NORMALIZED TIGHT VS. GENERAL FRAMES IN SAMPLING PROBLEMS

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ABSTRACT. We present a new approach to sampling theory using the operator theory framework. We use a replacement operator and replace general frames of the sampling and reconstruction subspaces by normalized tight frames. The replacement can be done in a numerically stable and efficient way. The approach enables us to unify the standard consistent reconstruction results with the results for quasiconsistent reconstruction. Our approach naturally generalizes to consistent and quasiconsistent reconstructions from several samples. Not only we can handle sampling problems in a more efficient way, we also answer questions that seem to be open so far.

1. INTRODUCTION

There is a large variety of applications of sampling theory that include signal and image processing, communication engineering, information theory, and many others. The central idea of this theory is to recover a continuous-time function from a discrete set of samples. The most often cited historical beginnings go back to Cauchy [7], Whittaker [29], Kotel'nikov [21], and Shannon [25]. In this paper we tackle the problem from the Hilbert space point of view which has been gaining more and more attention recently. A comprehensive introduction into the

approach can be found in, say, Eldar [10]. For a survey of sampling theory see e. g. [27].

There are papers that consider some special approaches such as Eldar and Dvorkind [11], where the authors focus on generalized sampling, and Antezana and Corach [3], where the authors consider the problem of best error estimates for general type sampling. However, most of the authors are working in the consistent sampling setting using frames, sampling subspace $\mathcal{S}$ and reconstruction subspace $\mathcal{W}$. One of the most important features of the approach, the concept of consistent sampling is introduced in Unser and Aldroubi [28]. The authors of that paper present an applications point of view and consider only the specific setup and applications. Eldar [10] generalizes the approach to general Hilbert space setup. She works in finite dimensional Hilbert space and assumes that

$$\text{(E)} \qquad\qquad\qquad \mathcal{S}^{\perp} \cap \mathcal{W} = 0.$$

With Werther [12] they generalize the results of [10] to infinite dimensional case still using Condition (E); while with Dvorkind [9] they work in finite dimensions but drop the Condition (E). Hirabayashi and Unser [19] drop Condition (E) but retain finite dimensionality. They introduce certain other assumptions for reconstruction that are motivated by applications. Corach and Giribet [8] work in general setup without either of the two above mentioned conditions. They use the tools of operator theory and focus on the inner product. To make the projectors orthogonal they replace it with a suitable semi-definite inner product. It seems that the problem of reconstruction for more than one sample was first studied in Arias and Conde [4].

In the Hilbert space framework the samples of the original signal $f \in \mathcal{H}$, where $\mathcal{H}$ is a separable Hilbert space over the real numbers, are seen as the inner products of f with a set of sampling vectors, which span the sampling subspace $\mathcal{S}$. Next, a reconstruction of f, $\tilde{f}$, is obtained as a linear combination of a set of reconstruction vectors that span the reconstruction subspace $\mathcal{W}$. One usually assumes that the coefficients of such a reconstruction are obtained by a bounded linear transformation of the samples called a filter. Observe that the classical approach to sampling theory can be seen as a special case of the Hilbert space framework [23].

The authors that are using Hilbert space approach are using projections of various kinds and pseudoinverse theory, especially the pseudoinverses of the Moore-Penrose type, as a main tool to get the reconstruction out of a sample and to solve other problems. Our main goal is to introduce some other Hilbert space techniques that seem more natural. We introduce an operator that we are calling a replacement operator which exchanges general frames of the sampling space and reconstruction space with normalized tight frames, the procedure called symmetric approximation by Frank, Paulsen, and Tiballi [14]. Using this operator we are able to apply the theory of relations and operations on orthogonal projections as studied in Hladnik and Omladič [20] and in Omladič [24]. One of the main contributions of our techniques is that we are able to use theoretically the same procedure to get an optimal consistent reconstruction (in the case that there are

more than one such solutions to the problem) and to get a quasiconsistent reconstruction (in the case that the problem has no solution). Also, we are able to handle easier some other problems of the sampling theory and answer some questions on simultaneous reconstruction from more than one sample that seem to be open so far.

The sampling vectors are assumed to form a frame $\mathcal{F}_\mathcal{S} = \{s_k\}_{k\in\Gamma}$ of the sampling space $\mathcal{S}$, a (closed) subspace of the Hilbert space $\mathcal{H}$. The index set Γ is either finite or countably infinite depending on the cardinality of the dimension of $\mathcal{S}$. The sample of the original signal is either considered to be a member f of $\mathcal{H}$ or the set of inner products $\{\langle f, s_k\rangle\}_{k\in\Gamma}$, which is an element of the Hilbert space $l^2 = l^2(\Gamma)$. The mapping between the two presentations of a sample is called the *synthesis operator* $S \in L(l^2, \mathcal{H})$. It is defined by $S^*f = \{\langle f, s_k\rangle\}_{k\in\Gamma}$.

