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EXISTENCE RESULTS FOR APPROXIMATE SET-VALUED EQUILIBRIUM PROBLEMS

MALEK ABBASI[1] and MAHBOUBEH REZAEI[2]*

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Abstract. This paper studies the generalized approximate set-valued equilibrium problems and furnishes some new existence results. The existence results for solutions are derived by using the celebrated KKM theorem and some concepts associated with the semi-continuity of the set-valued mappings such as outer-semicontinuity, inner-semicontinuity, upper-semicontinuity and so forth. The results achieved in this paper generalize and improve the works of many authors in references.

1. Introduction and preliminaries

The notion of set-valued equilibrium problems has been discussed by many authors. (see for instance [2] and the references therein) Indeed, equilibrium problems in a variety of disciplines such as market equilibrium problems, economic equilibrium problems, traffic network equilibrium problems, mathematical programming, complementarity problems and so on, play vital roles; see [3, 4, 5, 6, 18]. It is completely understood that many concepts and problems in nonlinear analysis such as variational inequality problems, optimization problems, inverse optimization problems and fixed point problems are just special cases of equilibrium problems. So it will be of high importance to generalize or improve the results achieved by the authors who have worked in these fields. In

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*Corresponding author.
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this direction, many valuable achievements have been produced. Among these productions, the work of [2] is an illustrative one. Indeed, the authors of the mentioned paper have discussed two equilibrium problems with the following wording:
(SVEP):

$$\text{find } \bar{x} \in C \text{ such that}$$

$$\Phi(\bar{x}, y) \subset \mathbb{R}_+ \ \forall y \in C,$$

and the following weaker one:
(SVEP(W)):

$$\text{find } \bar{x} \in C \text{ such that}$$

$$\Phi(\bar{x}, y) \cap \mathbb{R}_+ \neq \emptyset \ \forall y \in C,$$

where C is a subset of a Hausdorff topological space, $\Phi : C \times C \rightrightarrows \mathbb{R}$ is a set-valued mapping and $\mathbb{R}_+$ denotes the set of nonnegative real numbers. The following paper proceeds to move in this direction. Indeed, we discuss four approximate set-valued equilibrium problems with the following wording:

(1) ($+\epsilon$-SVEP(W)): find $\bar{x} \in X$ such that

$$\Phi(\bar{x}, x) + \epsilon\|\bar{x} - x\| \nsubseteq -\text{int}K \ \forall x \in X,$$

(2) ($-\epsilon$-SVEP(W)): find $\bar{x} \in X$ such that

$$\Phi(\bar{x}, x) - \epsilon\|\bar{x} - x\| \nsubseteq -\text{int}K \ \forall x \in X,$$

(3) ($+\epsilon$-SVEP): find $\bar{x} \in X$ such that

$$\Phi(\bar{x}, x) + \epsilon\|\bar{x} - x\| \subset K \ \forall x \in X,$$

(4) ($-\epsilon$-SVEP): find $\bar{x} \in X$ such that

$$\Phi(\bar{x}, x) - \epsilon\|\bar{x} - x\| \subset K \ \forall x \in X,$$

where, in these four problems X, Y are two Banach spaces, $\Phi : X \times X \rightrightarrows Y$ is a set-valued mapping, K is a pointed closed convex cone in Y with nonempty interior and ϵ is a fixed point in cone K. Notice that in these problems the sum is in the sense of the usual Minkowski sum of sets and we will obey this role throughout the paper. Also notice that the following implications hold.

- $\bar{x}$ is a solution of problem (2) $\Longrightarrow$ $\bar{x}$ is a solution of problem (1);
- $\bar{x}$ is a solution of problem (4) $\Longrightarrow$ $\bar{x}$ is a solution of problem (3);
- $\bar{x}$ is a solution of problem (3) $\Longrightarrow$ $\bar{x}$ is a solution of problem (1);
- $\bar{x}$ is a solution of problem (4) $\Longrightarrow$ $\bar{x}$ is a solution of problem (2).

The following formulation unifies problems (1) and (2) into a single one. We do this unification just to ease referencing.
($\pm\epsilon$-SVEP(W)):

$$\text{find } \bar{x} \in X \text{ such that}$$

$$\Phi(\bar{x}, x) \pm \epsilon\|\bar{x} - x\| \nsubseteq -\text{int}K \ \forall x \in X.$$

The same statement holds for problems (3) and (4). Each problem is of interest and will be discussed in this context but we are concerned mostly with the second and third ones.
By using the celebrated KKM theorem (or Ky Fan's theorem) and some concepts associated with the continuity of set-valued mappings (inner-semicontinuity, outer-semicontinuity, upper-semicontinuity and so forth), we follow some existence results consisting of some sufficient conditions guaranteeing the solvability of the mentioned problems. Let us verify these problems more precisely in the next section and conclude this section adding only a paragraph summarizing the organization of the paper.
In section 2 we present definitions and notations needed in addressing our study. We will also formulate the approximate set-valued equilibrium problems we aim to discuss here. In section 3 we study some existence theorems for approximate set-valued equilibrium problems ($\pm\epsilon$-(SVEP(W))). In section 4 we discuss the two last equilibrium problems ($\pm\epsilon$-(SVEP)) and achieve some new results. In this way, Ky Fan's theorem is our main weapon. This tool together with some concepts and notions of semi-continuity of set-valued mappings establish the main conclusions of this study. We hope the reader will find something of interest in this article.

2. Preliminaries

Throughout this paper, unless otherwise specified, let X, Y be two Banach spaces. Let $K \subset Y$ be a closed convex pointed cone with $\text{int}K \neq \emptyset$, where $\text{int}K$ denotes the topological interior of K. Let $X^\star$ denote the dual of X. If $F : X \rightrightarrows Y$ is a set-valued mapping, then the graph of F, denoted $\text{gph}(F)$, is defined as

$$\text{gph}(F) = \{(x, y) \in X \times Y : y \in F(x)\}.$$

The projection of $\text{gph}(F)$ onto its first argument is regarded as the domain of F, denoted $\text{dom}(F)$ and given by

$$\text{dom}(F) = \{x \in X : F(x) \neq \emptyset\}.$$

Given $x, y \in Y$, we sometime use the following ordering relations on the space Y [13]:

$$\begin{aligned} y <_K x \quad &\Leftrightarrow \quad y - x \in -\text{int}K; \\ y \nless_K x \quad &\Leftrightarrow \quad y - x \notin -\text{int}K; \\ y \leq_K x \quad &\Leftrightarrow \quad y - x \in -K; \\ y \nleq_K x \quad &\Leftrightarrow \quad y - x \notin -K. \end{aligned}$$

