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CONSTRUCTION OF A NEW CLASS OF QUANTUM MARKOV FIELDS

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Abstract. In the present paper, we propose a new construction of quantum Markov fields on arbitrary connected, infinite, locally finite graphs. The construction is based on a specific tessellation on the considered graph, that allows us to express the Markov property for the local structure of the graph. Our main result concerns the existence and uniqueness of quantum Markov field over such graphs.

1. Introduction

One of the basic open problem in quantum probability is to develop a theory of quantum Markov fields, which are conventionally quantum Markov processes with multi-dimensional index set. Here Quantum Markov fields are noncommutative extensions of the classical Markov fields (see [4, 8, 11]). On the other hand, these quantum fields can be considered as extensions of quantum Markov chains [1, 7] to general graphs.

In [3, 10] the first attempts to construct quantum analogues of classical Markov chains have been carried out. In [3] quantum Markov fields were considered over integer lattices, unfortunately there was not given any non trivial examples of such fields. In [5, 6], quantum Markov chains (fields) on the tree like graphs (like

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Cayley tree) have been constructed and investigated, but the proposed construction does not work for general graphs.

A main aim of the present paper is to provide a construction of new class of quantum Markov fields on arbitrary connected, infinite, locally finite graphs. The construction is based on a specific tessellation on the considered graphs, it allows us to express the Markov property for the local structure of the graph. Our main result is the existence and uniqueness of quantum Markov field over such graphs. We note that even in the classical case, the proposed construction gives a new ways to define Markov fields (see [12, 13]).

2. Graphs

Let $G = (V, E)$ be a (non-oriented simple) graph, that is, L is a nonempty set and E is identified as a subset of an ordered pairs of V, i.e.

$$E \subseteq \{\{x, y\} : x, y \in E, x \neq y\}$$

Elements of V and E are called, respectively, *vertices* and *edges*. Two vertices x and y are said to be *nearest neighbors* if there exist an edge joining them (i.e. $\{x, y\} \in E$) and we denote them by $x \sim y$. For any vertex $y \in V$ we denote its nearest neighbors by

$$N_y := \{x \in V : y \sim x\}. \tag{2.1}$$

Notice that $x \notin N_x$. The set $\{y\} \cup N_y$ is said to be *interaction domain* or *plaquette* at y. If for every $x \in V$ one has $|N_x| < \infty$ then the graph is called *locally finite*. An *edge path* or *walk* joining two vertices x and y is a finite sequence of edges $x = x_0 \sim x_1 \sim \ldots x_{d-1} \sim x_d = y$. In this case d is the length of the edge path. The graph is said to be *connected* if every two disjoint vertices can be joined by an edge path. In the sequel, we assume that the graph G is infinite, connected and locally finite. Note that in this case the set V is automatically countable. Now for any nonempty $\Lambda \subseteq V$ we associate its following parts

- *complement*:
$$\Lambda^c := V \setminus \Lambda$$
- *boundary*:
$$\partial\Lambda := \{x \in \Lambda : \exists y \in \Lambda^c; \quad x \sim y\}$$
- *interior*:
$$\overset{\circ}{\Lambda} := \Lambda \setminus \partial\Lambda$$
- *external boundary*:
$$\vec{\partial}\Lambda := \{y \in \Lambda^c : \quad \exists x \in \Lambda; \quad x \sim y\}$$
- *closure*:
$$\overline{\Lambda} := \Lambda \cup \vec{\partial}\Lambda.$$

By $\mathcal{F}$ we denote the net of all finite subsets of V, i.e.

$$\mathcal{F} := \{\Lambda \subseteq V : |V| < \infty\}, \tag{2.2}$$

where $|\cdot|$ denotes the cardinality of a set.

3. Tessellations on graphs

In this section we propose a tessellation on the considered graphs, which will play a key role in the construction. Therefore, the resulting quantum Markov field will depend also on the tessellation.

Fix a "root" $y_1 \in V$ and define by induction the following sets

$$V_{0,1} := \{y_1\}.$$

Having defined $V_{0,n}$, put

$$V_n := \bigcup_{y \in V_{0,n}} (\{y\} \cup N_y) \quad \text{and} \quad V_{0,n+1} := V_{0,n} \cup \vec{\partial} V_n.$$

Define the following set of vertices: $V_0 := \bigcup_{n \geq 1} V_{0,n}$.

From now on, elements of V_0 will be called *vertices*, any other element of V belongs to some plaquette at a certain element of V_0. Notice that in this construction, for every n, the inner boundary ∂V_n of each V_n contains no vertex:

$$\partial V_n \cap V_0 = \emptyset.$$

Since $|V| = +\infty$ and, by assumption, V is connected, one has

$$|V_{n+1}| \geq |\overline{V}_n| + 1 \geq |V_n| + 2, \qquad |V_{0,n+1}| \geq |V_{0,n}| + 1$$

It follows that, if Λ is any finite set, there exists $N \in \mathbb{N}$ such that

$$\Lambda \subseteq V_N$$

Therefore $\{V_n\}$ is an exhaustive sequence of finite subsets recovering the all the vertices set V.

One can check that

$$V_0 := \{y_1\} \cup \bigcup_{n \geq 1} \vec{\partial} V_n \tag{3.1}$$

and

$$V = \bigcup_{y \in V_0} \{y\} \cup N_y.$$

Remark 3.1. (i) For each $x \in V \setminus V_0$, there exists at least one $y \in V_0$ such that x belongs to the plaquette at y.

(ii) Each $y \in V_0$ belongs to its plaquette (i.e. the plaquette $\{y\} \cup N_y$) but no other one with center in V_0.