Similarly, we consider the reconstruction subspace $\mathcal{W}$ and its frame $\mathcal{F}_\mathcal{W} = \{w_k\}_{k\in\Gamma'}$ with synthesis operator $W \in L(l^2, \mathcal{H})$. The reconstruction of f is of the form $\tilde{f} = \sum_{k\in\Gamma'} c_k w_k$, where $\{c_k\}_{k\in\Gamma'}$ is in l^2. We call a reconstruction $\tilde{f}$ *consistent* if $S^*\tilde{f} = S^*f$ and we call $\tilde{f}$ a *quasiconsistent reconstruction* if $\|S^*\tilde{f} - S^*f\| = \min\|S^*g - S^*f\|$, where g runs through all reconstructions.

Our main results hold under the assumption that the sum of the sampling and the reconstruction subspaces $\mathcal{S} + \mathcal{W}$ is a closed subspace of $\mathcal{H}$. This is equivalent to invertibility of certain operator and enables us to give a consistent or a quasiconsistent reconstruction explicitly in terms of the samples. Observe that this condition is always satisfied in the case of finite-dimensional $\mathcal{S}$ and $\mathcal{W}$.

In Section 2 we present our new approach using normalized tight frames via the replacement operator. The main relations between orthogonal projections that are used in the sequel are also introduced. In Section 3 we present the solution of reconstruction problem using the new approach. Simultaneous reconstructions of more than one sample are considered in Section 4.

2. Preliminaries

2.1. **Frames.** Let $\mathcal{S}$ be a (closed) linear subspace of the Hilbert space $\mathcal{H}$. A sequence of vectors $\{s_k\}_{k\in\Gamma}$, where Γ is an either finite or countably infinite index set, is called a *frame* for $\mathcal{S}$ if there exist constants C_L and C_U such that

$$C_L \|x\|^2 \leqslant \sum_{k\in\Gamma} |\langle x, s_k\rangle|^2 \leqslant C_U \|x\|^2, \tag{2.1}$$

for all $x \in \mathcal{S}$. The fact that $\mathcal{S}$ is closed follows from Condition (2.1) by well-known results in Hilbert space theory even in the case that Γ is infinite. The frame is called *tight* if $C_L = C_U$ and it is called *normalized tight* if $C_L = C_U = 1$. Observe that an orthonormal basis is a special case of a normalized tight frame due to the Parseval's identity. Two frames $\{s_k\}_{k\in\Gamma}$ and $\{t_k\}_{k\in\Gamma}$ of closed subspaces $\mathcal{S}$ and $\mathcal{T}$, respectively, are called *weakly similar* if there exists an invertible bounded operator $A : \mathcal{S} \to \mathcal{T}$ such that $As_k = t_k$ for all k. They are called *similar* if they are weakly similar and $\mathcal{S} = \mathcal{T}$.

There are several natural ways to associate a normalized tight frame to a given frame. A set of possibilities is given by generalizations of different orthonormalization procedures. The Löwdin orthogonalization for bases [1] was generalized to frames by Frank, Paulsen and Tiballi in [14] (see also [26]). The Löwdin orthogonalization has better properties for numerical computation as opposed to the better known Gram-Schmidt orthogonalization. In [14], the authors consider

$$\inf \sum_{k \in \Gamma} \|t_k - s_k\|^2, \tag{2.2}$$

where the infimum is taken over all normalized tight frames $\{t_k\}_{k \in \Gamma}$ weakly similar to a given frame. If S is the synthesis operator of the frame $\mathcal{F} = \{s_k\}_{k \in \Gamma}$ then we denote by P the orthogonal projection of l^2 onto the range of S^*S. Furthermore, suppose that $S = U|S|$ is the polar decomposition of S and that $\{e_k\}_{k \in \Gamma}$ is the standard orthonormal basis of l^2. Then the infimum in (2.2) exists if and only if the operator $P - |S|$ is a Hilbert-Schmidt operator. This is always the case if Γ is finite. The infimum, when it exists, is obtained for the normalized tight frame $\widetilde{\mathcal{F}} = \{Ue_k\}_{k \in \Gamma}$, which is similar to $\mathcal{F}$. The frame $\widetilde{\mathcal{F}}$ called a *symmetric approximation* of $\mathcal{F}$. See [14, Thm. 2.3] for all of these.

