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kTH-ORDER SLANT TOEPLITZ OPERATORS ON THE FOCK SPACE

SHIVAM KUMAR SINGH [1*] and ANURADHA GUPTA [2]

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ABSTRACT. The notion of slant Toeplitz operators B_ϕ and kth-order slant Toeplitz operators B_ϕ^k on the Fock space is introduced and some of its properties are investigated. The Berezin transform of slant Toeplitz operator B_ϕ is also obtained. In addition, the commutativity of kth-order slant Toeplitz operators with co-analytic and harmonic symbols is discussed.

1. INTRODUCTION

O. Toeplitz [16] in 1911 introduced the notion of Toeplitz operator T_ϕ for bounded measurable function ϕ with applications in prediction theory, wavelet analysis and differential equations. Later on Toeplitz operators on Hardy spaces have been studied extensively. In particular authors like Brown, Halmos [4] and Douglas [5] have done the remarkable study of this operators. Then, in the year 1995, Ho [8, 9, 10, 11], introduced slant Toeplitz operator having the property that its matrix with respect to standard orthonormal basis could be obtained by eliminating every alternate row of the matrix of the corresponding Toeplitz operator. After the introduction of class of slant Toeplitz operators, the study has gained voluminous importance due to its multidirectional applications as these class of operators have played major roles in wavelet analysis, dynamical system and in curve and surface modelling ([6, 7, 13, 14, 17]). Many mathematicians (for e.g. [1, 2, 3]) generalized the notion of slant Toeplitz operators to different spaces such as Hardy spaces, Bergman space and studied its properties. Motivated by

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*Corresponding author.
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the work of these researchers, here we introduce slant Toeplitz operators on the Fock space and study some of its algebraic properties.

Let the space $L^2(\mathbb{C}, d\mu)$ denote the Hilbert space of all Lebesgue measurable square integrable functions f on $\mathbb{C}$ with the norm

$$\|f\| = \left(\int_{\mathbb{C}} |f(z)|^2 d\mu(z)\right)^{1/2}$$

where the measure $d\mu(z) = e^{-|z|^2} dA(z)$ and dA denotes the Lebesgue area measure on complex plane. We denote the Fock space by $\mathbb{F}^2$ which consists of all entire functions in $L^2(\mathbb{C}, d\mu)$ and is a closed subspace of $L^2(\mathbb{C}, d\mu)$. The space $\mathbb{F}^2$ is a Hilbert space with the inner product inherited from $L^2(\mathbb{C}, d\mu)$ as

$$\langle f, g\rangle = \int_{\mathbb{C}} f(z)\overline{g(z)} d\mu(z) \quad \text{where } f \text{ and } g \text{ are in } \mathbb{F}^2. \tag{1.1}$$

The set of all polynomials in the complex variable z is denoted by $\mathbb{P}[z]$ which is contained in the Fock space and moreover it is dense. For $n \geq 0$, let $e_n(z) = \frac{z^n}{\sqrt{\pi n!}}$, then the set $\{e_n\}_{n\geq 0}$ forms the orthonormal basis for $\mathbb{F}^2$ (see [18]). Let $P : L^2(\mathbb{C}, d\mu) \longrightarrow \mathbb{F}^2$ be the orthogonal projection. Then for $f \in L^2(\mathbb{C}, d\mu)$, we have

$$\begin{aligned} P(f(z)) &= \langle Pf(w), K_z(w)\rangle = \langle f(w), P(K_z(w))\rangle \\ &= \langle f(w), K_z(w)\rangle = \frac{1}{\pi}\int_{\mathbb{C}} f(w)e^{z\bar{w}} d\mu(w), \end{aligned}$$

where $K_z(w) = \overline{K(z, w)} = \frac{1}{\pi}e^{w\bar{z}}$ is the Fock kernel and the normalized reproducing kernel is given by

$$k_z(w) = \frac{K_z(w)}{\|K_z\|} = \frac{K(w, z)}{\sqrt{K(z, z)}} = \frac{1}{\sqrt{\pi}} e^{w\bar{z} - \frac{|z|^2}{2}}.$$

Let $L^\infty(\mathbb{C})$ denote the set of all essentially bounded measurable functions in the entire complex plane, then for $\phi \in L^\infty(\mathbb{C})$, the multiplication operator on $L^2(\mathbb{C}, d\mu)$ is defined by $M_\phi(f) = \phi \cdot f$ for all $f \in L^2(\mathbb{C}, d\mu)$ and the Toeplitz operator T_ϕ on $\mathbb{F}^2$ is defined by $T_\phi(f) = P(\phi \cdot f)$ for all $f \in \mathbb{F}^2$.

In this paper, we have introduced the notion of slant Toeplitz operators B_ϕ and the kth-order slant Toeplitz operators B_ϕ^k on the Fock space and have studied its properties. In particular, we have given the explicit expression for Berezin transform of slant Toeplitz operator B_ϕ and also obtained the conditions for boundedness, compactness of B_ϕ. In addition, we have shown that the necessary and sufficient conditions for the commutativity of kth-order slant Toeplitz operators on the Fock space are that their symbols functions must be linearly dependent.