The set V_0 given by (3.1) (or equivalently the family $\{V_{0,n}; \quad n = 1, 2, \cdots\}$) is called *tessellation* on the graph G.

4. Quantum Markov Fields

In this section we propose a definition for backward Markov fields, for the same graph $G = (V, E)$ with the given tessellation $\{V_{0,n} : \quad n = 1, 2, \cdots\}$.

The map

$$x \in V \longrightarrow \mathcal{H}_x \text{ “ state space on } x \text{ ”}$$

defines a bundle on V whose fiber is a finite dimensional Hilbert space $\mathcal{H}_x$. Denote $\mathcal{A}_x := \mathcal{B}(\mathcal{H}_x), x \in V$. Define for any finite subset $\Lambda \subseteq V$ the algebra

$$\mathcal{A}_\Lambda := \bigotimes_{x\in\Lambda} \mathcal{A}_x.$$

then one get on a canonical way, the quasi-local algebra

$$\mathcal{A}_V := \bigotimes_{x\in V} \mathcal{A}_x$$

defined as the closure of the local algebra

$$\mathcal{A}_{V,loc} := \bigcup_{\Lambda\in\mathcal{F}} \mathcal{A}_\Lambda,$$

where $\mathcal{F}$ is given by (2.2).

Analogously, one can define for any subset $\Lambda' \subseteq V$, the algebra $\mathcal{A}_{\Lambda'} := \bigotimes_{x\in\Lambda'} \mathcal{A}_x$. Notice that for $\Lambda \subseteq \Lambda' \subseteq V$ one can see $\mathcal{A}_\Lambda$ as C^*-subalgebra of $\mathcal{A}_{\Lambda'}$ through the following embedding

$$\mathcal{A}_\Lambda \equiv \mathcal{A}_\Lambda \otimes \mathbb{1}_{\Lambda'\setminus\Lambda} \subseteq \mathcal{A}_{\Lambda'}.$$

Definition 4.1. Consider a triplet $C \subseteq B \subseteq A$ of unital C^*-algebras. Recall [2] that a *quasi-conditional expectation* with respect to the given triplet is a completely positive (CP), unital linear map $\mathcal{E} : A \to B$ such that $\mathcal{E}(ca) = c\mathcal{E}(a)$, for all $a \in A$, $c \in C$.

We give the definition of general of backward quantum Markov field, which is independent of the tessellation.

Definition 4.2. A state φ on $\mathcal{A}_V$ is said to be *backward quantum Markov field* if for any sequence $\{\Lambda_n\}_{n=0}^\infty$ of finite subsets of V satisfying $\Lambda_n \subseteq \Lambda_{n+1}$, there exists a pair $(\varphi_{\Lambda_0}, \{E_{\Lambda_n,\Lambda_{n+1}}\}_{n=0}^\infty\})$ with φ_{Λ_0} is a state on $\mathcal{A}_{\Lambda_0}$ and $E_{\Lambda_n,\Lambda_{n+1}}$ is a quasi-conditional expectation with respect to the triplet $\mathcal{A}_{\Lambda_n} \subseteq \mathcal{A}_{\bar{\Lambda}_n} \subseteq \mathcal{A}_{\Lambda_{n+1}}$ such that

$$\varphi = \lim_{n\to\infty} \varphi_{\Lambda_0} \circ E_{\Lambda_0,\Lambda_1} \circ \cdots \circ E_{\Lambda_n,\Lambda_{n+1}}, \tag{4.1}$$

where the limit is taken in the weak-*-topology.

Remark 4.3. In Definition 4.2, the condition $\Lambda_n \subseteq \Lambda_{n+1}$ for every $n \in \mathbb{N}$ implies that $\Lambda_n \uparrow V$ and the limit state obtained by the right side of the equation (4.1) is defined on the full algebra $\mathcal{A}_V$,

If φ is a *backward quantum Markov field* in the sense of Definition 4.2, then it satisfies Definition 4.2 of [5] for any increasing sequence $\{\Lambda_n\}_{n=0}^\infty$ of finite subsets of V such that $\bar{\Lambda}_n = \Lambda_{n+1}$, to get such a sequence of subsets, we consider $\Lambda_0 \in \mathcal{F}$, and for $\in n \geq 1$ put

$$\Lambda_n = \bar{\Lambda}_{n-1}.$$

Clearly one has $\Lambda_n \subseteq \Lambda_{n+1}$ and $\Lambda_n \uparrow V$.

Now we introduce a class of *backward quantum Markov field* that depends on the tessellation $\{V_{0,n}, \quad n = 1, 2, \cdots\}$

Definition 4.4. A state φ on $\mathcal{A}_V$ is said to be *backward quantum Markov field* with respect to the tessellation $\{V_{0,n}, \quad n = 1, 2, \cdots\}$, (or V_0-*backward quantum Markov field*) if for any sequence $\{\Lambda_n\}_{n=0}^{\infty}$ of finite subsets satisfying

$$\Lambda_n \subseteq \Lambda_{n+1}, \quad \vec{\partial}\Lambda \cap V_0 = \emptyset \tag{4.2}$$

there exists a pair $(\varphi_{\Lambda_0}, \{E_{\Lambda_n,\Lambda_{n+1}}\}_{n=0}^{\infty})$ with φ_{Λ_0} is a state on $\mathcal{A}_{\Lambda_0}$ and $E_{\Lambda_n,\Lambda_{n+1}}$ is a quasi-conditional expectation with respect to the triplet $\mathcal{A}_{\Lambda_n} \subseteq \mathcal{A}_{\bar{\Lambda}_n} \subseteq \mathcal{A}_{\Lambda_{n+1}}$ such that

$$\varphi = \lim_{n\to\infty} \varphi_{\Lambda_0} \circ E_{\Lambda_0,\Lambda_1} \circ \cdots \circ E_{\Lambda_n,\Lambda_{n+1}},$$

where the limit is taken in the weak-*-topology.