Another normalized tight frame associated to a given frame is the so-called *associated tight frame* given by $\widehat{\mathcal{F}} = \{T^{-\frac{1}{2}}s_k\}_{k \in \Gamma}$, where $T : \mathcal{S} \to \mathcal{S}$ is given by $T = SS^*|_{\mathcal{S}}$. It is defined in [2, (2.12)] and its properties are discussed in [5, p. 2354].

Observe that both constructions of normalized tight frames described above can be performed numerically by standard algorithms for matrix functions and polar decomposition (see e. g. [17, 18]).

We assume that a (general) frame $\{s_k\}_{k \in \Gamma}$ and a normalized tight frame $\{\sigma_k\}_{k \in \Gamma}$ are similar for the subspace $\mathcal{S}$. The normalized tight frame may be obtained from the general one via procedures described above or otherwise. Then the synthesis operator $S \in L(l^2, \mathcal{H})$ is given by $S^*x = \{\langle x, s_k \rangle\}_{k \in \Gamma}$ and the *normalized synthesis operator* $\Sigma \in L(l^2, \mathcal{H})$ by $\Sigma^*x = \{\langle x, \sigma_k \rangle\}_{k \in \Gamma}$. We introduce an operator $R \in L(l^2, l^2) =: L(l^2)$ by $R : \{\langle x, s_k \rangle\}_{k \in \Gamma} \mapsto \{\langle x, \sigma_k \rangle\}_{k \in \Gamma}$. Since S and Σ have the same range and the same kernel it follows by [13, Cor. 1] that R exists, is bounded and boundedly invertible. Moreover, Condition (2.1) applied to $x \in \mathcal{S}$ implies that

$$\|R\|^2 \leqslant C_L^{-1} \quad \text{and} \quad \|R^{-1}\|^2 \leqslant C_U.$$

Furthermore, we have that

$$\Sigma^* = RS^*$$

by definition, so that the role of operator R is to "replace" the starting frame by a normalized tight frame. This is why we will call it the *replacement operator*.

One of the advantages of using a normalized tight frame is the fact that the orthogonal projection with range $\mathcal{S}$ is a natural operator associated to the frame. The following result follows easily from [16, Prop. 1.1] or [23, pp. 211-212]. There it is proved that each normalized tight frame is a dilation of an orthonormal basis

and that the reconstruction formula

$$x = \sum_{k\in\Gamma} \langle x, \sigma_k\rangle \sigma_k$$

holds for each $x \in \mathcal{S}$.

Proposition 2.1. *Suppose that $\{\sigma_k\}_{k\in\Gamma}$ is a normalized tight frame of $\mathcal{S}$. Then, the orthogonal projection with range $\mathcal{S}$ is equal to the product $\Sigma\Sigma^*$.*

Proof: By the reconstruction formula above we have

$$x = \sum_{k\in\Gamma} \langle x, \sigma_k\rangle \sigma_k = \Sigma\Sigma^* x$$

for $x \in \mathcal{S}$ and

$$0 = \sum_{k\in\Gamma} \langle x, \sigma_k\rangle \sigma_k = \Sigma\Sigma^* x$$

for $x \in \mathcal{S}^\perp$.   □

2.2. **Projections.** Let $\mathcal{S}$ and $\mathcal{T}$ be two closed linear subspaces of the Hilbert space $\mathcal{H}$. If the latter is a direct sum of them, we write $\mathcal{H} = \mathcal{S} \oplus \mathcal{T}$. The (oblique) projection on $\mathcal{S}$ along $\mathcal{T}$ will be denoted by $P_{\mathcal{S},\mathcal{T}}$. In case that $\mathcal{T} = \mathcal{S}^\perp$, where $\mathcal{S}^\perp$ is the orthogonal complement of $\mathcal{S}$, we call this projection the (orthogonal) projection on $\mathcal{S}$ and denote it simply by $P_{\mathcal{S}}$.

Suppose that $\{\sigma_k\}_{k\in\Gamma}$ is a normalized tight frame for $\mathcal{S}$. By Proposition 2.1 the action of the orthogonal projection $P_{\mathcal{S}}$ on an arbitrary $x \in \mathcal{H}$ is given by the product $\Sigma\Sigma^*$, i. e.,

$$P_{\mathcal{S}} x = \sum_{k\in\Gamma} \langle x, \sigma_k\rangle \sigma_k,$$

where the sum on the right hand side, if infinite, converges in the Hilbert space norm.