In this work we aim to discuss the following approximate set-valued equilibrium problems:
($\pm\epsilon$-SVEP(W)): find $\bar{x} \in X$ such that

$$\Phi(\bar{x}, x) \pm \epsilon\|\bar{x} - x\| \nsubseteq -\text{int}K \ \forall x \in X,$$

and
($\pm\epsilon$-SVEP): find $\bar{x} \in X$ such that

$$\Phi(\bar{x}, x) \pm \epsilon\|\bar{x} - x\| \subset K \ \forall x \in X,$$

where $\Phi : X \times X \rightrightarrows Y$ is a set-valued mapping and $\epsilon \in K$ is a fixed point. In other words $\bar{x} \in X$ is a solution of the ($\pm\epsilon$-SVEP(W)) if for any $x \in X$ there exists some $y \in \Phi(\bar{x}, x)$ so that

$$y \pm \epsilon\|\bar{x} - x\| \not<_K 0.$$

Similarly $\bar{x} \in X$ is a solution of the ($\pm\epsilon$-SVEP) if for any $x \in X$ and $y \in \Phi(\bar{x}, x)$ one has

$$0 \leq_K y \pm \epsilon\|\bar{x} - x\|.$$

In the special case that Y is a finite dimensional Banach space ($Y = \mathbb{R}^m$ for some $m \in \mathbb{N}$), the approximate set-valued equilibrium problems ($\mp\epsilon$-SVEP(W)) and ($\pm\epsilon$-SVEP) are also called the approximate vector set-valued equilibrium problems. The problem ($+\epsilon$-SVEP(W)) reduces to:

(1) a set-valued equilibrium problem (SVEP(W)), discussed in [2], whenever $Y = \mathbb{R}$, $K = \mathbb{R}^1_+ = \mathbb{R}_+$ and $\epsilon = 0$;
(2) a generalized vector equilibrium problem introduced and studied by Li and Zhao [16], whenever $\epsilon = 0$, $\Phi(x, y) = \{h(x, y) + f(x) - f(y)\}$, $Y = \mathbb{R}^m$ and $K = \mathbb{R}^m_+ = \{x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m : x_i \geq 0, i = 1, 2, \ldots, m\}$, where $h : X \times X \to Y$ and $f : X \to Y$ are two vector-valued mappings satisfying $h(x, x) = 0$ for all $x \in X$;
(3) a vector equilibrium problem [7, 8, 9, 12], whenever $\epsilon = 0$, $\Phi(x, y) = \{h(x, y)\}$, $Y = \mathbb{R}^m$ and $K = \mathbb{R}^m_+$, where $h : X \times X \to Y$ is a vector-valued mapping satisfying $h(x, x) = 0$ for all $x \in X$;
(4) a generalized ϵ-vector variational inequality [14, 15], whenever $\Phi(x, y) = \{\langle T(x), y - x\rangle + f(x) - f(y)\}$, $Y = \mathbb{R}^m$ and $K = \mathbb{R}^m_+$ where $T : X \to L(X, Y)$ maps each $x \in X$ to a continuous linear operator from X to Y and $f : X \to Y$ is a vector-valued map.

Analogously the approximate set-valued equilibrium problems ($\pm\epsilon$-SVEP) reduce to the set-valued equilibrium problem (SVEP), discussed in [2], by letting $Y = \mathbb{R}$, $K = \mathbb{R}^1_+ = \mathbb{R}_+$ and $\epsilon = 0$.

We continue our study by stating some notions related to the continuity and semicontinuity of set-valued mappings. Let us first recall the two concepts of $\lim\mathrm{int}$ and $\lim\mathrm{ext}$ associated with a net of sets [10]. Let $\{C_i\}_{i\in I}$ be a net of sets in a topological space V. We write $x \in \lim\mathrm{ext}_{i\in I} C_i$ if and only if for each neighborhood W of x one has $W \cap C_i \neq \emptyset$ for i in a cofinal subset of I. Analogously we may write $x \in \lim\mathrm{int}_{i\in I} C_i$ if and only if for each neighborhood W of x we have $W \cap C_i \neq \emptyset$ for i in a terminal subset of I. We say the net $\{C_i\}_{i\in I}$ converges to the set C if and only if $\lim\mathrm{ext}_{i\in I} C_i = \lim\mathrm{int}_{i\in I} C_i = C$. Now assume that V, H are Hausdorff topological spaces and $F : V \rightrightarrows H$ is a set-valued mapping. We now recall the followings notions of continuity and semicontinuity [10].
The set-valued mapping F is said to be:

(1) outer-semicontinuous at $v \in \mathrm{dom}(F)$ if whenever a net $\{v_i\}_{i\in I} \subset V$ converges to v, then $\lim\mathrm{ext}_{i\in I} F(v_i) \subset F(v)$;
(2) inner-semicontinuous at $v \in \mathrm{dom}(F)$ if whenever a net $\{v_i\}_{i\in I} \subset V$ converges to v, then $F(v) \subset \lim\mathrm{int}_{i\in I} F(v_i)$;
(3) upper-semicontinuous at $v \in \mathrm{dom}(F)$ if whenever S is an open set containing $F(v)$ then there exists an open set, say U, containing v so that $F(U) \subset S$;
(4) continuous at $v \in \mathrm{dom}(F)$ if it is both outer-semicontinuous and inner-semicontinuous at v;
(5) K-continuous at $v \in \mathrm{dom}(F)$ (K for Kuratowski) if it is both upper-semicontinuous and inner-semicontinuous at v.

Among these five kinds of continuity and semicontinuity for such a set-valued mapping the first and the second ones (i.e., outer-semicontinuity and inner-semicontinuity) are of interest in this paper.

In this study we also need the notions of convexity and concavity associated with the set-valued mappings. Suppose that $F : V \rightrightarrows H$ is a set-valued mapping with V, H two linear spaces. Let us remark once again that we follow Minkowski [17] in the definition of addition of set-valued mappings.