Now we fix the following product state

$$\varphi^0 := \bigotimes_{x\in V} \varphi_x^0$$

on the algebra $\mathcal{A}_V$, where φ_x^0 is a state on $\mathcal{A}_x$ for every $x \in V$. Denote for $\Lambda \subseteq V$,

$$\varphi_\Lambda^0 := \bigotimes_{x\in\Lambda} \varphi_x^0,$$

which is the restriction of the state φ_V^0 to $\mathcal{A}_\Lambda$.

We aim to construct a quantum Markov field on the algebra $\mathcal{A}_V$ through a perturbation of the product state φ_V^0.

5. Construction of conditional density amplitudes

It is well known from [2] that quasi-conditional expectations are more convenient than Umegaki conditional expectations (see Definition 5.1) to express the non-commutative Markov property, therefore we will perturb φ-conditional expectations (see [2]) to get quasi-conditional expectations using a commuting set of operators with the considered tessellation.

For any ordered pair $y \in V_0$ and $x \in N_y$, let be given an operator

$$\tilde{K}_{(x,y)} \in \mathcal{A}_{\{x,y\}}$$

such that it is invertible and the C^*-subalgebra

$$\mathcal{K} = \overline{\{\tilde{K}_{\{x,y\}}^*, \ \tilde{K}_{\{x,y\}} : y \in V_0, \ x \in N_y\}}^{C^*} \tag{5.1}$$

is commutative.

Definition 5.1. A *Umegaki conditional expectation* is a norm one projection from a C^*-algebra onto one of its C^*-subalgebra .

Definition 5.2. Let $\mathcal{A}_1, \mathcal{A}_2$ be two C^*-algebras with units respectively I_1 and I_2 and let $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$. An element $K \in \mathcal{A}$ is called a *conditional density amplitude* with respect to a state φ on $I_1 \otimes \mathcal{A}_2$ if

$$\mathbb{E}^{\varphi}(K^*K) = I_1,$$

where $\mathbb{E}^{\varphi}$ is the Umegaki conditional expectation from $\mathcal{A}$ onto $\mathcal{A}_1 \otimes I_2$ defined by linear extension of

$$\mathbb{E}^{\varphi}(a_1 \otimes a_2) = \varphi(I_1 \otimes a_2)a_1 \otimes I_2.$$

We also call K is a *conditional density amplitude* for the φ-conditional expectation $\mathbb{E}^{\varphi}$.

For each $x \in V$ by $\mathbb{E}^0_{\{x\}^c}$ we denote the Umegaki conditional expectation from the algebra $\mathcal{A}$ onto the algebra $\mathcal{A}_{\{x\}^c}$ defined on localized elements $a = \bigotimes_{z\in V} a_z = a_x \otimes a_{\{x\}^c}$ by

$$\mathbb{E}^0_{\{x\}^c}(a_x \otimes a_{\{x\}^c}) = \varphi^0_x(a_x)a_{\{x\}^c}.$$

One can prove the following

Lemma 5.3. *For every pair of vertices $(x, y) \in V^2$*

$$[\mathbb{E}^0_{\{x\}^c}, \mathbb{E}^0_{\{y\}^c}] = 0.$$

Lemma 5.4. *For any $\Lambda \in \mathcal{F}$*

$$\mathbb{E}^0_{\Lambda^c} := \prod_{x\in\Lambda} \mathbb{E}^0_{\{x\}^c}$$

is well defined, moreover it is a Umegaki conditional expectation from $\mathcal{A}_V$ onto $\mathcal{A}_{\Lambda^c}$ and one has for $a_\Lambda \in \mathcal{A}_\Lambda, a_{\Lambda^c} \in \mathcal{A}_{\Lambda^c}$

$$\mathbb{E}^0_{\Lambda^c}(a_\Lambda \otimes a_{\Lambda^c}) = \varphi^0_\Lambda(a_\Lambda)a_{\Lambda^c}. \tag{5.2}$$

Remark 5.5. The map $\mathbb{E}^0_{\Lambda^c}$ can be defined, through the equation (5.2), for an arbitrary part (not necessarily finite) Λ of V and it is still Umegaki conditional expectation from $\mathcal{A}_V$ onto $\mathcal{A}_{\Lambda^c}$

Proposition 5.6. *Let $y \in V_0$, the operator*

$$B_{N_y} := \mathbb{E}^0_{\{y\}^c}\left(\left|\prod_{x\in N_y} \tilde{K}_{\{x,y\}}\right|^2\right) \in \mathcal{A}_{N_y}$$

is invertible.

Proof. Consider $B_{\{y\}\cup N_y} := \left|\prod_{x\in N_y} \tilde{K}_{\{x,y\}}\right|^2 \in \mathcal{A}_{\{y\}\cup N_y}$ and denote its spectrum by $\sigma(B_{\{y\}\cup N_y})$, which is a closed subset of the complex field, while the operator $B_{\{y\}\cup N_y}$ is positive definite then $sigma(B_{ycupN_y} \subseteq]0, \|\widetilde{K}_{\{y\}\cup N_y}\|]$. Now the spectrum $\sigma(B_{\{y\}\cup N_y})$ is closed and does not contain zero then there is $\varepsilon > 0$ such that $\sigma(B_{\{y\}\cup N_y}) \subseteq [\varepsilon, \|B\|]$, therefore

$$B_{\{y\}\cup N_y} \geq \varepsilon \mathbb{1}.$$

Using the positivity and identity preservation of the map Umegaki-conditional expectation $\mathbb{E}^0_{\{y\}^c}$, one gets $\mathbb{E}^0_{\{y\}^c}(B) \geq \varepsilon \mathbb{1}$, which means $B_{N_y} = \mathbb{E}^0_{\{y\}^c}$ is invertible. □

In the sequel, we assume that for every $y \in V_0$ the operator $\mathcal{B}_{N_y}$ belongs to the commutant $\mathcal{K}'$ of the algebra $\mathcal{K}$ defined by (5.1) in the full algebra $\mathcal{A}_V$, note that under this condition the the operators $B^{pm1/2}_{N_y}$ belongs also to $\mathcal{K}'$.