We now present a slight modification of known results that simplifies proofs of the results to follow. Let P and Q be any orthogonal projections. Furthermore, denote by $\mathcal{C}(P,Q)$ the "commuting subspace" of the two projections $\mathcal{C}(P,Q) = \{x \in \mathcal{H}; PQx = QPx\}$ and its orthogonal complement by $\mathcal{NC}(P,Q) = \mathcal{C}(P,Q)^\perp$. Observe that $\mathcal{C}(P,Q)$ is invariant for both P and Q, and consequently the same holds for $\mathcal{NC}(P,Q)$. The restrictions of P to subspaces $\mathcal{C}(P,Q)$ respectively $\mathcal{NC}(P,Q)$ will be denoted by P_1 respectively P_2. Similarly, restrictions of Q to these subspaces will be denoted by Q_1 respectively Q_2. The proof of the following proposition follows easily from Halmos' Two Projections Theorem [15]. See also [20, 24]. For details and proofs, we refer the reader to [6, Sect. 1 and Ex. 3.2].

Proposition 2.2.   (i) *Projections P_1 and Q_1 commute on $\mathcal{H}_1 = \mathcal{C}(P,Q)$.*
   (ii) *On $\mathcal{H}_1$ we have that:*
        − $\mathcal{R}(P_1) + \mathcal{R}(Q_1) = \mathcal{H}_1$ *if and only if* $(I - P_1)(I - Q_1) = 0$
        − $\mathcal{R}(P_1) \cap \mathcal{R}(Q_1) = 0$ *if and only if* $P_1 Q_1 = 0$
   (iii) *On $\mathcal{H}_2 = \mathcal{NC}(P,Q)$ it always holds that*
        − $\mathcal{R}(P_2) + \mathcal{R}(Q_2) = \mathcal{H}_2,$
        − $\mathcal{R}(P_2) \cap \mathcal{R}(Q_2) = 0.$

(iv) *There exists a decomposition of $\mathcal{H}_2$ and a hermitian operator T, which has trivial kernel and its spectrum is contained in the interval $\left[0, \frac{\pi}{2}\right]$, such that P_2 and Q_2 are of the form*

$$P_2 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \quad and \quad Q_2 = \begin{pmatrix} \cos^2 T & \cos T \sin T \\ \cos T \sin T & \sin^2 T \end{pmatrix} \tag{2.3}$$

Operator T is invertible if and only if $\mathcal{R}(P) + \mathcal{R}(Q)$ is a closed subspace of $\mathcal{H}$.

3. Normalized tight frames

In this section we assume that the frames of both the sampling subspace $\mathcal{S}$ and the reconstruction subspace $\mathcal{W}$ are normalized tight frames. So, $\mathcal{F}_\mathcal{S} = \{s_k\}_{k \in \Gamma}$ is a normalized tight frame of $\mathcal{S}$ and $\mathcal{F}_\mathcal{W} = \{w_k\}_{k \in \Gamma'}$ is a normalized tight frame of $\mathcal{W}$. Now, if this were not so, we could replace the starting frames by normalized tight ones using, say, the procedure of symmetric approximation or the associated tight frame as described in Subsection 2.1. There, we also point out which of these ways are more important from theoretical point of view and which ones are better in numerical applications. Theoretically we might think of using the proposed replacement operator on a general frame to get a normalized tight one. And this may and will be assumed throughout the paper.

Observe that adjoints of the synthesis operators $S^* \in L(\mathcal{H}, l^2)$ and $W^* \in L(\mathcal{H}, l^2)$, as well as, the original synthesis operators $S \in L(l^2, \mathcal{H})$ and $W \in L(l^2, \mathcal{H})$ are all partial isometries under our assumptions. Therefore, the (necessarily orthogonal) projections on the corresponding spaces are given by $P_\mathcal{S} = SS^* \in L(\mathcal{H}, \mathcal{H})$ and $P_\mathcal{W} = WW^* \in L(\mathcal{H}, \mathcal{H})$. In this case, a consistent reconstruction exists always if and only if to any $f \in \mathcal{H}$ we can find an $\tilde{f} \in \mathcal{W}$ such that $S^*(\tilde{f} - f) = 0$ or equivalently $\tilde{f} - f \in \mathcal{N}(P_\mathcal{S})$. So, we have:

Lemma 3.1. *In the case of normalized tight frames we have that:*

(i) *A consistent reconstruction $\tilde{f} \in \mathcal{W}$ exists for every original signal $f \in \mathcal{H}$ if and only if*

$$\mathcal{H} = \mathcal{W} + \mathcal{N}(P_\mathcal{S}). \tag{3.1}$$

Suppose that a consistent reconstruction exists. Then the following assertions hold:

(ii) *A consistent reconstruction is unique up to addition of a member of $\mathcal{W} \cap \mathcal{N}(P_\mathcal{S})$, i.e., if $\tilde{f}$ and $\hat{f}$ are consistent reconstructions then $\tilde{f} - \hat{f} \in \mathcal{W} \cap \mathcal{N}(P_\mathcal{S})$.*

(iii) *A consistent reconstruction is unique if and only if*

$$\mathcal{W} \cap \mathcal{N}(P_\mathcal{S}) = 0, \tag{3.2}$$

i.e., a consistent reconstruction is unique if and only if the sum in (3.1) is direct.

Proof: The existence part of the lemma was shown above. If $\tilde{f}, \hat{f}$ are two consistent reconstructions, then $\tilde{f} - \hat{f}$ belongs to both $\mathcal{W}$ and $\mathcal{N}(P_\mathcal{S})$ and consequently to the intersection in (3.2) thus finishing the proof of the lemma. $\qquad \square$

Note that the assertions of Lemma 3.1 can be generalized to the frames that are not necessarily normalized tight. Since we work only with normalized tight frames we do not include more general statements.

In case that Condition (3.2) holds, the solution $\tilde{f}$ is unique. If not, we want to find $\tilde{f}$ such that

$$\|S^*(f - \tilde{f})\| = \min \|S^*(f - g)\|, \tag{3.3}$$

when g runs through all reconstructions. A reconstruction satisfying Condition (3.3) is called a *quasiconsistent reconstruction*.

Note that every consistent reconstruction is also quasiconsistent. In the case that (3.2) fails, there can be more than one consistent reconstruction and we would like to find a consistent reconstruction with minimal norm. Similarly, quasiconsistent reconstruction may not be unique. We call a consistent reconstruction with minimal norm the *optimal consistent reconstruction* and we call the quasiconsistent reconstruction with minimal norm among all quasiconsistent reconstructions the *optimal quasiconsistent reconstruction*. In either of the cases we denote it by $\tilde{f}_{opt}$.

Since S is a partial isometry it follows that $\|P_{\mathcal{S}} f\| = \|S^* f\|$. So, the value of $\|S^*(f - g)\|$ in (3.3) is minimal if and only if the value of $\|P_{\mathcal{S}}(f - g)\|$ is minimal. Hence, our definition of a quasiconsistent reconstruction is equivalent to the one used in [4].

Our approach yields both types of reconstructions in formally the same way. In the following Lemma we denote $P = P_{\mathcal{W}}$ and $Q = I - P_{\mathcal{S}}$. We use the notation and apply the results of Proposition 2.2.

In the rest of the section we assume that the sum $\mathcal{S} + \mathcal{W}$ is a closed subspace of $\mathcal{H}$. This assumption is equivalent to the fact that the operator T in (2.3) is invertible. This is a classical result in operator theory first stated by Krein, Krasnoselski and Milman [22]. We refer to [6, Ex. 3.2] for details and proof.

Lemma 3.2. *For $f = f_1 + f_2$, $\tilde{f}_{opt} = \tilde{f}_1 + \tilde{f}_2$ it holds that*

    (i) $\tilde{f}_1 = P_1(I - Q_1)f_1$ *and*
    (ii) $\tilde{f}_2 = (I - P_2 Q_2)^{-1} P_2 (I - Q_2) f_2 = P_2 f_2 + \cot T (I - P_2) f_2$.

Proof: Since the direct sum $\mathcal{H}_1 \oplus \mathcal{H}_2$ is orthogonal it follows that (3.3) with g running over $\mathcal{W}$ holds for $\tilde{f}_{opt}$ if and only if

$$\|f_1 - \tilde{f}_1\| = \min \|f_1 - g_1\| \quad \text{and} \quad \|f_2 - \tilde{f}_2\| = \min \|f_2 - g_2\|,$$

where g_1 runs over $\mathcal{R}(P_1) = \mathcal{H}_1 \cap \mathcal{W}$ and g_2 over $\mathcal{R}(P_2) = \mathcal{H}_2 \cap \mathcal{W}$. Since P_1 and Q_1 commute it follows easily that $\tilde{f}_1 = P_1(I - Q_1)f_1$. By applying *(iv)* to Q and $I - Q$, in place of Q, it follows that under our closedness assumption each block of Q_2 in (2.3) is invertible. This implies that $\tilde{f}_2$ is given by $\tilde{f}_2 = (I - P_2 Q_2)^{-1} P_2 (I - Q_2) f_2 = P_2 f_2 + \cot T (I - P_2) f_2$. $\square$