The set-valued mapping F is said to be: (see [13, 2])

(1) convex if
$$\lambda F(v) + (1-\lambda)F(u) \subset F(\lambda v + (1-\lambda)u),$$
for all $u, v \in V$ and $\lambda \in [0, 1]$;
(2) concave if the converse inclusion holds. That is
$$F(\lambda v + (1-\lambda)u) \subset \lambda F(v) + (1-\lambda)F(u),$$
for all $u, v \in V$ and $\lambda \in [0, 1]$.

We finally state the definition of a KKM mapping and then the Ky Fan's lemma for easy reference.

Definition 2.1. [11, 19] Let V be a Hausdorff topological vector space and let A be a nonempty subset of V. A set-valued mapping $F : A \rightrightarrows V$ is called a KKM map if
$$\mathrm{conv}\{v_1, v_2, \cdots, n_n\} \subseteq \bigcup_{k=1}^{n} F(v_k),$$
for each finite subset $\{v_1, v_2, \cdots, v_n\} \subseteq A$, where $\mathrm{conv}\{v_1, v_2, \cdots, v_n\}$ denotes the convex hull of the points $\{v_1, v_2, \cdots, v_n\}$.

Lemma 2.2. [11, 19] *Let V be a Hausdorff topological vector space and let $A \subseteq V$ be an arbitrary set. Let $F : A \rightrightarrows V$ be an KKM mapping. If F has closed values and $F(\bar{v})$ is compact for at least one $\bar{v} \in A$, then*
$$\bigcap_{v\in A} F(v) \neq \emptyset.$$

In some situations, verifying the condition that a given set-valued mapping is a KKM one may seem somewhat difficult. The following corollary, as a direct consequence of Ky Fan's theorem, provides another suitable tool. (see [1] for a direct proof)

Corollary 2.3. *Let V be a Hausdorff topological vector space and $A \subseteq V$ be a convex set. Let $F : A \rightrightarrows V$ be a given set-valued mapping which has closed values and $F(\bar{v})$ is compact for at least one $\bar{v} \in A$. If furthermore F satisfies the following two conditions:*

(1) *$v \in F(v)$ for each $v \in A$;*

(2) *$F(\lambda v + (1-\lambda)u) \subseteq F(v) \bigcup F(u)$ for any $\lambda \in (0,1)$ and $v, u \in A$,*

then

$$\bigcap_{v \in A} F(v) \neq \emptyset.$$

3. Existence Theorems for Approximate Set-Valued Equilibrium Problem ($\pm\epsilon$-SVEP(W))

In this section, we establish some existence results for solutions of the generalized approximate set-valued equilibrium problems ($\pm\epsilon$-SVEP(W)) formulated in the previous section. We begin with the following simple result. The terminologies and notations are all as above.

Theorem 3.1. *Suppose that Y is a finite dimensional Banach space. Suppose that the following assumptions hold true:*

(1) *$\Phi(x,x) \nsubseteq -intK$ for each $x \in X$;*

(2) *the set-valued mappings $x \mapsto \Phi(x,y)$ is outer-semicontinuous and bounded for each $y \in X$;*

(3) *for any $x \in X$, the set-valued mapping $y \mapsto \Phi(x,y)$ is concave;*

(4) *there exist a nonempty compact subset C and $c \in C$ such that for any $y \in X \setminus C$,*

$$\Phi(y,c) + \epsilon\|y-c\| \subset -intK.$$

Then the approximate set-valued equilibrium problem ($+\epsilon$-SVEP(W)) has a solution.

Proof. Associated with this problem we define a set-valued mapping $\Gamma : X \to X$ by

$$\Gamma(y) = \{x \in X : \Phi(x,y) + \epsilon\|x-y\| \nsubseteq -\mathrm{int}K\}, \quad \forall y \in X.$$

Obviously $\bar{x} \in X$ is a solution of ($+\epsilon$-SVEP(W)) if $\bar{x} \in \bigcap_{y \in X} \Gamma(y)$. We therefore set out to check the conditions of Corollary 2.3 to complete the proof. The first assumption of theorem guarantees that $y \in \Gamma(y)$ for all $y \in X$. This implies $\Gamma(y) \neq \emptyset$ for all $y \in X$. This also implies that the first condition of Corollary 2.3 holds true. Let us check the second condition of the mentioned corollary. Let $\lambda \in (0,1)$ and $x, y \in X$. Let $w \in \Gamma(\lambda x + (1-\lambda)y)$. By way of contradiction assume that $w \notin \Gamma(x) \bigcup \Gamma(y)$. This implies

$$\Phi(w,x) + \epsilon\|w-x\| \subset -\mathrm{int}K,$$
$$\Phi(w,y) + \epsilon\|w-y\| \subset -\mathrm{int}K.$$

Multiplying both sides of the first inclusion by λ and the second inclusion by $(1-\lambda)$ and then summing up we follow

$$\lambda\Phi(w,x)+(1-\lambda)\Phi(w,y)+\epsilon\|\lambda w-\lambda x\|+\epsilon\|(1-\lambda)w-(1-\lambda)y\| \subset -\mathrm{int}K. \tag{3.1}$$

Since $\epsilon \in K$ and

$$\|w-(\lambda x+(1-\lambda)y)\| - (\|\lambda w-\lambda x\| + \|(1-\lambda)w-(1-\lambda)y\|),$$

is a nonpositive real number it follows that

$$\epsilon\|w-(\lambda x+(1-\lambda)y\| - (\epsilon\|\lambda w-\lambda x\| + \epsilon\|(1-\lambda)w-(1-\lambda)y\|) \in -K.$$

