{k,n+1}| = 0$ for each $k \in \{1, 2, 3, \cdots, M\}$,

P_C is a projection operator from H_1 onto C. Suppose that $\Omega = \{p \in F : Ap \in EP(\mathcal{G})\} \neq \emptyset$. Assume $\forall \ n \in \mathbb{N}$ for any bounded set $D \subset C$ the relation

$$\limsup_{n \to \infty} \|T_{[n+1]}x - T_{[n]}x\| = 0 \tag{2.6}$$

holds, then $x_n \rightarrow p \in \Omega$, where p is the unique solution of the variational inequality problem

$$\langle \gamma f(p) - \mu Gp, \hat{x} - p \rangle \leq 0, \ \forall \hat{x} \in \Omega. \tag{2.7}$$

Proof. From the choice of γ and μ, $(\mu G - \gamma f)$ is strongly monotone, then the variational inequality (2.7) has a unique solution in Ω. Now we show that $\{x_n\}$ is bounded. Let $p \in \Omega$, then

$$\|T_{[n]r_n}y_n - T_{[n]r_n}p\| \leq \|y_n - p\|. \tag{2.8}$$

By taking $\mathfrak{J}_n^k = T_{s_{k,n}}^{g_k} T_{s_{k-1,n}}^{g_{k-1}} \cdots T_{s_{2,n}}^{g_2} T_{s_{1,n}}^{g_1}$, for each $k \in \{1, 2, 3, \cdots, M\}$ and $\mathfrak{J}_n^0 = I$ for all n, from the nonexpansive of $T_{s_{k,n}}^{g_k}$ for each $k = 1, 2, \cdots, M$ implies

that $\mathfrak{J}_n^k$ is nonexpansive, it follows that

$$||\mathfrak{J}_n^k Ax_n - Ap|| \le ||Ax_n - Ap||. \tag{2.9}$$

And also, from (b) of Lemma 1.5, we have

$$
\begin{aligned}
2\lambda \langle x_n - p, B(\mathfrak{J}_n^M - I)Ax_n \rangle &= 2\lambda \langle A(x_n - p), (\mathfrak{J}_n^M - I)Ax_n \rangle \\
&= 2\lambda \langle A(x_n - p) + (\mathfrak{J}_n^M - I)Ax_n - (\mathfrak{J}_n^M - I)Ax_n, (\mathfrak{J}_n^M - I)Ax_n \rangle \\
&= 2\lambda \langle (\mathfrak{J}_n^M Ax_n - Ap) - (\mathfrak{J}_n^M - I)Ax_n, (\mathfrak{J}_n^M - I)Ax_n \rangle \\
&= 2\lambda \left(\langle \mathfrak{J}_n^M Ax_n - Ap, (\mathfrak{J}_n^M - I)Ax_n \rangle - ||(\mathfrak{J}_n^M - I)Ax_n||^2 \right) \\
&= 2\lambda \left(\frac{1}{2}||\mathfrak{J}_n^M Ax_n - Ap||^2 + \frac{1}{2}||(\mathfrak{J}_n^M - I)Ax_n||^2 \right. \\
&\quad \left. - \frac{1}{2}||Ax_n - Ap||^2 - ||(\mathfrak{J}_n^M - I)Ax_n||^2 \right) \\
&\le 2\lambda \left(\frac{1}{2}||Ax_n - Ap||^2 + \frac{1}{2}||(\mathfrak{J}_n^M - I)Ax_n||^2 \right. \\
&\quad \left. - \frac{1}{2}||Ax_n - Ap||^2 - ||(\mathfrak{J}_n^M - I)Ax_n||^2 \right) \\
&= -\lambda ||(\mathfrak{J}_n^M - I)Ax_n||^2. \tag{2.10}
\end{aligned}
$$

We also have

$$||B(\mathfrak{J}_n^M - I)Ax_n||^2 \le ||B||^2 ||(\mathfrak{J}_n^M - I)Ax_n||^2. \tag{2.11}$$

By using (2.8)-(2.11), and (b) of Lemma 1.5 we obtain

$$
\begin{aligned}
||y_n - p||^2 &= ||P_C(x_n + \lambda B(\mathfrak{J}_n^M - I)Ax_n) - p||^2 \\
&\le ||x_n + \lambda B(\mathfrak{J}_n^M - I)Ax_n - p||^2 \\
&= ||x_n - p||^2 + ||\lambda B(\mathfrak{J}_n^M - I)Ax_n||^2 + 2\lambda \langle x_n - p, B(\mathfrak{J}_n^M - I)Ax_n \rangle \\
&\le ||x_n - p||^2 + ||\lambda B(\mathfrak{J}_n^M - I)Ax_n||^2 - \lambda ||(\mathfrak{J}_n^M - I)Ax_n||^2 \\
&= ||x_n - p||^2 - \lambda(1 - \lambda ||B||^2)||(\mathfrak{J}_n^M - I)Ax_n||^2 \\
&\le ||x_n - p||^2. \tag{2.12}
\end{aligned}
$$