Lemma 5.7. *The operator*

$$K_{\{y\}\cup N_y} := \left(\prod_{x \in N_y} \widetilde{K}_{\{x,y\}} \right) B^{-1/2}_{N_y}$$

is a $\varphi^0_{\{y\}}$-conditional density amplitude in the algebra $\mathcal{A}_{\{y\}\cup N_y}$.

Proof. Using the commutativity of the algebra $\mathcal{K}$ we get

$$\begin{aligned}
&\mathbb{E}^0_{\{y\}^c}\left(K^*_{\{y\}\cup N_y} K_{\{y\}\cup N_y}\right) \\
&\quad = \mathbb{E}^0_{\{y\}^c}\left(B^{-1/2}_{N_y} \left(\prod_{x\in N_y} \tilde{K}_{\{x,y\}} \right)^* \left(\prod_{y\in N_y} \tilde{K}_{\{x,y\}} \right) B^{-1/2}_{N_y} \right) \\
&\quad = (B^{-1/2}_{N_y})^* E^0_{\{y\}^c} \left(\prod_{x\in N_y} \tilde{K}^*_{\{x,y\}} \tilde{K}_{\{x,y\}} \right) B^{-1/2}_{N_y} \\
&\quad = (B^{-1/2}_{N_y})^* B_{N_y} B^{-1/2}_{N_y} \\
&\quad = \mathbb{1}.
\end{aligned}$$

□

Now, for each $\Lambda \in \mathcal{F}$, we define

$$\vec{\partial}_0 \Lambda := \bigcup_{y \in \partial\Lambda \cap V_0} N_y.$$

By construction the family

$$\{K^*_{\{y\}\cup N_y} \ , \ K_{\{y\}\cup N_y} \ : \ x \sim y \in V\}$$

is commutative, therefore the following operator is well defined

$$K_{\Lambda\cup\vec{\partial}_0\Lambda} := \prod_{y\in\Lambda\cap V_0} K_{\{y\}\cup N_y} \in \mathcal{A}_{\Lambda\cup\vec{\partial}_0\Lambda} \subseteq \mathcal{A}_{\bar{\Lambda}} \quad \text{for every} \quad \Lambda \in \mathcal{F}. \tag{5.3}$$

Remark 5.8.

1. In general, it is possible that $\mathcal{A}_{\Lambda\cup\vec{\partial}_0\Lambda}$ is a proper sub-algebra of $\mathcal{A}_{\bar{\Lambda}}$. Since, by the construction of the tessellation, the set $\Lambda \cup \vec{\partial}_0\Lambda$ cannot contain elements of V_0.
2. If $\Lambda \cap V_0 = \emptyset$, we convent that $K_{\Lambda\cup\vec{\partial}_0\Lambda} = \mathbb{1}$.

Theorem 5.9. *For any $\Lambda \in \mathcal{F}$, the operator $K_{\Lambda\cup\vec{\partial}_0\Lambda}$ defined by (5.3) is a conditional density amplitude for the Umegaki conditional expectation $\mathbb{E}^0_{(\Lambda\cap V_0)^c}$.*

Proof. By construction the family $\{K_{\{y\}\cup N_y}, K^*_{\{y\}\cup N_y} : y \in \Lambda\cap V_0\}$ is commutative, then one can write

$$K^*_{\Lambda\cup\vec{\partial}_0\Lambda}K_{\Lambda\cup\vec{\partial}_0\Lambda} = \prod_{y\in\Lambda\cap V_0} K^*_{\{y\}\cup N_y}K_{\{y\}\cup N_y}$$

and using the following property of the tessellation *for disjoint elements y and z of V_0 the plaquette at y does not contain z* , we conclude that $K_{\{y\}\cup N_y}$ is localized in $\{z\}^c$. Then, by Lemma 5.4, one gets

$$\mathbb{E}^0_{\{z\}^c}(K^*_{\{y\}\cup N_y}K_{\{y\}\cup N_y}) = K^*_{\{y\}\cup N_y}K_{\{y\}\cup N_y}$$

then after a small iteration, we obtain

$$\mathbb{E}^0_{(\Lambda\cap V_0)^c}\left(K^*_{\Lambda\cup\vec{\partial}_0\Lambda}K_{\Lambda\cup\vec{\partial}_0\Lambda}\right) = \prod_{y\in\Lambda\cap V_0}\mathbb{E}^0_{\{y\}^c}(K^*_{\{y\}\cup N_y}K_{\{y\}\cup N_y}).$$

By Lemma 5.7, one has $\mathbb{E}^0_{\{y\}^c}(K^*_{\{y\}\cup N_y}K_{\{y\}\cup N_y}) = \mathbb{1}$, hence we get

$$\mathbb{E}^0_{(\Lambda\cap V_0)^c}\left(K^*_{\Lambda\cup\vec{\partial}_0\Lambda}K_{\Lambda\cup\vec{\partial}_0\Lambda}\right) = \mathbb{1}$$

□

The following auxiliary results can be easily proved.