Corollary 3.3. *There exists a consistent reconstruction $\tilde{f} \in \mathcal{W}$ for every original signal $f \in \mathcal{H}$ if and only if $\mathcal{H}_1 = \mathcal{R}(P_1) + \mathcal{R}(Q_1)$.*

4. Reconstructions for two or more samples for normalized tight frames

Suppose we are given two synthesis operators $S_i \in L(l^2, \mathcal{H})$, $i = 1, 2$, corresponding to normalized tight frames $\mathcal{F}_{\mathcal{S}_i} = \{s_n^i\}_{n \in \Gamma}$ of sampling spaces $\mathcal{S}_i$, defined by $S_i^* f = \{\langle f, s_n^i \rangle\}_{n \in \Gamma}$. We are interested in *simultaneous consistent or quasiconsistent reconstructions*, i.e., we are searching for a reconstruction $\tilde{f} \in \mathcal{W}$ such that the value of

$$\left\| P_{\mathcal{S}_1}(f - \tilde{f}) \right\|^2 + \left\| P_{\mathcal{S}_2}(f - \tilde{f}) \right\|^2 \tag{4.1}$$

is minimal possible. If the value of (4.1) is 0 then $\tilde{f}$ is called a *consistent reconstruction*. If the minimal value of (4.1) is positive then $\tilde{f}$ with the minimal value of (4.1) is called a *quasiconsistent reconstruction*.

Since S_i, $i = 1, 2$, are partial isometries it follows that $\|P_{\mathcal{S}_i} f\| = \|S_i^* f\|$. Then similarly, as in the reconstruction from one sample, the value of (4.1) is minimal if and only if the value of

$$\left\| S_1^*(f - \tilde{f}) \right\|^2 + \left\| S_2^*(f - \tilde{f}) \right\|^2$$

is minimal. Thus, our definition of a quasiconsistent reconstruction is equivalent to the one used in [4].

We denote by $\mathcal{N}$ the intersection $\mathcal{N}(P_{\mathcal{S}_1}) \cap \mathcal{N}(P_{\mathcal{S}_2})$ and by $\mathcal{W}_1$ the intersection $\mathcal{N} \cap \mathcal{W}$. Since the projector $P_{\mathcal{W}}$ is orthogonal it follows that there is a subspace $\mathcal{W}_2 \subset \mathcal{W}$ such that $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$ is an orthogonal direct sum and $\mathcal{W}_2$ is $P_{\mathcal{W}}$ invariant. Similarly, $\mathcal{W}_1$ and its orthogonal complement $\mathcal{W}_1^{\perp} \subset \mathcal{H}$ are $P_{\mathcal{S}_i}$, $i = 1, 2$, invariant.

With respect to the orthogonal direct sum decomposition $\mathcal{H} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}^{\perp}$ we write

$$P_{\mathcal{W}} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ P_{\mathcal{S}_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & P_{11} & P_{12} \\ 0 & P_{12}^* & P_{13} \end{pmatrix} \text{ and } P_{\mathcal{S}_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & P_{21} & P_{22} \\ 0 & P_{22}^* & P_{23} \end{pmatrix}. \tag{4.2}$$

Recall that a simultaneous consistent reconstruction is possible for every $f \in \mathcal{H}$ if and only if $\mathcal{H} = \mathcal{W} + \mathcal{N}$, and it is unique if and only if $\mathcal{H} = \mathcal{W} \oplus \mathcal{N}$ is a direct sum (cf. [4, Thm. 4.1 and Prop. 4.3]).

To obtain an explicit expression for the quasiconsistent reconstruction in terms of the sample, we assume that the sums $\mathcal{S}_1 + \mathcal{W}$ and $\mathcal{S}_2 + \mathcal{W}$ are closed subspaces in $\mathcal{H}$.

From all these it can be easily proved that:

Lemma 4.1. *The image of $P_{\mathcal{W}}(P_{\mathcal{S}_1} + P_{\mathcal{S}_2})$ is a subspace of $\mathcal{W}_2$. The restriction $A = P_{\mathcal{W}}(P_{\mathcal{S}_1} + P_{\mathcal{S}_2}) P_{\mathcal{W}}$ to $\mathcal{W}$ is positive semi-definite and $\langle Aw, w \rangle > 0$ for $w \in \mathcal{W}_2$, $w \neq 0$. In particular, the restriction*

$$A_2 : \mathcal{W}_2 \to \mathcal{W}_2$$

of A is invertible.