Notice $\lambda \in (0, \frac{1}{||B||^2})$, $\lambda(1 - \lambda ||B||^2) > 0$. It follows from (2.12) that

$$||y_n - p|| \le ||x_n - p||,$$

and

$$
\begin{aligned}
||z_n - p|| &= \beta ||y_n - p|| + (1 - \beta)||T_{[n]r_n} y_n - p|| \\
&\le ||y_n - p|| \le ||x_n - p||. \tag{2.13}
\end{aligned}
$$

Using (2.13), we obtain

$$
\begin{aligned}
||x_{n+1} - p|| &= ||\alpha_n \gamma f(x_n) + \delta_n x_n + ((1 - \delta_n)I - \alpha_n \mu G)z_n - p|| \\
&= ||\alpha_n \gamma f(x_n) - \alpha_n \mu G(p) + \alpha_n \mu G(p) + \delta_n x_n - \delta_n p + \delta_n p \\
&\quad + ((1 - \delta_n)I - \alpha_n \mu G)z_n - p|| \\
&= ||\alpha_n \gamma f(x_n) - \alpha_n \mu G(p) + \delta_n x_n - \delta_n p + ((1 - \delta_n)I - \alpha_n \mu G)z_n \\
&\quad + \alpha_n \mu G(p) - p + \delta_n p|| \\
&= ||\alpha_n \gamma f(x_n) - \alpha_n \mu G(p) + \delta_n x_n - \delta_n p + ((1 - \delta_n)I - \alpha_n \mu G)z_n \\
&\quad - ((1 - \delta_n)I - \alpha_n \mu G)p|| \\
&\leq \alpha_n ||\gamma f(x_n) - \mu G(p)|| + \delta_n ||x_n - p|| \\
&\quad + (1 - \delta_n)||(I - \frac{\alpha_n}{(1 - \delta_n)}\mu G)(z_n) - (I - \frac{\alpha_n}{(1 - \delta_n)}\mu G)(p)|| \\
&\leq \alpha_n ||\gamma f(x_n) - \gamma f(p) + \gamma f(p) - \mu G(p)|| + \delta_n ||x_n - p|| \\
&\quad + (1 - \delta_n - \alpha_n \tau)||z_n - p|| \\
&\leq \alpha_n \gamma \alpha ||x_n - p|| + \alpha_n ||\gamma f(p) - \mu G(p)|| + \delta_n ||x_n - p|| \\
&\quad + (1 - \delta_n - \alpha_n \tau)||z_n - p|| \\
&\leq \alpha_n \gamma \alpha ||x_n - p|| + \alpha_n ||\gamma f(p) - \mu G(p)|| + (1 - \alpha_n \tau)||x_n - p|| \\
&= (1 - \alpha_n(\tau - \alpha\gamma))||x_n - p|| + \alpha_n(\tau - \alpha\gamma)\frac{||\gamma f(p) - \mu G(p)||}{\tau - \alpha\gamma} \\
&\leq \max\left\{||x_n - p||, \frac{||\gamma f(p) - \mu G(p)||}{\tau - \alpha\gamma}\right\}.
\end{aligned}
$$

By induction, we obtain

$$
||x_n - p|| \leq \max\left\{||x_1 - p||, \frac{||\gamma f(p) - \mu G(p)||}{\tau - \alpha\gamma}\right\}, \ n \geq 1.
$$

Hence $\{x_n\}$ is bounded, also $\{y_n\}, \{Ax_n\}, \{z_n\}$ and $\{T_{[n]r_n}y_n\}$ are all bounded. Next show that $\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0$.

From Lemma 1.3, using the method of Step 2 in Colao et al. [6], since $\mathfrak{J}_n^k Ax_n \in K$ for $k \in \{1, 2, \cdots, M\}$, we obtain

$$
g_k(\mathfrak{J}_n^k Ax_n, y) + \frac{1}{s_{k,n}}\langle y - \mathfrak{J}_n^k Ax_n, \mathfrak{J}_n^k Ax_n - Ax_n \rangle \geq 0, \ \forall y \in K,
$$

and

$$
g_k(\mathfrak{J}_{n+1}^k Ax_n, y) + \frac{1}{s_{k,n+1}}\langle y - \mathfrak{J}_{n+1}^k Ax_n, \mathfrak{J}_{n+1}^k Ax_n - Ax_n \rangle \geq 0, \ \forall y \in K,
$$

In particular, we have

$$
\begin{aligned}
g_k(\mathfrak{J}_n^k Ax_n, &\mathfrak{J}_{n+1}^k Ax_n) \\
&+ \frac{1}{s_{k,n+1}}\langle \mathfrak{J}_{n+1}^k Ax_n - \mathfrak{J}_n^k Ax_n, \mathfrak{J}_n^k Ax_n - Ax_n \rangle \geq 0,
\end{aligned} \tag{2.14}
$$

and

$$g_k(\mathfrak{J}_{n+1}^k Ax_n, \mathfrak{J}_n^k Ax_n)$$
$$+ \quad \frac{1}{s_{k,n+1}} \langle \mathfrak{J}_n^k Ax_n - \mathfrak{J}_{n+1}^k Ax_n, \mathfrak{J}_{n+1}^k Ax_n - Ax_n \rangle \geq 0, \qquad (2.15)$$