Applying this conclusion in (3.1), using the third condition of theorem and Lemma 2.3.4, p. 22 of [13], we deduce that

$$\begin{aligned} & \Phi(w,\lambda x+(1-\lambda)y)+\epsilon\|w-(\lambda x+(1-\lambda)y\| \\ \subset\ & \lambda\Phi(w,x)+(1-\lambda)\Phi(w,y)+\epsilon\|w-(\lambda x+(1-\lambda)y\| \\ \subset\ & -\mathrm{int}K-K \\ \subset\ & -\mathrm{int}K, \end{aligned}$$

which is absurd because it violates $w \in \Gamma(\lambda x+(1-\lambda)y)$. It follows that the second condition of Corollary 2.3 holds too. We now prove that $\Gamma(y)$ is closed for any $y \in X$. So let $y \in X$ and (x_n) be a sequence in $\Gamma(y)$ which converges to some $x \in X$. Hence

$$\Phi(x_n,y)+\epsilon\|x_n-y\| \nsubseteq -\mathrm{int}K,$$

for all $n \in \mathbb{N}$. It follows that for any $n \in \mathbb{N}$ there exists $w_n \in \Phi(x_n,y)$ so that

$$w_n+\epsilon\|x_n-y\| \notin -\mathrm{int}K. \tag{3.2}$$

Without any loss of generality we may assume that (w_n) converges to some w which belongs to $\lim\mathrm{ext}_{n\in\mathbb{N}}\Phi(x_n,y)$. Letting $n\to\infty$ in (3.2) we deduce that

$$w+\epsilon\|x-y\| \notin -\mathrm{int}K. \tag{3.3}$$

By the hypothesis, the set-valued mappings $x \mapsto \Phi(x,y)$ is outer-semicontinuous thus $w \in \Phi(x,y)$. This by virtue of (3.3) yields

$$\Phi(x,y)+\epsilon\|x-y\| \nsubseteq -\mathrm{int}K,$$

from which we deduce that $\Gamma(y)$ is closed. Finally the last condition of the theorem guarantees that $\Gamma(c)$ as a closed subset of the compact set C is compact too. This completes the proof. $\square$

The following theorem guarantees that under some mild conditions the approximate set-valued equilibrium problem ($-\epsilon$-SVEP(W)) has a solution. The details are as follows. Let us remark that for a set-valued mapping $F: V \rightrightarrows H$, where V, H are tow Hausdorff topological vector spaces, we say F is weakly outer-semicontinuous (respectively, weakly inner-semicontinuous) at $v \in V$ whenever F is outer-semicontinuous (respectively, inner-semicontinuous) at v with respect to the weak topology of V.

Theorem 3.2. *Let Y be a finite dimensinal Banach space. Suppose that the following assumptions hold:*

(1) $\Phi(x,x) \nsubseteq -intK$ *for each* $x \in X$*;*

(2) *the set-valued mappings* $x \mapsto \Phi(x, y)$ *is weakly outer-semicontinuous and bounded for each* $y \in X$*;*
(3) *for any* $x \in X$ *the set* $H_x = \{y \in X : \Phi(x, y) - \epsilon\|x - y\| \subset -intK\}$ *is convex;*
(4) *there exist a nonempty weakly compact subset* C*,* $c \in C$ *and* $\bar{y}^\star \in X^\star$ *such that for any* $y \in X \setminus C$*,*

$$\Phi(y, c) - \epsilon|\bar{y}^\star(y - c)| \subset -intK.$$

Then the approximate set-valued equilibrium problem ($-\epsilon$*-SVEP(W)) has a solution.*

Proof. We define a set-valued mapping $\Gamma : X \to X$ by

$$\Gamma(y) = \{x \in X : \Phi(x, y) - \epsilon\|x - y\| \nsubseteq -\text{int}K\}, \quad \forall y \in X.$$

Obviously $\bar{x} \in X$ is a solution of ($-\epsilon$-SVEP) if $\bar{x} \in \bigcap_{y \in X} \Gamma(y)$. We set out to check the conditions of Fan's lemma to complete the proof. The first assumption of theorem guarantees that $y \in \Gamma(y)$ for all $y \in X$ and therefore $\Gamma(y) \neq \emptyset$ for each $y \in Y$. Let us prove that Γ is a KKM mapping. Suppose, on the contrary, that Γ is not a KKM mapping. Then, there exists a finite subset $\{y_1, y_2, \ldots, y_n\}$ of X such that

$$\text{conv}\{y_1, y_2, \ldots, y_n\} \not\subset \bigcup_{i=1}^{n} \Gamma(y_i).$$

Hence, there exists $y \in \text{conv}\{y_1, y_2, \ldots, y_n\}$ such that

$$y \notin \bigcup_{i=1}^{n} \Gamma(y_i).$$

So, for any $i \in \{1, 2, \ldots, n\}$, we have

$$\Phi(y, y_i) - \epsilon\|y_i - y\| \subset -\text{int}K.$$

Hence, $\{y_1, y_2, \ldots, y_n\} \subset H_y$. Since H_y is convex, we deduce that

$$\text{conv}\{y_1, y_2, \ldots, y_n\} \subset H_y.$$

Since $y \in \text{conv}\{y_1, y_2, \ldots, y_n\}$, we have $y \in H_y$. This implies that

$$\Phi(y, y) - \epsilon\|y - y\| \subset -\text{int}K,$$

which violates the first assumption of the theorem. Therefore Γ is a KKM mapping. We now define a set-valued mapping $\Lambda : X \times X^\star \rightrightarrows X$

$$\Lambda(y, y^\star) = \{x \in X : \Phi(x, y) - \epsilon|y^\star(x - y)| \nsubseteq -\text{int}K\}, \quad \forall y \in X, y^\star \in X^\star.$$

We claim that $\Lambda(y, y^\star)$ is weakly closed for each $y \in X$ and $y^\star \in X^\star$. Take $(y, y^\star) \in X \times X^\star$. Let $\{x_i\}_{i \in I}$ be a net in $\Lambda(y, y^\star)$ which converges to some $x \in X$. Hence

$$\Phi(x_i, y) - \epsilon|y^\star(x_i - y)| \nsubseteq -\text{int}K,$$

for all $i \in I$. It follows that for any $i \in I$ there exists $w_i \in \Phi(x_i, y)$ so that

$$w_i - \epsilon|y^\star(x_i - y)| \notin -\text{int}K. \tag{3.4}$$

Without any loss of generality we may assume that $\{w_i\}_{i\in I}$ converges to some w which belongs to $\lim\mathrm{ext}_{i\in I}\Phi(x_i, y)$. We deduce from (3.4) that

$$w - \epsilon|y^\star(x-y)| \notin -\mathrm{int}K. \tag{3.5}$$

Since, by the hypothesis, the set-valued mappings $x \mapsto \Phi(x, y)$ is weakly outer-semicontinuous thus $w \in \Phi(x, y)$. This property together with (3.5) yield

$$\Phi(x, y) - \epsilon|y^\star(x-y)| \nsubseteq -\mathrm{int}K,$$

from which we deduce that $\Lambda(y, y^\star)$ is weakly closed. We now assert that the following equality holds:

$$\Gamma(y) = \bigcap_{y^\star \in U_{X^\star}} \Lambda(y, y^\star), \tag{3.6}$$

where $U_{X^\star}$ denotes the boundary of the unite ball in $X^\star$. To see this let $x \in \Gamma(y)$. Thus

$$\Phi(x, y) - \epsilon\|x-y\| \nsubseteq -\mathrm{int}K.$$

If now $x \notin \bigcap_{y^\star\in U_{X^\star}} \Lambda(y, y^\star)$, thus there exists some $y^\star \in U_{X^\star}$ so that $x \notin \Lambda(y, y^\star)$. Therefore

$$\Phi(x, y) - \epsilon|y^\star(x-y)| \subset -\mathrm{int}K.$$

On the other hand $|y^\star(x-y)| \leq \|x-y\|$ from which we deduce that $\epsilon|y^\star(x-y)| - \epsilon\|x-y\| \in -K$. These two last inequalities imply

$$\Phi(x, y) - \epsilon\|x-y\| \subset -\mathrm{int}K,$$

which is absurd. Conversely let $x \in \bigcap_{y^\star\in U_{X^\star}} \Lambda(y, y^\star)$. This implies

$$\Phi(x, y) - \epsilon|y^\star(x-y)| \nsubseteq -\mathrm{int}K,$$

for all $y^\star \in U_{X^\star}$. By the Hahn–Banach theorem there exists some $y^\star \in U_{X^\star}$ so that $y^\star(x-y) = \|x-y\|$, from which we deduce that

$$\Phi(x, y) - \epsilon\|x-y\| \nsubseteq -\mathrm{int}K.$$

Thus equality (3.6) holds, proving the claim. We continue the proof of the theorem. The last hypothesis of theorem implies that $\Lambda(c, \bar{y}^\star)$ is weakly compact and thus $\bigcap_{y^\star\in U_{X^\star}} \Lambda(c, y^\star)$ is weakly compact too. Hence $\Gamma(c)$ is weakly compact. By our discussion above (before the equality (3.6)), and using the mentioned equality we know that $\Gamma(y)$ is weakly closed for all $y \in X$. We see that the whole conditions of Fan's theorem hold. This completes the proof. □

Theorem 3.3. *Let Y be a finite dimensional Banach space. Assume that the following conditions are satisfied:*

1. $\Phi(x, x) \nsubseteq -\mathit{int}K$ *for each* $x \in X$*;*
2. *the set-valued mappings* $x \mapsto \Phi(x, y)$ *is weakly outer-semicontinuous and bounded for each* $y \in X$*;*
3. *for any* $y \in X$ *the set* $H_y = \{(x, y^\star) \in X \times B_{X^\star} : \Phi(y, x) - \epsilon y^\star(y-x) \subset -\mathit{int}K\}$ *is convex, where* $B_{X^\star}$ *denotes the unit ball in* X^**;*
4. *there exist a nonempty weakly compact subset* C*,* $c \in C$ *and* $\bar{y}^\star \in X^\star$ *such that for any* $y \in X \setminus C$*,*

$$\Phi(y, c) - \epsilon\bar{y}^\star(y-c) \subset -\mathit{int}K.$$

Then the approximate set-valued equilibrium problem ($-\epsilon$-SVEP(W)) has a solution.

Proof. The proof is similar to that of Theorem 3.2, we therefor give only a sketch of the proof. Let $\Theta = X \times B_{X^\star}$. Equip Θ with the product topology, being X equipped with the $weak(\sigma(X, X^\star))$ topology and $B_{X^\star}$ with the relative $weak^\star(\sigma(X^\star, X))$ topology. Define the set-valued map $\Gamma : \Theta \rightrightarrows \Theta$ by

$$\Gamma(y, y^\star) := \{x \in X : \Phi(x, y) - \epsilon y^\star(x - y) \nsubseteq -\text{int}K\} \times B_{X^\star},$$

for all $(y, y^\star) \in \Theta$. Let us verify that Γ is a KKM map. To this end suppose, on the contrary, that Γ is not a KKM mapping. So, there exists a finite subset

$$\{(y_1, y_1^\star), (y_2, y_2^\star), \ldots, (y_n, y_n^\star)\} \subset X \times B_{X^\star}$$

such that

$$\text{conv}\{(y_1, y_1^\star), (y_2, y_2^\star), \ldots, (y_n, y_n^\star)\} \not\subset \bigcup_{i=1}^{n} \Gamma(y_i, y_i^\star).$$

Hence, there exists some $(y, y^\star) \in \text{conv}\{(y_1, y_1^\star), (y_2, y_2^\star), \ldots, (y_n, y_n^\star)\}$ such that

$$(y, y^\star) \notin \bigcup_{i=1}^{n} \Gamma(y_i, y_i^\star).$$

Since $B_{X^\star}$ is convex, it follows that for any $i \in \{1, 2, \ldots, n\}$, we have

$$\Phi(y, y_i) - \epsilon y_i^\star(y - y_i) \subset -\text{int}K.$$

Hence, $\{(y_1, y_1^\star), (y_2, y_2^\star), \ldots, (y_n, y_n^\star)\} \subset H_y$. The convexity of H_y implies that

$$\text{conv}\{(y_1, y_1^\star), (y_2, y_2^\star), \ldots, (y_n, y_n^\star)\} \subset H_y.$$

We deduce that $(y, y^\star) \in H_y$ and therefore

$$\Phi(y, y) - \epsilon y^\star(y - y) \subset -\text{int}K.$$

This contradicts the first assumption of the theorem. Therefore $\Gamma(\cdot, \cdot)$ is a KKM mapping. By the Banach–Alaoglu theorem and in virtue of the last condition of theorem we know that Γ satisfies the conditions of Fan's theorem entirely. It follows that there exists some $(x, x^\star) \in \Theta$ so that $(x, x^\star) \in \Gamma(y, y^\star)$ for all $(y, y^\star) \in \Theta$. Thus

$$\Phi(x, y) - \epsilon y^\star(x - y) \nsubseteq -\text{int}K, \tag{3.7}$$

for all $y \in X$ and $y^\star \in B_{X^\star}$. By Hahn-Banach theorem we deduce that for any $y \in X$ there exists some $y_y^\star \in B_{X^\star}$ satisfying $y_y^\star(x - y) = \|x - y\|$. By (3.7) we have

$$\Phi(x, y) - \epsilon y_y^\star(x - y) \nsubseteq -\text{int}K,$$

for all $y \in X$, from which the desired conclusion follows. □

The following corollary is an immediate conclusion of Theorem 3.2.