Lemma 5.10. *For every $\Lambda_1 \subseteq \Lambda_2 \subseteq V$, one has*

$$\mathbb{E}^0_{\Lambda_1}\circ\mathbb{E}^0_{\Lambda_2} = \mathbb{E}^0_{\Lambda_1}.$$

Lemma 5.11. *For finite subsets of Λ, Λ' of V with $\bar{\Lambda}\cap\Lambda' = \emptyset$, one has*

$$K_{(\Lambda\cup\Lambda')\cup\vec{\partial}(\Lambda\cup\Lambda')} = K_{\Lambda\cup\vec{\partial}_0\Lambda}K_{\Lambda'\cup\vec{\partial}_0\Lambda'}. \tag{5.4}$$

Theorem 5.12. *For $\Lambda_0 \subseteq \bar{\Lambda}_0 \subseteq \Lambda$, one has*

(i) *For $z \in V_0 \cap (\Lambda \setminus \bar{\Lambda}_0)$*

$$\mathbb{E}^0_{\{z\}^c}(K^*_{\Lambda\cup\vec{\partial}_0\Lambda}a_{\Lambda_0}K_{\Lambda\cup\vec{\partial}_0\Lambda}) = K^*_{(\Lambda\setminus\{z\})\cup\vec{\partial}_0(\Lambda\setminus\{z\})}a_{\Lambda_0}K_{(\Lambda\setminus\{z\})\cup\vec{\partial}_0(\Lambda\setminus\{z\})} \tag{5.5}$$

for every $a_{\Lambda_0} \in \mathcal{A}_{\Lambda_0}$;

(ii)

$$\mathbb{E}^0_{(\Lambda\setminus\bar{\Lambda}_0)\cap V_0}(K^*_{\Lambda\cup\vec{\partial}_0\Lambda}a_{\Lambda_0}K_{\Lambda\cup\vec{\partial}_0\Lambda}) = K^*_{\bar{\Lambda}_0\cup\vec{\partial}_0(\bar{\Lambda}_0)}a_{\Lambda_0}K_{\bar{\Lambda}_0\cup\vec{\partial}_0(\bar{\Lambda}_0)}$$

for every $a_{\Lambda_0} \in \mathcal{A}_{\Lambda_0}$;

Proof. (i) For a general Λ_0, if $z \in (\Lambda \setminus \bar{\Lambda}_0)\cap V_0$, N_z can intersect $\vec{\partial}\Lambda_0$, but not Λ_0. Therefore, $K_{\{z\}\cup N_z}$ and a_{Λ_0} are localized on disjoint parts then they commute, and while the family $\{K_{\{y\}\cup N_y}, K^*_{\{y\}\cup N_y} : y \in \Lambda \cap V_0\}$ is commutative. It follows from (5.4) that

$$\begin{aligned}
&\mathbb{E}^0_{\{z\}^c}(K^*_{\Lambda\cup\vec{\partial}_0\Lambda}a_{\Lambda_0}K_{\Lambda\cup\vec{\partial}_0\Lambda})\\
&= \mathbb{E}^0_{\{z\}^c}(\prod_{y\in\Lambda\cap V_0}K^*_{\{y\}\cup N_y}a_{\Lambda_0}\prod_{y\in\Lambda\cap V_0}K_{\{y\}\cup N_y})\\
&= \mathbb{E}^0_{\{z\}^c}\left((K^*_{\{z\}\cup N_z}K_{\{y\}\cup N_y})\times(K^*_{(\Lambda\setminus\{z\})\cup\vec{\partial}_0(\Lambda\setminus\{z\})}a_{\Lambda_0}K_{(\Lambda\setminus\{z\})\cup\vec{\partial}_0(\Lambda\setminus\{z\})})\right)\\
&= \mathbb{E}^0_{\{z\}^c}\left(K^*_{\{z\}\cup N_z}K_{\{y\}\cup N_y}\right)K^*_{(\Lambda\setminus\{z\})\cup\vec{\partial}_0(\Lambda\setminus\{z\})}a_{\Lambda_0}K_{(\Lambda\setminus\{z\})\cup\vec{\partial}_0(\Lambda\setminus\{z\})}
\end{aligned}$$

and by Lemma 5.7 one has $\mathbb{E}^0_{\{z\}^c}\left(K^*_{\{z\}\cup N_z}K_{\{y\}\cup N_y}\right)=\mathbb{1}$. Hence, we get

$$\mathbb{E}^0_{\{z\}^c}(K^*_{\Lambda\cup\vec{\partial}_0\Lambda}a_{\Lambda_0}K_{\Lambda\cup\vec{\partial}_0\Lambda})=K^*_{(\Lambda\setminus\{z\})\cup\vec{\partial}_0(\Lambda\setminus\{z\})}a_{\Lambda_0}K_{(\Lambda\setminus\{z\})\cup\vec{\partial}_0(\Lambda\setminus\{z\})}$$

(ii) Iterating the procedure of (5.5) to cover all $z\in(\Lambda\setminus\bar{\Lambda}_0)\cap V_0$ one finds

$$\begin{aligned}\mathbb{E}^0_{(\Lambda\setminus\Lambda_0)\cap V_0}\left(K^*_{\Lambda\cup\vec{\partial}_0\Lambda}a_{\Lambda_0}K_{\Lambda\cup\vec{\partial}_0\Lambda}\right) &= \left(\prod_{z\in(\Lambda\setminus\bar{\Lambda}_0)\cap V_0}\mathbb{E}^0_{\{z\}^c}\right)\left(K^*_{\Lambda\cup\vec{\partial}_0\Lambda}a_{\Lambda_0}K_{\Lambda\cup\vec{\partial}_0\Lambda}\right)\\ &= K^*_{\bar{\Lambda}_0\cup\vec{\partial}_0(\bar{\Lambda}_0)}a_{\Lambda_0}K_{\bar{\Lambda}_0\cup\vec{\partial}_0(\bar{\Lambda}_0)}.\end{aligned}$$