Adding (2.14) and (2.15) and using the monotonicity of g_k for $k \in \{1, 2, \cdots, M\}$, we obtain

$$0 \leq \langle \mathfrak{J}_n^k Ax_n - \mathfrak{J}_{n+1}^k Ax_n, \frac{\mathfrak{J}_{n+1}^k Ax_n - Ax_n}{s_{k,n+1}} - \frac{\mathfrak{J}_n^k Ax_n - Ax_n}{s_{k,n}} \rangle.$$

This implies that

$$
\begin{aligned}
0 &\leq \langle \mathfrak{J}_n^k Ax_n - \mathfrak{J}_{n+1}^k Ax_n, \frac{\mathfrak{J}_{n+1}^k Ax_n - Ax_n}{s_{k,n+1}} - \frac{\mathfrak{J}_n^k Ax_n - Ax_n}{s_{k,n}} \rangle \\
&= \langle \mathfrak{J}_n^k Ax_n - \mathfrak{J}_{n+1}^k Ax_n, \mathfrak{J}_{n+1}^k Ax_n - Ax_n - \frac{s_{k,n+1}}{s_{k,n}} \left(\mathfrak{J}_n^k Ax_n - Ax_n \right) \rangle \\
&= \langle \mathfrak{J}_n^k Ax_n - \mathfrak{J}_{n+1}^k Ax_n, \mathfrak{J}_{n+1}^k Ax_n \\
&\quad - \mathfrak{J}_n^k Ax_n + \mathfrak{J}_n^k Ax_n - Ax_n - \frac{s_{k,n+1}}{s_{k,n}} \left(\mathfrak{J}_n^k Ax_n - Ax_n \right) \rangle \\
&= \langle \mathfrak{J}_n^k Ax_n - \mathfrak{J}_{n+1}^k Ax_n, \mathfrak{J}_{n+1}^k Ax_n - \mathfrak{J}_n^k Ax_n \\
&\quad + \left(1 - \frac{s_{k,n+1}}{s_{k,n}} \right) \left(\mathfrak{J}_n^k Ax_n - Ax_n \right) \rangle \\
&\leq \left| 1 - \frac{s_{k,n+1}}{s_{k,n}} \right| \|\mathfrak{J}_n^k Ax_n - \mathfrak{J}_{n+1}^k Ax_n\| (\|\mathfrak{J}_n^k Ax_n\| + \|Ax_n\|) \\
&\quad - \|\mathfrak{J}_n^k Ax_n - \mathfrak{J}_{n+1}^k Ax_n\|^2.
\end{aligned}
$$

Thus

$$\|\mathfrak{J}_n^k Ax_n - \mathfrak{J}_{n+1}^k Ax_n\| \leq \left| \frac{s_{k,n} - s_{k,n+1}}{s_{k,n}} \right| (\|\mathfrak{J}_n^k Ax_n\| + \|Ax_n\|).$$

Hence from (C2), we obtain

$$\lim_{n \to \infty} \|\mathfrak{J}_n^k Ax_n - \mathfrak{J}_{n+1}^k Ax_n\| = 0. \qquad (2.16)$$

Set $\theta_n := \frac{\alpha_n}{1 - \delta_n}$ and $u_n := \theta_n \gamma f(x_n) + (I - \theta_n \mu G) z_n$. Then

$$x_{n+1} = \delta_n x_n + (1 - \delta_n) u_n \text{ and } \lim_{n \to \infty} \theta_n = 0. \qquad (2.17)$$

Therefore

$$
\begin{aligned}
\|u_{n+1} - u_n\| &= \|\theta_{n+1} \gamma f(x_{n+1}) + (I - \theta_{n+1} \mu G) z_{n+1} \\
&\quad - \theta_n \gamma f(x_n) - (I - \theta_n \mu G) z_n\| \\
&\leq \theta_{n+1}(\|\gamma f(x_{n+1})\| + \|\mu G(z_{n+1})\|) \\
&\quad + \theta_n(\|\gamma f(x_n)\| + \|\mu G(z_n)\|) \\
&\quad + \|z_{n+1} - z_n\| \qquad (2.18)
\end{aligned}
$$

and

$$
\begin{aligned}
||z_{n+1} - z_n|| &= \beta||y_{n+1} - y_n|| + (1-\beta)||T_{[n+1]r_{n+1}}y_{n+1} - T_{[n]r_n}y_n|| \\
&\leq \beta||y_{n+1} - y_n|| + (1-\beta)\Big[||y_n - y_{n+1}|| \\
&\quad + ||T_{[n+1]r_{n+1}}y_n - T_{[n+1]r_n}y_n|| + ||T_{[n+1]r_n}y_n - T_{[n]r_n}y_n||\Big] \\
&\leq ||y_{n+1} - y_n|| + \frac{|r_n - r_{n+1}|}{r_n}||T_{[n]r_n}y_n - y_n|| \\
&\quad + ||T_{[n+1]r_n}y_n - T_{[n]r_n}y_n||
\end{aligned}
\tag{2.19}
$$