Proof: Since $P_{\mathcal{S}_1}$ and $P_{\mathcal{S}_2}$ are orthogonal it follows that their ranges are contained in the subspace $\mathcal{W}_1^\perp$. This subspace is invariant for both projections and thus also for their sum. Thus, we have that

$$P_{\mathcal{W}}(P_{\mathcal{S}_1} + P_{\mathcal{S}_2})\mathcal{W}_1^\perp \subset P_{\mathcal{W}}(\mathcal{W}_1^\perp) = \mathcal{W}_2.$$

For $w \in \mathcal{W}$ we have a decomposition $w = w_1 + w_2$ with respect to the orthogonal sum $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$. Then

$$\langle Aw, w \rangle = \langle (P_{\mathcal{S}_1} + P_{\mathcal{S}_2})w, w \rangle = \|P_{\mathcal{S}_1}w_2\|^2 + \|P_{\mathcal{S}_2}w_2\|^2 \geq 0.$$

Therefore, if $w = w_2$ is nonzero then $0 < \langle Aw, w \rangle >= \langle A_2 w_2, w_2 \rangle >$ and A_2 is positive definite. In particular, it is invertible. Observe that $\mathcal{W}_2 = (\mathcal{W}_2 \cap \mathcal{S}_1) \oplus (\mathcal{W}_2 \cap \mathcal{S}_1^\perp) \oplus \mathcal{W}_3$ is an orthogonal direct sum for some subspace $\mathcal{W}_3$ of $\mathcal{W}_2$. With respect to this sum the operator P_{11} is of the form

$$P_{11} = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \cos^2 T_1 \end{pmatrix},$$

where T_1 is the operator given in (2.3). Similarly, we can also decompose (with respect to a different decomposition) the operator P_{21} as

$$P_{21} = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \cos^2 T_2 \end{pmatrix}$$

for some operator T_2. Now, the special form of P_{11} and P_{21} and the fact that the spectra of T_1 and T_2 are both bounded away from 0 imply that the sum $A_2 = P_{11} + P_{21}$ is invertible. $\qquad\square$

Theorem 4.2. *Reconstruction $\tilde{f}$ is simultaneous consistent or quasiconsistent if and only if*

$$P_{\mathcal{W}}(P_{\mathcal{S}_1} + P_{\mathcal{S}_2})P_{\mathcal{W}}\tilde{f} = P_{\mathcal{W}}(P_{\mathcal{S}_1} + P_{\mathcal{S}_2})f. \tag{4.3}$$

Moreover,

$$\tilde{f}_{opt} = T_2^{-1}P_{\mathcal{W}}(P_{\mathcal{S}_1} + P_{\mathcal{S}_2})f$$

is the simultaneous consistent or quasiconsistent reconstruction with minimal norm. Writing $\varphi_i = S_i^* f$, $i = 1, 2$, for the original samples we have

$$\tilde{f}_{opt} = A_2^{-1}P_{\mathcal{W}}(S_1\varphi_1 + S_2\varphi_2). \tag{4.4}$$

Proof: For an element $\tilde{f} \in \mathcal{W}$ we write

$$\varphi_f(\tilde{f}) = \left\| P_{\mathcal{S}_1}(f - \tilde{f}) \right\|^2 + \left\| P_{\mathcal{S}_2}(f - \tilde{f}) \right\|^2.$$

With respect to the orthogonal direct sum $\mathcal{H} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}^\perp$ we write $f = f_1 + f_2 + f_3$ and $\tilde{f} = \tilde{f}_1 + \tilde{f}_2$. Using decompositions (4.2) we obtain after a straightforward computation that $\varphi_f(\tilde{f})$ is equal to

$$\langle (P_{11} + P_{21})(f_2 - \tilde{f}_2), f_2 - \tilde{f}_2 \rangle + 2\langle (P_{12} + P_{22})^*(f_2 - \tilde{f}_2), f_3 \rangle + \langle (P_{13} + P_{23})f_3, f_3 \rangle. \tag{4.5}$$

Recall from Lemma 4.1 that the operator $A_2 = P_{11} + P_{21}$ is invertible. Note that

$$\tilde{f}_{opt} = f_2 + A_2^{-1}(P_{12} + P_{22})f_3 \tag{4.6}$$

is an element of $\mathcal{W}_2$. Then a direct calculation shows that

$$\varphi_f(\tilde{f}_{opt}) = \langle U_2 f_3, f_3 \rangle, \tag{4.7}$$

where $U_2 = P_{13} + P_{23} - (P_{12} + P_{22})^* A_2^{-1}(P_{12} + P_{22})$.