□

Remark 5.13. Keeping the notations of Theorem 5.12 if $\vec{\partial}\Lambda_0\cap V_0=\emptyset$ then using the same argument one gets

$$\mathbb{E}^0_{\Lambda\setminus\Lambda_0}(K^*_{\Lambda\cup\vec{\partial}_0\Lambda}a_{\Lambda_0}K_{\Lambda\cup\vec{\partial}_0\Lambda})=K^*_{\Lambda_0\cup\vec{\partial}_0\Lambda_0}a_{\Lambda_0}K_{\Lambda_0\cup\vec{\partial}_0\Lambda_0}$$

for every $a_0\in\mathcal{A}_{\Lambda_0}$.

6. Main result

In this section, we prove a main result of the paper. First we need an auxiliary result.

Proposition 6.1. *Let $\Lambda_1,\Lambda_2\in\mathcal{F}$ with $\Lambda_1\subseteq\Lambda_2$. Define*

$$E_{\Lambda_1,\Lambda_2}(a)=\mathbb{E}^0_{(\Lambda_2\setminus\bar{\Lambda}_1)^c}\left(K^*_{(\Lambda_2\setminus\bar{\Lambda}_1)\cup\vec{\partial}_0(\Lambda_2\setminus\bar{\Lambda}_1)}aK_{(\Lambda_2\setminus\bar{\Lambda}_1)\cup\vec{\partial}_0(\Lambda_2\setminus\bar{\Lambda}_1)}\right)\tag{6.1}$$

for $a\in\mathcal{A}_V$. Then E_{Λ_1,Λ_2} is a quasi-conditional expectation with respect to the following triplet $\mathcal{A}_{\Lambda_1}\subseteq\mathcal{A}_{\bar{\Lambda}_1}\subseteq\mathcal{A}_{\Lambda_2}$.

Proof. The map E_{Λ_1,Λ_2} is clearly linear and valued in $\mathcal{A}_{\bar{\Lambda}_1}$. **Unitality**: using commutativity of the family $\{\mathbb{E}_{\{z\}^c}:z\in(\Lambda_2\setminus\Lambda_1)\cap V_0\}$ (by Lemma 5.3), one can write

$$\mathbb{E}^0_{(\Lambda_2\setminus\Lambda_1)^c}=\mathbb{E}^0_{((\Lambda_2\setminus\bar{\Lambda}_1)\cap V_0^c)^c}\circ\mathbb{E}^0_{((\Lambda_2\setminus\Lambda_1)\cap V_0)^c}\tag{6.2}$$

and using Theorem 5.9 for $\Lambda=\Lambda_2\setminus\bar{\Lambda}_1$ we obtain

$$\mathbb{E}^0_{((\Lambda_2\setminus\bar{\Lambda}_1)\cap V_0)^c}\left(K^*_{(\Lambda_2\setminus\bar{\Lambda}_1)\cup\vec{\partial}_0(\Lambda_2\setminus\bar{\Lambda}_1)}K_{(\Lambda_2\setminus\bar{\Lambda}_1)\cup\vec{\partial}_0(\Lambda_2\setminus\bar{\Lambda}_1)}\right)=\mathbb{1}$$

then using (6.2) one finds

$$\mathbb{E}^0_{(\Lambda_2\setminus\bar{\Lambda}_1)^c}\left(K^*_{(\Lambda_2\setminus\bar{\Lambda}_1)\cup\vec{\partial}_0(\Lambda_2\setminus\bar{\Lambda}_1)}K_{(\Lambda_2\setminus\bar{\Lambda}_1)\cup\vec{\partial}_0(\Lambda_2\setminus\bar{\Lambda}_1)}\right)=\mathbb{E}^0_{((\Lambda_2\setminus\bar{\Lambda}_1)\cap V_0^c)^c}(\mathbb{1})=\mathbb{1}$$

hence,

$$E_{\Lambda_1,\Lambda_2}(\mathbb{1})=\mathbb{1}$$

. **Complete positivity**: One can check that for any $y\in V_0$ the map

$$a\mapsto E_{\{y\}^c}(a):=\mathbb{E}^0_{\{y\}^c}(K^*_{\{y\}\cup N_y}aK_{\{y\}\cup N_y})$$

is completely positive. Now using the commutativity of the operators $K_{\{y\}\cup N_y}, y \in V_0$ one gets

$$E_{\Lambda_1,\Lambda_2} = \mathbb{E}^0_{(\Lambda_2\setminus\bar{\Lambda}_1)\cap V_0} \circ \left(\prod_{y\in(\Lambda_2\setminus\bar{\Lambda}_1)\cap V_0} E_{\{y\}^c} \right)$$

then E_{Λ_1,Λ_2} is the composition of completely positive maps, so it is completely positive.