Also

$$
\begin{aligned}
||y_{n+1} - y_n|| &= ||(x_{n+1} + \lambda B(\mathfrak{J}_{n+1}^M - I)Ax_{n+1}) - (x_n + \lambda B(\mathfrak{J}_n^M - I)Ax_n)|| \\
&= ||x_{n+1} - x_n - \lambda BA(x_{n+1} - x_n) + \lambda B(\mathfrak{J}_{n+1}^M Ax_{n+1} - \mathfrak{J}_n^M Ax_n)|| \\
&\leq ||x_{n+1} - x_n - \lambda BA(x_{n+1} - x_n)|| + \lambda||B||||\mathfrak{J}_{n+1}^M Ax_{n+1} - \mathfrak{J}_n^M Ax_n|| \\
&= \Big(||x_{n+1} - x_n||^2 - 2\lambda||BA||||x_{n+1} - x_n||^2 + \lambda^2||BA||^2||x_{n+1} - x_n||^2\Big)^{\frac{1}{2}} \\
&\quad + \lambda||B||||\mathfrak{J}_{n+1}^M Ax_{n+1} - \mathfrak{J}_n^M Ax_n|| \\
&= \Big(1 - 2\lambda||BA|| + \lambda^2||BA||^2\Big)^{\frac{1}{2}}||x_{n+1} - x_n|| \\
&\quad + \lambda||B||||\mathfrak{J}_{n+1}^M Ax_{n+1} - \mathfrak{J}_n^M Ax_n|| \\
&= (1 - \lambda||BA||)||x_{n+1} - x_n|| + \lambda||B||||\mathfrak{J}_{n+1}^M Ax_{n+1} - \mathfrak{J}_n^M Ax_n|| \\
&\leq (1 - \lambda||BA||)||x_{n+1} - x_n|| + \lambda||B||||\mathfrak{J}_{n+1}^M Ax_{n+1} - \mathfrak{J}_{n+1}^M Ax_n|| \\
&\quad + \lambda||B||||\mathfrak{J}_{n+1}^M Ax_n - \mathfrak{J}_n^M Ax_n|| \\
&\leq (1 - \lambda||BA||)||x_{n+1} - x_n|| + \lambda||B||||A||||x_{n+1} - x_n|| \\
&\quad + \lambda||B||||\mathfrak{J}_{n+1}^M Ax_n - \mathfrak{J}_n^M Ax_n|| \\
&= ||x_{n+1} - x_n|| + \lambda||B||||\mathfrak{J}_{n+1}^M Ax_n - \mathfrak{J}_n^M Ax_n||.
\end{aligned}
\tag{2.20}
$$

From (2.18), (2.19) and (2.20) we obtain

$$
\begin{aligned}
||u_{n+1} - u_n|| - ||x_{n+1} - x_n|| &\leq \theta_{n+1}(||\gamma f(x_{n+1})|| + ||\mu G(z_{n+1})||) \\
&\quad + \theta_n(||\gamma f(x_n)|| + ||\mu G(z_n)||) + \lambda||B||||\mathfrak{J}_{n+1}^M Ax_n - \mathfrak{J}_n^M Ax_n|| \\
&\quad + \frac{|r_n - r_{n+1}|}{r_n}\Big(||T_{[n]r_n}y_n - y_n|| + (||T_{[n]r_n}y_n|| + ||y_n||)\Big) \\
&\quad + ||T_{[n+1]r_n}(y_n) - T_{[n]r_n}(y_n)||.
\end{aligned}
$$

this implies from (C1), (C2), (2.6) and (2.16), that

$$
\limsup_{n \to \infty}(||u_{n+1} - u_n|| - ||x_{n+1} - x_n||) \leq 0
$$

from Lemma 1.10, we obtain

$$
\lim_{n \to \infty} ||u_n - x_n|| = 0.
\tag{2.21}
$$

From (2.17) and (2.21), we obtain

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{2.22}$$

But

$$||x_{n+N} - x_n|| \leq ||x_{n+N} - x_{n+N-1}|| + \cdots + ||x_{n+1} - x_n|| \to 0$$

as $n \to \infty$. Hence

$$\lim_{n\to\infty} ||x_{n+N} - x_n|| = 0. \tag{2.23}$$

Also, from (2.5), we obtain

$$
\begin{aligned}
||z_n - x_n|| &\leq ||x_{n+1} - x_n|| + ||x_{n+1} - z_n|| \\
&= ||x_{n+1} - x_n|| + ||\alpha_n \gamma f(x_n) + \delta_n x_n + ((1 - \delta_n)I - \alpha_n \mu G)z_n - z_n|| \\
&\leq ||x_{n+1} - x_n|| + \delta_n ||x_n - z_n|| + \alpha_n(||\mu G(z_n)|| + ||\gamma f(x_n)||).
\end{aligned}
$$