Corollary 3.4. *Let Y be a finite dimensinal Banach space. Suppose that the following assumptions hold:*

(1) $\Phi(x, x) \nsubseteq -intK$ *for each* $x \in X$*;*

(2) *the set-valued mappings* $x \mapsto \Phi(x, y)$ *is weakly outer-semicontinuous and bounded for each* $y \in X$;
(3) *for any* $x \in X$ *the set-valued mapping* $y \mapsto \Phi(x, y) - \epsilon\|y - x\|$ *is concave;*
(4) *there exist a nonempty weakly compact subset* C, $c \in C$ *and* $\bar{y}^\star \in X^\star$ *such that for any* $y \in X \setminus C$,

$$\Phi(y, c) - \epsilon|\bar{y}^\star(y - c)| \subset -intK.$$

Then the approximate set-valued equilibrium problem ($-\epsilon$*-SVEP(W)) has a solution.*

Proof. Since the set-valued mappings $x \mapsto \Phi(x, y) - \epsilon\|y - x\|$ is concave thus the third condition of Theorem 3.2 holds. The proof is over. □

These results yield some conclusions we state below. Let us first recall some well-known results about the semi-continuity of the set-valued mappings.

Theorem 3.5. *([10]) Assume that* V, H *are Hausdorff topological spaces and take a set-valued mapping* $F : V \rightrightarrows H$. *Then the following statements are equivalent.*
(1) $gph(F)$ *is closed.*
(2) F *is outer-semicontinuous.*

Theorem 3.6. *([10]) Assume that* V, H *are Hausdorff topological spaces and take a set-valued mapping* $F : V \rightrightarrows H$. *Assume further that* H *is regular (i.e., (T3)) and* F *is a closed-value set-valued mapping (i.e.,* $F(v)$ *is closed for each* $v \in V$*). If* F *is upper-semicontinuous, then* F *is outer-semicontinuous.*

Combining these results with Theorems 3.1, 3.2, 3.3 and Corollary 3.4 we can produce some new results. Here we only state two of them.

Corollary 3.7. *Let* Y *be a finite dimensional Banach space. Assume that the following conditions are satisfied:*
(1) $\Phi(x, x) \nsubseteq -intK$ *for each* $x \in X$;
(2) *the set-valued mappings* $x \mapsto \Phi(x, y)$, *denoted* Φ_y, *is bounded and* $gph(\Phi_y)$ *is weakly closed for each* $y \in X$;
(3) *for any* $y \in X$ *the set* $H_y = \{(x, y^\star) \in (X \times B_{X^\star} : \Phi(y, x) - \epsilon y^\star(y - x) \subset -intK\}$ *is convex;*
(4) *there exist a nonempty weakly compact subset* C, $c \in C$ *and* $\bar{y}^\star \in X^\star$ *such that for any* $y \in X \setminus C$,

$$\Phi(y, c) - \epsilon\bar{y}^\star(y - c) \subset -intK.$$

Then the approximate set-valued equilibrium problem ($-\epsilon$*-SVEP(W)) has a solution.*

Corollary 3.8. *Let* Y *be a finite dimensional Banach space. Assume that the following conditions are satisfied:*
(1) $\Phi(x, x) \nsubseteq -intK$ *for each* $x \in X$;
(2) *the set-valued mappings* $x \mapsto \Phi(x, y)$ *is weakly upper-semicontinuous, weakly closed-value and bounded for each* $y \in X$;
(3) *for any* $y \in X$ *and* $y^\star \in B_{X^\star}$ *the set* $H_y = \{(x, y^\star) \in X \times B_{X^\star} : \Phi(y, x) - \epsilon y^\star(y - x) \subset -intK\}$ *is convex;*

(4) *there exist a nonempty weakly compact subset* C, $c \in C$ *and* $\bar{y}^\star \in X^\star$ *such that for any* $y \in X \setminus C$,

$$\Phi(y,c) - \epsilon \bar{y}^\star(y-c) \subset -intK.$$

Then the approximate set-valued equilibrium problem ($-\epsilon$*-SVEP(W)) has a solution.*

Remark 3.9. In the case that the Banach space Y is not finite dimensional, if the set-valued mapping $x \mapsto \Phi(x,y)$ is relatively compact-valued for each $y \in X$(i.e., $\Phi(x,y)$ is relatively compact in Y for each $y \in X$), then the whole of the above conclusions still hold.

This remark together with Corollary 3.8 now yield the following result for the case that the image space Y is not finite dimensional.

Corollary 3.10. *Let* Y *be an arbitrary Banach space. Assume that the following conditions are satisfied:*

(1) $\Phi(x,x) \nsubseteq -intK$ *for each* $x \in X$*;*
(2) *the set-valued mappings* $x \mapsto \Phi(x,y)$ *is weakly upper-semicontinuous and weakly compact-valued for each* $y \in X$*;*
(3) *for any* $y \in X$ *the set* $H_y = \{(x,y^\star) \in X \times B_{x^\star} : \Phi(y,x) - \epsilon y^\star(y-x) \subset -intK\}$ *is convex;*
(4) *there exist a nonempty weakly compact subset* C, $c \in C$ *and* $\bar{y}^\star \in X^\star$ *such that for any* $y \in X \setminus C$,

$$\Phi(y,c) - \epsilon \bar{y}^\star(y-c) \subset -intK.$$

Then the approximate set-valued equilibrium problem ($-\epsilon$*-SVEP(W)) has a solution.*

Proof. Since, according to the second hypothesis, the set-valued mappings $x \mapsto \Phi(x,y)$ is weakly compact-valued for each $y \in X$ thus it is weakly closed-value and bounded for each $y \in X$. On the other hand every Banach space is regular. One may apply Corollary 3.8 to complete the proof. □

4. Existence Theorems for Approximate Set-Valued Equilibrium Problem ($\pm\epsilon$-SVEP)

In this section, we establish some existence results for solutions of the generalized approximate set-valued equilibrium problems ($\pm\epsilon$-SVEP) with the following formulation (we repeat once again):
find $\bar{x} \in X$ such that

$$\Phi(\bar{x},x) \pm \epsilon\|\bar{x}-x\| \subset K \ \forall x \in X,$$

where $\Phi : X \times X \rightrightarrows Y$ is a set-valued mapping and $\epsilon \in K$ is a fixed point. In other words $\bar{x} \in X$ is a solution of any such problem if for any $x \in X$ and $y \in \Phi(\bar{x},x)$ one has

$$0 \leq_K y \pm \epsilon\|\bar{x}-x\|.$$

As we have already mentioned, this approximate set-valued equilibrium problem reduces to the set-valued equilibrium problem (SVEP), discussed in [2], by letting

$Y = \mathbb{R}$, $K = \mathbb{R}^1_+ = \mathbb{R}_+$ and $\epsilon = 0$. We remark that the assumptions and notations are all as above. To continue we start with the following theorem.