Let $a \in \mathcal{A}_{\Lambda_2}, c \in \mathcal{A}_{\Lambda_1}$, while $K^*_{(\Lambda_2\setminus\Lambda_1)\cup\vec{\partial}_0(\Lambda_2\setminus\Lambda_1)} \in \mathcal{A}_{\bar{(}\Lambda_2\setminus\Lambda_1)}$ then it commutes with c, then using the fact that

$$\mathbb{E}^0_{(\Lambda_2\setminus\Lambda_1)^c}(cd) = c\mathbb{E}^0_{(\Lambda_2\setminus\Lambda_1)^c}(d)$$

for every $d \in \mathcal{A}$, one gets

$$\begin{aligned} E_{\Lambda_1,\Lambda_2}(ca) &= \mathbb{E}^0_{(\Lambda_2\setminus\bar{\Lambda}_1)^c}\left(K^*_{(\Lambda_2\setminus\bar{\Lambda}_1)\cup\vec{\partial}_0(\Lambda_2\setminus\bar{\Lambda}_1)} caK_{(\Lambda_2\setminus\bar{\Lambda}_1)\cup\vec{\partial}_0(\Lambda_2\setminus\bar{\Lambda}_1)}\right) \\ &= \mathbb{E}^0_{(\Lambda_2\setminus\bar{\Lambda}_1)^c}\left(cK^*_{(\Lambda_2\setminus\bar{\Lambda}_1)\cup\vec{\partial}_0(\Lambda_2\setminus\bar{\Lambda}_1)} aK_{(\Lambda_2\setminus\bar{\Lambda}_1)\cup\vec{\partial}_0(\Lambda_2\setminus\bar{\Lambda}_1)}\right) \\ &= c\mathbb{E}^0_{(\Lambda_2\setminus\bar{\Lambda}_1)^c}\left(K^*_{(\Lambda_2\setminus\bar{\Lambda}_1)\cup\vec{\partial}_0(\Lambda_2\setminus\bar{\Lambda}_1)} aK_{(\Lambda_2\setminus\bar{\Lambda}_1)\cup\vec{\partial}_0(\Lambda_2\setminus\bar{\Lambda}_1)}\right) \\ &= cE_{\Lambda_1,\Lambda_2}(a). \end{aligned}$$

Hence, E_{Λ_1,Λ_2} is a quasi-conditional expectation with respect to the given triplet. □

Now we pass to our main result.

Theorem 6.2. *For each $\Lambda \in \mathcal{F}$ define the state $\widetilde{\varphi}_\Lambda$ on $\mathcal{A}$ by*

$$\widetilde{\varphi}_\Lambda(a) := \varphi^0(K^*_{\Lambda\cup\vec{\partial}_0\Lambda} aK_{\Lambda\cup\vec{\partial}_0\Lambda}).$$

Then the net $\{\widetilde{\varphi}_\Lambda\}_{\Lambda\in\mathcal{F}}$ converges in the weak--topology, moreover the limiting state φ is a backward Markov field on $\mathcal{A}_V$ with respect to the tessellation V_0.*

Proof. First we prove the existence of the limit. Due to the density argument, it is sufficient to establish the existence of the limit in the local algebra $\mathcal{A}_{V,loc}$.

Let $a \in \mathcal{A}_{V,loc}$ then $a \in \mathcal{A}_{\Lambda_0}$ for some $\Lambda_0 \in \mathcal{F}$. For $\Lambda \in \mathcal{F}$ with $\Lambda_0 \subseteq \Lambda$, we have

$$\widetilde{\varphi}_\Lambda(a) = \varphi^0\left(K^*_{\Lambda\cup\vec{\partial}_0\Lambda} aK_{\Lambda\cup\vec{\partial}_0\Lambda}\right) = \varphi^0 \circ \mathbb{E}_{(\Lambda\setminus\bar{\Lambda}_0)^c}\left(K^*_{\Lambda\cup\vec{\partial}_0\Lambda} aK_{\Lambda\cup\vec{\partial}_0\Lambda}\right)$$

and by Theorem 5.12 one gets

$$\mathbb{E}_{(\Lambda\setminus\bar{\Lambda}_0)^c}\left(K^*_{\Lambda\cup\vec{\partial}_0\Lambda} aK_{\Lambda\cup\vec{\partial}_0\Lambda}\right) = K^*_{\bar{\Lambda}_0\cup\vec{\partial}_0\bar{\Lambda}_0} aK_{\bar{\Lambda}_0\cup\vec{\partial}_0\bar{\Lambda}_0},$$

so

$$\widetilde{\varphi}_\Lambda(a) = \varphi^0(K^*_{\bar{\Lambda}_0\cup\vec{\partial}_0\bar{\Lambda}_0} aK_{\bar{\Lambda}_0\cup\vec{\partial}_0\bar{\Lambda}_0}) = \widetilde{\varphi}_{\bar{\Lambda}_0}(a).$$

As $\Lambda \to V$, we find that $\Lambda_0 \subseteq \Lambda$ up to some order, hence the net $\{\widetilde{\varphi}(a)\}_{\Lambda\in\mathcal{F};\Lambda_0\subseteq\Lambda}$ is stationary. This means that

$$\lim_{\Lambda\to V;\Lambda_0\subseteq\Lambda} \widetilde{\varphi}_\Lambda(a) = \varphi_{\bar{\Lambda}_0}(a) =: \varphi(a).$$

Therefore the limit exist on the local algebra, and yet it exists on the full algebra $\mathcal{A}_V$.

Now we establish that the state φ is a quantum Markov field.