It follows that

$$||x_n - z_n|| \leq \frac{1}{1 - \delta_n}\Big(||x_{n+1} - x_n|| + \alpha_n(||\mu G(z_n)|| + ||\gamma f(x_n)||)\Big)$$

which implies, from (2.22) that

$$\lim_{n\to\infty} ||x_n - z_n|| = 0. \tag{2.24}$$

Next we show that $\lim_{n\to\infty} ||\mathfrak{J}_n^{k+1} Ax_n - \mathfrak{J}_n^k Ax_n|| = 0, \quad \forall k \in \{0, 1, 2, ..., M - 1\}$. Let $p \in F$ and $k \in \{0, 1, 2, ..., M - 1\}$. Since $T_{s_{k+1,n}}^{g_{k+1}}$ is firmly nonexpansive, we obtain

$$
\begin{aligned}
||\mathfrak{J}_n^{k+1} Ax_n - Ap||^2 &= ||T_{s_{k+1,n}}^{g_{k+1}} \mathfrak{J}_n^k Ax_n - T_{s_{k+1,n}}^{g_{k+1}} Ap||^2 \\
&\leq \langle T_{s_{k+1,n}}^{g_{k+1}} \mathfrak{J}_n^k Ax_n - Ap, \mathfrak{J}_n^k Ax_n - Ap \rangle \\
&= \frac{1}{2}\Big(||T_{s_{k+1,n}}^{g_{k+1}} \mathfrak{J}_n^k Ax_n - Ap||^2 + ||\mathfrak{J}_n^k Ax_n - Ap||^2 \\
&\quad - ||T_{s_{k+1,n}}^{g_{k+1}} \mathfrak{J}_n^k Ax_n - \mathfrak{J}_n^k Ax_n||^2\Big) \\
&= \frac{1}{2}\Big(||\mathfrak{J}_n^{k+1} Ax_n - Ap||^2 + ||\mathfrak{J}_n^k Ax_n - Ap||^2 \\
&\quad - ||\mathfrak{J}_n^{k+1} Ax_n - \mathfrak{J}_n^k Ax_n||^2\Big).
\end{aligned}
$$

It follows that

$$||\mathfrak{J}_n^{k+1} Ax_n - Ap||^2 \leq ||Ax_n - Ap||^2 - ||\mathfrak{J}_n^{k+1} Ax_n - \mathfrak{J}_n^k Ax_n||^2. \tag{2.25}$$

Also, from (2.5) and using (c) of Lemma 1.5, we obtain

$$
\begin{aligned}
||z_n - p||^2 &= ||\beta(y_n - p) + (1 - \beta)(T_{[n]r_n} y_n - p)||^2 \\
&= \beta ||y_n - p||^2 + (1 - \beta)||T_{[n]r_n} y_n - p||^2 \\
&\quad - \beta(1 - \beta)||T_{[n]r_n} y_n - y_n||^2 \\
&\leq ||y_n - p||^2 - \beta(1 - \beta)||T_{[n]r_n} y_n - y_n||^2 \\
&\leq ||x_n - p||^2 - \beta(1 - \beta)||T_{[n]r_n} y_n - y_n||^2
\end{aligned}
$$

this implies

$$\beta(1-\beta)||T_{[n]r_n}y_n - y_n||^2 \leq (||x_n - p|| + ||z_n - p||)||x_n - z_n||$$

since $\beta(1-\beta) > 0$ from (2.24), we obtain

$$\lim_{n\to\infty} ||T_{[n]r_n}y_n - y_n|| = 0. \tag{2.26}$$

Also, from (2.5), we have

$$\begin{aligned}
||x_n - y_n|| &\leq ||x_n - z_n|| + ||z_n - y_n|| \\
&\leq ||x_n - z_n|| + ||\beta y_n + (1-\beta)T_{[n]r_n}y_n - y_n|| \\
&= ||x_n - z_n|| + (1-\beta)||T_{[n]r_n}y_n - y_n||
\end{aligned}$$

from (2.24) and (2.26), we obtain

$$\lim_{n\to\infty} ||x_n - y_n|| = 0. \tag{2.27}$$

Using the same argument of (2.12), we obtain

$$\begin{aligned}
\lambda(1 - \lambda||B||^2)||\mathfrak{J}_n^{k+1}Ax_n - Ax_n|| &\leq ||x_n - p||^2 - ||y_n - p||^2 \\
&\leq (||x_n - p|| + ||y_n - p||)||x_n - y_n||
\end{aligned}$$

since $\lambda(1 - \lambda||B||^2) > 0$, it follows from (2.27) that

$$\lim_{n\to\infty} ||\mathfrak{J}_n^{k+1}Ax_n - Ax_n|| = 0. \tag{2.28}$$

Also, from (2.25), we obtain

$$\begin{aligned}
||\mathfrak{J}_n^{k+1}Ax_n - \mathfrak{J}_n^k Ax_n||^2 &\leq ||Ax_n - Ap||^2 - ||\mathfrak{J}_n^{k+1}Ax_n - Ap||^2 \\
&\leq (||Ax_n - Ap|| + ||\mathfrak{J}_n^{k+1}Ax_n - Ap||)||\mathfrak{J}_n^{k+1}Ax_n - Ax_n||
\end{aligned}$$

it follows from (2.28) that

$$\lim_{n\to\infty} ||\mathfrak{J}_n^{k+1}Ax_n - \mathfrak{J}_n^k Ax_n|| = 0. \tag{2.29}$$