Theorem 4.1. *Suppose that Y is an arbitrary Banach space. Suppose that the following assumptions hold true:*

(1) *$\Phi(x, x) \subset K$ for each $x \in X$;*
(2) *the set-valued mappings $x \mapsto \Phi(x, y)$ is inner-semicontinuous for each $y \in X$;*
(3) *for any $x \in X$, the set-valued mapping $y \mapsto \Phi(x, y)$ is convex;*
(4) *for all $x, y \in X$*
$$\|x - y\| < \inf_{u \notin \Phi(x,y)} \|u\|;$$
(5) *there exist a nonempty compact subset C and $c \in C$ such that for any $y \in X \setminus C$,*
$$\Phi(y, c) + \epsilon\|y - c\| \not\subseteq K.$$

Suppose further that $\|\epsilon\| \leq 1$. Then the approximate set-valued equilibrium problem (+ϵ-SVEP) has a solution.

Proof. We define a set-valued mapping $\Gamma : X \rightrightarrows X$ given by
$$\Gamma(y) = \{x \in X : \Phi(x, y) + \epsilon\|x - y\| \subset K\}.$$
We assert that Γ satisfies the conditions of Corollary 2.3 entirely. The details are as follows. For all $y \in X$ one has $y \in \Gamma(y)$, as a direct consequence of the first assumption of theorem. So the condition (1) of the mentioned lemma holds. We now prove that $\Gamma(y)$ is closed for all $y \in X$. To see this let $y \in X$. Let (x_n) be a sequence in $\Gamma(y)$ converging to some $x \in X$. So
$$\Phi(x_n, y) + \epsilon\|x_n - y\| \subset K,$$
for all $n \in \mathbb{N}$. The closeness of K implies that
$$\lim\text{int}_{n\in\mathbb{N}}\{\Phi(x_n, y) + \epsilon\|x_n - y\|\} \subset K,$$
from which we deduce that
$$\lim\text{int}_{n\in\mathbb{N}}\{\Phi(x_n, y)\} + \epsilon\|x - y\| \subset K.$$
Since the set-valued mappings $x \mapsto \Phi(x, y)$ is inner-semicontinuous, thus
$$\Phi(x, y) + \epsilon\|x - y\| \subset K.$$
Hence $x \in \Gamma(y)$ and therefore $\Gamma(y)$ is closed.
Obviously $\Gamma(c)$ is compact. This is an easy consequence of the last condition of theorem and the conclusion of the previous section. We finally prove that
$$\Gamma(\lambda x + (1 - \lambda)y) \subset \Gamma(x) \cup \Gamma(y),$$
for all $x, y \in X$ and $\lambda \in (0, 1)$. For convenience let $\beta = 1 - \lambda$. The proof of this assertion goes upon the following steps. First for $u \in X$ and $u^\star \in U_{X^\star}$ consider the following settings:
$$H_u = \{y \in X : \Phi(u, y) + \epsilon\|u - y\| \not\subseteq K\},$$
$$H_{u,u^\star} = \{y \in X : \Phi(u, y) + \epsilon u^\star(u - y) \not\subseteq K\},$$

where as we have already stated $U_{X^\star}$ denotes the boundary of the unit ball in $X^\star$.

Step(1): for all $u, y \in X$ and $u^\star \in U_{X^\star}$ we have $\epsilon u^\star(y-u) \in \Phi(u,y)$. If not, then there exists $u, y \in X$ and $u^\star \in U_{X^\star}$ such that $\epsilon u^\star(y-u) \notin \Phi(u,y)$. It follows that $\|\epsilon u^\star(y-u)\| > \|y-u\|$, by the condition (4) of theorem, which is absurd since $u^\star \in U_{X^\star}$ and $\|\epsilon\| \le 1$.

Step(2): $H_{u,u^\star}$ is a convex set for all $u \in X$ and $u^\star \in U_{X^\star}$. To see this let $y_1, y_2 \in H_{u,u^\star}$ and λ, β as above. Suppose that $\lambda y_1 + \beta y_2 \notin H_{u,u^\star}$. It follows that

$$\Phi(u, \lambda y_1 + \beta y_2) + \epsilon u^\star(u - (\lambda y_1 + \beta y_2)) \subset K.$$

By condition (3) we know that the set-valued mapping $y \mapsto \Phi(u,y)$ is convex. Thus

$$\lambda\left(\Phi(u,y_1) + \epsilon u^\star(u-y_1)\right) + \beta\left(\Phi(u,y_2) + \epsilon u^\star(u-y_2)\right) \subset K.$$

By virtue of *Step(1)* we deduce that

$$\lambda\left(\epsilon u^\star(y_1-u) + \epsilon u^\star(u-y_1)\right) + \beta\left(\Phi(u,y_2) + \epsilon u^\star(u-y_2)\right) \subset K,$$

from which we follow

$$\Phi(u,y_2) + \epsilon u^\star(u-y_2) \subset K.$$

This means $y_2 \notin H_{u,u^\star}$ which is absurd.