Let $\{\Lambda_n \mid n \in \mathbb{N}\}_{n\in\mathbb{N}}$ be a family of subset of $\mathcal{F}$ satisfying

$$\Lambda_n \subseteq \Lambda_{n+1}, \quad \vec{\partial}\Lambda_n \cap V_0 = \emptyset$$

Let $E_{\Lambda_n,\Lambda_n+1}$ be given by (6.1). Then, for $a \in \mathcal{A}_{V_n}$, we have

$$\begin{aligned}
&\widetilde{\varphi}_{\Lambda_n} \circ E_{\Lambda_n,\Lambda_{n+1}}(a)\\
&= \varphi^0 \left(K^*_{\Lambda_n\cup\vec{\partial}_0\Lambda_n} E_{\Lambda_n,\Lambda_{n+1}}(a) K_{\Lambda_n\cup\vec{\partial}_0\Lambda_n} \right)\\
&= \varphi^0 \left(K^*_{\Lambda_n\cup\vec{\partial}_0\Lambda_n} \mathbb{E}^0_{(\Lambda_{n+1}\setminus\bar{\Lambda}_n)^c} \left(K^*_{(\Lambda_{n+1}\setminus\bar{\Lambda}_n)\cup\vec{\partial}_0(\Lambda_{n+1}\setminus\bar{\Lambda}_n)} a K_{(\Lambda_{n+1}\setminus\bar{\Lambda}_n)\cup\vec{\partial}_0(\Lambda_{n+1}\setminus\bar{\Lambda}_n)} \right) K_{\Lambda_n\cup\vec{\partial}_0\Lambda_n} \right)
\end{aligned}$$

Since $K_{\Lambda_n\cup\vec{\partial}_0\Lambda_n} \in \mathcal{A}_{\bar{\Lambda}_n} \subseteq \mathcal{A}_{(\Lambda_{n+1}\setminus\bar{\Lambda}_n)^c}$ and $\mathbb{E}^0_{(\Lambda_{n+1}\setminus\bar{\Lambda}_n)^c}$ is a Umegaki conditional expectation from $\mathcal{A}_V$ onto $\mathcal{A}_{(\Lambda_{n+1}\setminus\bar{\Lambda}_n)^c}$ then one finds

$$\begin{aligned}
&\widetilde{\varphi}_{\Lambda_n} \circ E_{\Lambda_n,\Lambda_{n+1}}(a)\\
&= \varphi^0 \mathbb{E}^0_{(\Lambda_{n+1}\setminus\bar{\Lambda}_n)^c} \left(K^*_{\Lambda_n\cup\vec{\partial}_0\Lambda_n} K^*_{(\Lambda_{n+1}\setminus\bar{\Lambda}_n)\cup\vec{\partial}_0(\Lambda_{n+1}\setminus\bar{\Lambda}_n)} a K_{(\Lambda_{n+1}\setminus\bar{\Lambda}_n)\cup\vec{\partial}_0(\Lambda_{n+1}\setminus\bar{\Lambda}_n)} K_{\Lambda_n\cup\vec{\partial}_0\Lambda_n} \right)
\end{aligned}$$

and by the assumption (4.2) one has

$$\vec{\partial}\Lambda_n \cap V_0 = \emptyset$$

then $\bar{\Lambda}_n \cap V_0 = \Lambda_n \cap V_0$ and

$$K_{\Lambda_n\cup\vec{\partial}_0\Lambda_n} = \prod_{y\in\Lambda_n\cap V_0} K_{\{y\}\cup N_y} = \prod_{y\in\bar{\Lambda}_n\cap V_0} K_{\{y\}\cup N_y} = K_{\bar{\Lambda}_n\cup\vec{\partial}_0\bar{\Lambda}_n}.$$

From Lemma 5.4 it follows that

$$K_{\Lambda_{n+1}\cup\vec{\partial}_0\Lambda_{n+1}} = K_{\bar{\Lambda}_n\cup\bar{\Lambda}_1} K_{(\Lambda_{n+1}\setminus\bar{\Lambda}_n)\cup\vec{\partial}_0(\Lambda_{n+1}\setminus\bar{\Lambda}_n)}$$

then we obtain

$$\widetilde{\varphi}_{\Lambda_n} \circ E_{\Lambda_n,\Lambda_{n+1}}(a) = \varphi^0 \circ \mathbb{E}^0_{(\Lambda_{n+1}\setminus\bar{\Lambda}_n)^c}(K^*_{\Lambda_{n+1}\cup\vec{\partial}_0\Lambda_{n+1}} a K_{\Lambda_{n+1}\cup\vec{\partial}_0\Lambda_{n+1}})$$

Hence, by construction one gets

$$\varphi^0_V = \varphi^0_V \circ \mathbb{E}_{(\Lambda_{n+1}\setminus\bar{\Lambda}_n)^c}$$

so

$$\widetilde{\varphi}_{\Lambda_n} \circ E_{\Lambda_n,\Lambda_{n+1}}(a) = \varphi^0_V(K^*_{\Lambda_{n+1}\cup\vec{\partial}_0\Lambda_{n+1}} a K_{\Lambda_{n+1}\cup\vec{\partial}_0\Lambda_{n+1}}) = \widetilde{\varphi}_{\Lambda_{n+1}}(a). \tag{6.3}$$

Now iterating the equation (6.3), we obtain

$$\widetilde{\varphi}_n = \widetilde{\varphi}_{\Lambda_0} \circ E_{\Lambda_0,\Lambda_1} \circ \cdots \circ E_{\Lambda_{n-1},\Lambda_n}.$$

Therefore

$$\varphi_V = \lim \varphi_{\Lambda_0} \circ E_{\Lambda_0,\Lambda_1} \circ \cdots \circ E_{\Lambda_{n-1},\Lambda_n},$$

where $\varphi_{\Lambda_0} = \widetilde{\varphi}_{\Lambda_0}\lceil_{\mathcal{A}_{\Lambda_0}}$. □

The provided construction allows us to produce a lot of interesting examples of quantum Markov fields on arbitrary connected, infinite, locally finite graphs. Note that the construction is based on a specific tessellation on the considered graph, it allows us to express the Markov property for the local structure of the graph. We note that even in the classical case, the proposed construction gives other ways to define Markov fields different to the existing ones (see [12]). This construction opens new perspectives in the theory of phase transitions in the scheme of quantum Markov fields (comp. [9]).

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