Since

$$\begin{aligned}
||T_{[n]r_n}x_n - x_n|| &\leq ||T_{[n]r_n}x_n - T_{[n]r_n}y_n|| + ||T_{[n]r_n}y_n - y_n|| \\
&\quad + ||y_n - x_n|| \\
&\leq 2||x_n - y_n|| + ||T_{[n]r_n}y_n - y_n||
\end{aligned}$$

then, from (2.26) and (2.27), we obtain

$$\lim_{n\to\infty} ||T_{[n]r_n}x_n - x_n|| = 0. \tag{2.30}$$

Also, using the fact that $T_{(i)r}$ is nonexpansive for $r > 0$ and each i, we obtain

$$\begin{aligned}
||x_n - T_{[n+1]r_{n+1}}x_n|| &\leq ||x_n - x_{n+1}|| + ||x_{n+1} - T_{[n+1]r_{n+1}}x_{n+1}|| \\
&\quad + ||T_{[n+1]r_{n+1}}x_{n+1} - T_{[n+1]r_{n+1}}x_n|| \\
&\leq 2||x_n - x_{n+1}|| + ||x_{n+1} - T_{[n+1]r_{n+1}}x_{n+1}||
\end{aligned}$$

from (2.22), (2.30), we have

$$\lim_{n\to\infty} ||x_n - T_{[n+1]r_{n+1}}x_n|| = 0.$$

Also, from (2.22) and (2.30), we obtain

$$\lim_{n\to\infty} ||x_{n+1} - T_{[n+1]r_{n+1}} x_n|| = 0.$$

Using the nonexpansivity of $T_{(i)r}$ for each i and $r > 0$, we obtain the following finite table

$$x_{n+N} - T_{(n+N)r_{n+N}} x_{n+N-1} \to 0 \text{ as } n \to \infty$$

$$T_{(n+N)r_{n+N}} x_{n+N-1} - T_{(n+N)r_{n+N}} T_{(n+N-1)r_{n+N-1}} x_{n+N-2} \to 0 \text{ as } n \to \infty$$

$$\vdots$$

$$T_{(n+N)r_{n+N}} \cdots T_{(n+2)r_{n+2}} x_{n+1} - T_{(n+N)r_{n+N}} \cdots T_{(n+1)r_{n+1}} x_n \to 0 \text{ as } n \to \infty$$

and adding up the table yields

$$x_{n+N} - T_{(n+N)r_{n+N}} T_{(n+N-1)r_{n+N-1}} \cdots T_{(n+1)r_{n+1}} x_n \to 0 \text{ as } n \to \infty.$$

Using this and (2.23), we obtain

$$\lim_{n\to\infty} ||x_n - T_{(n+N)r_{n+N}} T_{(n+N-1)r_{n+N-1}} \cdots T_{(n+1)r_{n+1}} x_n|| = 0. \qquad (2.31)$$

Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle (\gamma f - \mu G)p, x_n - p \rangle = \lim_{m\to\infty} \langle (\gamma f - \mu G)p, x_{n_m} - p \rangle$$

Since $\{x_n\}$ is bounded, without loss of generality, we may assume that $x_{n_m} \rightharpoonup \hat{x}$ for some $\hat{x} \in H_1$. Since the pool of mappings of $T_{[n]r_n}$ is finite, passing to a further subsequence if necessary, we may further assume that, for some $i \in \{1, 2, \cdots, N\}$,

$$T_{(n_i)r_{n_i}} \equiv T_{(i)r_{n_i}}, \quad \text{for all } i \geq 1.$$

It follows from (2.31) that

$$x_{n_i} - T_{(i+N)r_{n_{i+N}}} \cdots T_{(i+1)r_{n_{i+1}}} x_{n_i} \to 0 \text{ as } i \to \infty.$$

It follows from Lemma 1.7 that $T_{(i)r_{n_i}}$ for each $i \in \{1, 2, \cdots, N\}$ is firmly nonexpansive and hence nonexpansive, then demiclosedness principle of nonexpansive Lemma 1.9 ensures that the weak limit $\hat{x}$ of $\{x_{n_i}\}$ is a fixed point point of the mapping $T_{(i+N)r_{i+N}} \cdots T_{(i+1)r_{i+1}}$, this implies that $\hat{x} \in \bigcap_{i=1}^{N} F(T_{ir_n}) = \bigcap_{i=1}^{N} F(T_i)$.