Step(3): for all $u \in X$ we have

$$H_u = \bigcap_{u^\star \in U_{X^\star}} H_{u,u^\star}.$$

To see this first let $y \in \bigcap_{u^\star \in U_{X^\star}} H_{u,u^\star}$. Hence

$$\Phi(u,y) + \epsilon u^\star(u-y) \nsubseteq K, \tag{4.1}$$

for all $u^\star \in U_{X^\star}$. By the Hahn–Banach theorem there exists some $\bar{u}^\star \in U_{X^\star}$ so that $\bar{u}^\star(u-y) = \|u-y\|$. Applying this in (4.1) we follow

$$\Phi(u,y) + \epsilon \bar{u}^\star(u-y) \nsubseteq K.$$

Hence $y \in H_u$. Conversely let $y \in H_u$. If $y \notin \bigcap_{u^\star \in U_{X^\star}} H_{u,u^\star}$, then

$$\Phi(u,y) + \epsilon \bar{u}^\star(u-y) \subset K, \tag{4.2}$$

for some $\bar{u}^\star \in U_{X^\star}$. Since $\epsilon \in K$ thus $\epsilon\|u-y\| - \epsilon\bar{u}^\star(u-y) \in K$. Applying this in (4.2) it follows that

$$\begin{aligned} & \Phi(u,y) + \epsilon\|u-y\| \\ = \; & \Phi(u,y) + \epsilon\bar{u}^\star(u-y) + \epsilon\|u-y\| - \epsilon\bar{u}^\star(u-y) \\ \subset \; & K + K \\ = \; & K. \end{aligned}$$

In consequence $y \notin H_u$ which is absurd.

Step(4): H_u is convex for all $u \in X$. This is an easy consequence of *Step(2)* and *Step(3)*.

We now proceed to complete the final part of our proof. Toward this let $w \in \Gamma(\lambda x + \beta y)$, with λ, β as above. Hence

$$\Phi(w, \lambda x + \beta y) + \epsilon\|w - (\lambda x + \beta y)\| \subset K.$$

Consequently $\lambda x + \beta y \notin H_w$. Now assume that $w \notin \Gamma(x) \cup \Gamma(y)$. It follows that $x, y \in H_w$. On the other hand by *Step(4)* the set H_w is convex. This implies $\lambda x + \beta y \in H_w$ which contradicts the above conclusion. This completes the proof. □

Remark 4.2. Two hints about the proof of Theorem 4.1:

(1) a glimpse at the proof of Theorem 4.1 reveals that the condition (4) could be replaced by the following weaker one:
for any $u \in X$ the set
$$H_u = \{y \in X : \Phi(u, y) + \epsilon\|u - y\| \nsubseteq K\},$$
is convex. In light of this condition one may even remove the condition (3) and the conclusion of the theorem still holds.

(2) the technique used in *Step(2)* (i.e., using the duality argument) is just a trick for simplifying the proof. It increases the beauty of the proof too. Indeed, a direct proof by merely proving the convexity of the set H_u is also possible.

From Remark 4.2 we observe that it is of high importance to find the conditions that guarantee the convexity of the set H_u. The following corollaries simply state that the geometry of the cone K may give some sufficient conditions guaranteeing the convexity of H_u. The details are as follows:

Corollary 4.3. *Suppose that Y is an arbitrary Banach space. Suppose that the following assumptions hold true:*

(1) *$\Phi(x, x) \subset K$ for each $x \in X$;*
(2) *the set-valued mappings $x \mapsto \Phi(x, y)$ is inner-semicontinuous for each $y \in X$;*
(3) *for any $x \in X$, the set-valued mapping $y \mapsto \Phi(x, y)$ is convex;*
(4) *the cone K satisfies the following implication:*
$$a + b \in K \Longrightarrow a \in K \quad \text{or} \quad b \in K;$$
(5) *there exist a nonempty compact subset C and $c \in C$ such that for any $y \in X \setminus C$,*
$$\Phi(y, c) + \epsilon\|y - c\| \nsubseteq K.$$

Then the approximate set-valued equilibrium problem ($+\epsilon$-SVEP) has a solution.

Corollary 4.4. *Suppose that Y is an arbitrary Banach space. Suppose that the following assumptions hold true:*

(1) *$\Phi(x, x) \subset K$ for each $x \in X$;*
(2) *the set-valued mappings $x \mapsto \Phi(x, y)$ is inner-semicontinuous for each $y \in X$;*
(3) *for any $x \in X$, the set-valued mapping $y \mapsto \Phi(x, y)$ is convex;*
(4) *the complement of K, denoted K^c, is convex;*
(5) *there exist a nonempty compact subset C and $c \in C$ such that for any $y \in X \setminus C$,*
$$\Phi(y, c) + \epsilon\|y - c\| \nsubseteq K.$$

Then the approximate set-valued equilibrium problem ($+\epsilon$-SVEP) has a solution.

The following theorem uses weak topology and gives an existence result for the approximate set-valued equilibrium problem ($-\epsilon$-SVEP). The details are as follows:

Theorem 4.5. *Suppose that Y is an arbitrary Banach space. Suppose that the following assumptions hold true:*

(1) $\Phi(x,x) \subset K$ *for each* $x \in X$;
(2) *the set-valued mappings* $x \mapsto \Phi(x,y)$ *is weakly inner-semicontinuous for each* $y \in X$;
(3) *for all* $u \in X$, *the set* $H_u = \{y \in X : \Phi(u,y) - \epsilon\|u-y\| \nsubseteq K\}$ *is convex;*
(4) *there exist a nonempty weakly compact subset* C, $c \in C$ *and* $\bar{u}^\star \in U_{X^\star}$ *such that for any* $y \in X \setminus C$,

$$\Phi(y,c) - \epsilon\bar{u}^\star(y-c) \nsubseteq K.$$

Then the approximate set-valued equilibrium problem ($-\epsilon$-SVEP) has a solution.

Proof. Define a set-valued mapping $\Gamma : X \rightrightarrows X$ by

$$\Gamma(y) = \{x \in X : \Phi(x,y) - \epsilon\|x-y\| \subset K\}.$$

We assert that Γ satisfies the conditions of Corollary 2.3. One may easily by mimicking the proof of Theorem 4.1 check that Γ satisfies the conditions (1) and (2) of the mentioned corollary. For any $(y,y^\star) \in X \times U_{X^\star}$ define

$$\Lambda(y,y^\star) = \{x \in X : \Phi(x,y) - \epsilon y^\star(x-y) \subset K\}.$$

We claim that

$$\Gamma(y) = \bigcap_{y^\star \in U_{X^\star}} \Lambda(y,y^\star).$$

Similar to this case has been already verified we therefore do not prove this claim. This completes the proof. □

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[1]Department of Mathematics, University of Isfahan, P.O. Box 81745-163, Isfahan, Iran.

E-mail address: malek.abbasi@sci.ui.ac.ir

[2]Department of Mathematics, University of Isfahan, P.O. Box 81745-163, Isfahan, Iran.

E-mail address: mrezaie@sci.ui.ac.ir