Now, we choose an element $\tilde{f} \in \mathcal{W}$ and we want to show that

$$\varphi_f(\tilde{f}) - \varphi_f(\tilde{f}_{opt}) \geq 0. \tag{4.8}$$

Suppose that V_2 is a positive square root of A_2, i.e., $V_2^2 = A_2$. Then we use the expressions (4.5) and (4.7) to show that the difference in (4.8) is equal to

$$\langle A_2(f_2 - \tilde{f}_2), f_2 - \tilde{f}_2 \rangle + 2\langle A_2(f_2 - \tilde{f}_2), A_2^{-1}(P_{12} + P_{22})f_3 \rangle$$

$$+ \langle A_2^{-1}(P_{12} + P_{22})f_3, (P_{12} + P_{22})f_3 \rangle$$

$$= \left\| V_2(f_2 - \tilde{f}_2) + V_2^{-1}(P_{12} + P_{22})f_3 \right\|^2, \tag{4.9}$$

which is obviously nonnegative. Thus, $\tilde{f}_{opt}$ is a simultaneous quasiconsistent or consistent reconstruction and the minimal value of $\varphi_f(\tilde{f})$ is given by (4.7).

Suppose next that we are given $\tilde{f} = \tilde{f}_1 + \tilde{f}_2 \in \mathcal{W}_1 \oplus \mathcal{W}_2$. It is a simultaneous quasiconsistent or consistent reconstruction if and only if $\varphi_f(\tilde{f}) - \varphi_f(\tilde{f}_{opt}) = 0$. The latter condition holds if and only if the expression in (4.9) is equal to 0. Finally, this is equivalent to $\tilde{f}_2 = \tilde{f}_{opt}$ by (4.6) and consequently to Condition (4.3).

Since the direct sum $\mathcal{W}_1 \oplus \mathcal{W}_2$ is orthogonal it also follows that the norm of $\tilde{f}_{opt}$ is minimal among all of the norms of simultaneous consistent or quasiconsistent reconstructions. $\square$

Note that in the case of normalized tight frames Expression (4.4) gives an answer to the problem posed in [4, p. 748], which asks for an expression for a simultaneous consistent reconstruction in terms of the original samples φ_1 and φ_2.

We can generalize Theorem 4.2 to the case of more than two samples. Suppose that we are given k samples, $k \geq 3$. We write $S_i \in L(l^2, \mathcal{H})$, $i = 1, 2, \ldots, k$, for the synthesis operators corresponding to normalized tight frames $\mathcal{F}_{\mathcal{S}_i} = \{s_n^i\}_{n \in \Gamma}$ of sampling space $\mathcal{S}_i$. We are interested in *simultaneous consistent or quasiconsistent reconstructions*, i.e., we are searching for a reconstruction $\tilde{f} \in \mathcal{W}$ such that the value of $\sum_{i=1}^k \left\| P_{\mathcal{S}_i}(f - \tilde{f}) \right\|^2$ is minimal possible.

To obtain an explicit formula for quasiconsistent reconstruction in terms of samples we assume that the sums $\mathcal{S}_i + \mathcal{W}$ are closed subspaces of $\mathcal{H}$.

We denote by $\mathcal{N}$ the intersection $\bigcap_{i=1}^k \mathcal{N}(P_{\mathcal{S}_i})$ and by $\mathcal{W}_1$ the intersection $\mathcal{N} \cap \mathcal{W}$. We write $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$ for the orthogonal direct sum.

To prove the following theorem we use arguments very similar to those in the proof of Theorem 4.2. We replace the operator T by $T = P_{\mathcal{W}}\left(\sum_{i=1}^k P_{\mathcal{S}_i}\right)P_{\mathcal{W}}$. To simplify the notation we write $Q = \sum_{i=1}^k P_{\mathcal{S}_i}$.

Theorem 4.3. *A reconstruction $\tilde{f}$ is simultaneous consistent or quasiconsistent for k samples if and only if*

$$P_{\mathcal{W}} Q P_{\mathcal{W}} \tilde{f} = P_{\mathcal{W}} Q f.$$

Moreover,

$$\tilde{f}_{opt} = A_2^{-1} P_{\mathcal{W}} Q f$$

is a simultaneous consistent or quasiconsistent reconstruction with minimal norm. Writing $\varphi_i = S_i^* f$, $i = 1, 2, \ldots, k$, for the original samples, we have that

$$\tilde{f}_{opt} = A_2^{-1} P_{\mathcal{W}} \left(\sum_{i=1}^{k} S_i \varphi_i \right).$$

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