Moreover, note that by (A2) and given $Ay \in K$ and $k \in \{0, 1, \cdots, M-1\}$, we have

$$\frac{1}{s_{k+1,n}} \langle Ay - \mathfrak{J}_n^{k+1} Ax_n, \mathfrak{J}_n^{k+1} Ax_n - \mathfrak{J}_n^{k} Ax_n \rangle \geq g_{k+1}(Ay, \mathfrak{J}_n^{k+1} Ax_n).$$

Thus

$$\langle Ay - \mathfrak{J}_{n_m}^{k+1} Ax_{n_m}, \frac{\mathfrak{J}_{n_m}^{k+1} Ax_{n_m} - \mathfrak{J}_{n_m}^{k} Ax_{n_m}}{s_{k+1,n_m}} \rangle \geq g_{k+1}(Ay, \mathfrak{J}_{n_m}^{k+1} Ax_{n_m}). \qquad (2.32)$$

By condition (A4), $g_k(Ay, .)$, $\forall k$ is lower semicontinuous and convex, and thus weakly lower semicontinuous. From (2.29) and condition $\liminf_{n\to+\infty} s_{k,n} > 0$ we have that

$$\frac{\mathfrak{J}_{n_m}^{k+1} Ax_{n_m} - \mathfrak{J}_{n_m}^{k} Ax_{n_m}}{s_{k+1,n_m}} \to 0,$$

in norm. By definition of A and the fact that $x_{n_m} \rightharpoonup \hat{x}$, then $Ax_{n_m} \rightharpoonup A\hat{x} \in K$ as $m \to \infty$. Now set $\nu_{n_m}^{(k+1)} = Ax_{n_m} - \mathfrak{J}_{n_m}^{k+1} Ax_{n_m}$, it follows from (2.28) that $\nu_{n_m}^{(k+1)} \rightharpoonup 0$ for each $k = 1, 2, \cdots, M$ and $Ax_{n_m} - \nu_{n_m}^{(k+1)} = \mathfrak{J}_{n_m}^{k+1} Ax_{n_m}$. Therefore from (2.32), we have

$$(2.33)$$

$$\langle Ay - (Ax_{n_m} - \nu_{n_m}^{(k+1)}), \frac{\mathfrak{J}_{n_m}^{k+1} Ax_{n_m} - \mathfrak{J}_{n_m}^{k} Ax_{n_m}}{s_{k+1,n_m}} \rangle \geq g_{k+1}(Ay, Ax_{n_m} - \nu_{n_m}^{(k+1)}).$$

Therefore, letting $m \to \infty$ in (2.33), we obtain

$$g_{k+1}(Ay, A\hat{x}) \leq 0,$$

for all $Ay \in K$ and $k \in \{0, 1, 2, \cdots, M-1\}$. Replacing Ay with $Ay_t := tAy + (1-t)A\hat{x}$ with $t \in (0,1)$ and using (A1) and (A4), we obtain

$$0 = g_{k+1}(Ay_t, Ay_t) \leq tg_{k+1}(Ay_t, Ay) + (1-t)g_{k+1}(Ay_t, A\hat{x}) \leq tg_{k+1}(Ay_t, Ay).$$

Hence, $g_{k+1}(tAy + (1-t)A\hat{x}, Ay) \geq 0$, for all $t \in (0,1)$ and $Ay \in K$. Letting $t \to 0^+$ and using (A3), we conclude $g_{k+1}(A\hat{x}, Ay) \geq 0$, for all $Ay \in K$ and $k \in \{0, 1, 2, \cdots, M-1\}$. Therefore $A\hat{x} \in \bigcap_{k=1}^{M} F(T_{s_{k,n}}^{g_k})$.

Therefore

$$\begin{aligned}
\limsup_{n \to \infty} \langle (\gamma f - \mu G)p, x_n - p \rangle &= \lim_{m \to \infty} \langle (\gamma f - \mu G)p, x_{n_m} - p \rangle \\
&= \langle (\gamma f - \mu G)p, \hat{x} - p \rangle \leq 0.
\end{aligned}$$

We finally show that $x_n \to p$ as $n \to \infty$. From (2.5), we obtain

$$\begin{aligned}
||x_{n+1} - p||^2 &= \langle \alpha_n \gamma f(x_n) + \delta_n x_n + ((1-\delta_n)I - \alpha_n \mu G)z_n - p, x_{n+1} - p \rangle \\
&= \alpha_n \gamma \langle f(x_n) - f(p), x_{n+1} - p \rangle + \alpha_n \langle \gamma f(p) - \mu G(p), x_{n+1} - p \rangle \\
&\quad + \delta_n \langle x_n - p, x_{n+1} - p \rangle + \langle ((1-\delta_n)I - \alpha_n \mu G)(z_n - p), x_{n+1} - p \rangle \\
&\leq \alpha_n \gamma \alpha ||x_n - p|| ||x_{n+1} - p|| + \delta_n ||x_n - p|| ||x_{n+1} - p|| \\
&\quad + (1 - \delta_n - \alpha_n \tau)||z_n - p|| ||x_{n+1} - p|| \\
&\quad + \alpha_n \langle \gamma f(p) - \mu G(p), x_{n+1} - p \rangle \\
&\leq \alpha_n \gamma \alpha ||x_n - p|| ||x_{n+1} - p|| + (1 - \alpha_n \tau)||x_n - p|| ||x_{n+1} - p|| \\
&\quad + \alpha_n \langle \gamma f(p) - \mu G(p), x_{n+1} - p \rangle \\
&= [1 - \alpha_n(\tau - \gamma \alpha)] ||x_n - p|| ||x_{n+1} - p|| \\
&\quad + \alpha_n \langle \gamma f(p) - \mu G(p), x_{n+1} - p \rangle \\
&\leq [1 - \alpha_n(\tau - \gamma \alpha)] \frac{||x_n - p||^2 + ||x_{n+1} - p||^2}{2} \\
&\quad + \alpha_n \langle \gamma f(p) - \mu G(p), x_{n+1} - p \rangle
\end{aligned}$$

This implies

$$\begin{aligned}
||x_{n+1} - p||^2 &\leq \frac{[1 - \alpha_n(\tau - \gamma \alpha)]}{[1 + \alpha_n(\tau - \gamma \alpha)]} ||x_n - p||^2 \\
&\quad + \frac{2\alpha_n \langle \gamma f(p) - \mu G(p), x_{n+1} - p \rangle}{[1 + \alpha_n(\tau - \gamma \alpha)]}.
\end{aligned}$$

Thus

$$\begin{aligned}
||x_{n+1} - p||^2 &\leq \left[1 - \frac{2\alpha_n(\tau - \gamma\alpha)}{[1 + \alpha_n(\tau - \gamma\alpha)]}\right]||x_n - p||^2 \\
&\quad + \frac{2\alpha_n\langle\gamma f(p) - \mu G(p), x_{n+1} - p\rangle}{[1 + \alpha_n(\tau - \gamma\alpha)]}.
\end{aligned}$$

Let $b_n := \frac{2\alpha_n(\tau-\gamma\alpha)}{[1+\alpha_n(\tau-\gamma\alpha)]}$ and $\zeta_n := \frac{2\alpha_n\langle\gamma f(p)-\mu G(p),x_{n+1}-p\rangle}{[1+\alpha_n(\tau-\gamma\alpha)]}$.

But $\sum_{n=1}^{\infty}\alpha_n = \infty$ and $\limsup_{n\to\infty}\langle\gamma f(p) - \mu G(p), x_{n+1} - p\rangle \leq 0$.

It follows that $\sum_{n=1}^{\infty}b_n = \infty$ and $\limsup_{n\to\infty}\frac{\zeta_n}{\alpha_n} \leq 0$.

Applying Lemma 1.11, we conclude that $x_n \to p$ as $n \to \infty$.

$\square$

2.1. **Numerical Example.** Here, we discuss the direct application of Theorem 2.2 on a typical example on a real line. Consider the following:

$$\mathbb{H} = \mathbb{R}, C = [0, 1/2], g(z, y) = y^2 + yz - 2z^2, \ Gx = 2x, \ Tx = \frac{x}{2}, \ Ax = 3x = Bx,$$

$$T_s^g x = \{z \in C : g(z, y) + \frac{1}{s}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C\},$$

$$T_r x := \{z \in C : \langle y - z, Tz\rangle - \frac{1}{r}\langle y - z, (1 + r)z - x\rangle \leq 0, \ \forall y \in C\},$$

$$T_s^g x = \frac{1}{1 + 3s}x, \quad T_r x = \frac{2x}{2 + r}.$$

Choose $s = 1 = r, \alpha_n = \frac{1}{2n}, \delta_n = \frac{n}{2n+1}, f(x_n) = \frac{1}{4}x_n, \beta = \frac{1}{2}, \lambda = \frac{1}{10} \in (0, 1/||B||^2)$, $\kappa = 2, \eta = \frac{3}{2}, \mu = \frac{1}{2}, \gamma = \frac{1}{14}$, then the scheme (2.5) can be simplified as

$$y_n = \frac{13}{40}x_n,$$

$$z_n = \frac{13}{48}x_n,$$

$$x_{n+1} = \frac{112n^2 + 2n + 1}{112n(2n + 1)}x_n + \frac{13(2n^2 - 1)}{96n(2n + 1)}x_n.$$

Take the initial point $x_1 = 0.5$, the numerical experiment result using MATLAB is given in Figure 1, which shows the iteration process of the sequence $\{x_n\}$ converges to 0.
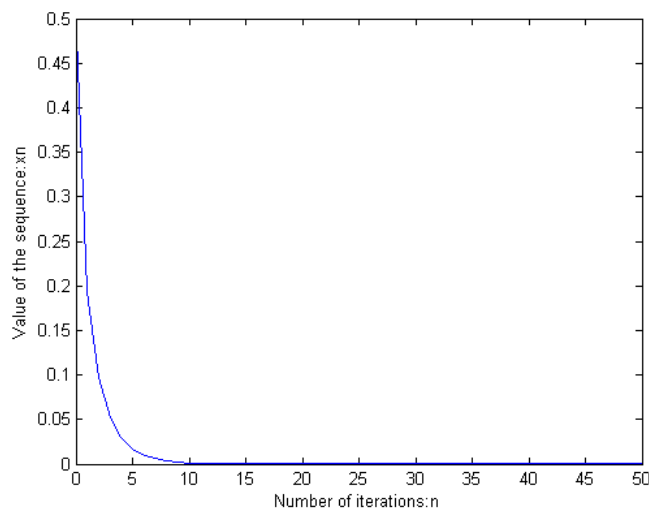
FIGURE 1. $x_1 = 0.5$, the convergence process of the sequence $\{x_n\}$.

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