2. kTH-ORDER SLANT TOEPLITZ OPERATORS ON THE FOCK SPACE

Lemma 2.1. *For non-negative integers s and t, we have*

$$\langle z^s, z^t\rangle = \begin{cases} \pi s! & \text{if } s = t \\ 0 & \text{otherwise} \end{cases}.$$

Proof. Indeed for $s = t$, using the equation (1.1) and the measure $d\mu(z)$, we get

$$\begin{aligned}\langle z^s, z^t\rangle &= \int_{\mathbb{C}} z^s \bar{z}^t e^{-|z|^2} dA(z) = \int_{r=0}^{\infty}\int_{\theta=0}^{2\pi} r^{s+t+1} e^{i(s-t)\theta} e^{-r^2} d\theta dr \\ &= 2\pi \int_{r=0}^{\infty} r^{2s+1} e^{-r^2} dr = \pi \int_{l=0}^{\infty} l^s e^{-l} dl = \pi\Gamma(s+1) = \pi s!\end{aligned}$$

Then $\langle z^s, z^t\rangle = \begin{cases} \pi s! & \text{if } s = t \\ 0 & \text{otherwise} \end{cases}$. ∎

Now in the next proposition we give the $(m,n)^{th}$ entry of the matrix of T_ϕ with respect to orthonormal basis $\{e_n\}_{n\geq 0}$ on the Fock space.

Proposition 2.2. *For the harmonic symbol $\phi(z) = \sum_{i=0}^{\infty} a_i z^i + \sum_{j=1}^{\infty} b_j \bar{z}^j$, the $(m,n)^{th}$ entry of matrix of T_ϕ with respect to orthonormal basis $\{e_n\}_{n\geq 0}$ of $\mathbb{F}^2$ is given by*

$$\langle T_\phi e_n, e_m\rangle = \begin{cases} \sqrt{\dfrac{m!}{n!}} a_{m-n} & \text{for } \; m \geq n \\ \sqrt{\dfrac{n!}{m!}} b_{n-m} & \text{for } \; n > m \end{cases}$$

where m and n are non-negative integers.

Proof. Here the $(m,n)^{th}$ entry of matrix of T_ϕ with respect to orthonormal basis $\{e_n\}_{n\geq 0}$ of $\mathbb{F}^2$ is given by

$$\begin{aligned}\langle T_\phi e_n, e_m\rangle &= \frac{1}{\sqrt{\pi n!}}\frac{1}{\sqrt{\pi m!}} \langle T_\phi z^n, z^m\rangle \\ &= \frac{1}{\pi\sqrt{n!}}\frac{1}{\sqrt{m!}} \left\langle P\left(\sum_{i=0}^{\infty} a_i z^{i+n} + \sum_{j=1}^{\infty} b_j \bar{z}^j z^n\right), z^m \right\rangle \\ &= \frac{1}{\pi\sqrt{n!}}\frac{1}{\sqrt{m!}} \left(\left\langle \sum_{i=0}^{\infty} a_i z^{i+n}, z^m\right\rangle + \left\langle \sum_{j=1}^{\infty} b_j \bar{z}^j z^n, z^m\right\rangle\right) \\ &= \frac{1}{\pi\sqrt{n!}}\frac{1}{\sqrt{m!}} \left(\sum_{i=0}^{\infty} a_i \left\langle z^{i+n}, z^m\right\rangle + \sum_{j=1}^{\infty} b_j \left\langle z^n, z^{j+m}\right\rangle\right). \qquad (2.1)\end{aligned}$$

For $m \geq n$, by Lemma 2.1 and by equation (2.1), it follows that

$$\begin{aligned}\langle T_\phi e_n, e_m\rangle &= \frac{1}{\sqrt{\pi n!}}\frac{1}{\sqrt{\pi m!}} \sum_{i=0}^{\infty} a_i \left\langle z^{i+n}, z^m\right\rangle \\ &= \frac{1}{\sqrt{\pi n!}}\frac{1}{\sqrt{\pi m!}} a_{m-n}\pi m! = \sqrt{\frac{m!}{n!}} a_{m-n}.\end{aligned}$$

For $n > m$, by Lemma 2.1 and by equation (2.1), it follows that

$$\langle T_\phi e_n, e_m\rangle = \frac{1}{\sqrt{\pi n!}}\frac{1}{\sqrt{\pi m!}}\sum_{j=1}^{\infty} b_j \left\langle z^n, z^{j+m}\right\rangle$$

$$= \frac{1}{\sqrt{\pi n!}}\frac{1}{\sqrt{\pi m!}} b_{n-m}\pi(n)! = \sqrt{\frac{n!}{m!}} b_{n-m}.$$

Thus $\langle T_\phi e_n, e_m\rangle = \begin{cases} \sqrt{\frac{m!}{n!}} a_{m-n} & \text{for} \quad m \geq n \\ \sqrt{\frac{n!}{m!}} b_{n-m} & \text{for} \quad n > m \end{cases}$

where m and n are non-negative integers. □

Hence the matrix of T_ϕ explicitly is given by

$$T_\phi = \begin{pmatrix} a_0 & b_1 & \sqrt{2}b_2 & \sqrt{6}b_3 & \dots \\ a_1 & a_0 & \sqrt{2}b_1 & \sqrt{6}b_2 & \dots \\ \sqrt{2}a_2 & \sqrt{2}a_1 & a_0 & \sqrt{3}b_1 & \dots \\ \sqrt{6}a_3 & \sqrt{6}a_2 & \sqrt{3}a_1 & a_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

and the adjoint of the matrix of T_ϕ is given by

$$T_\phi^\star = \begin{pmatrix} \bar{a}_0 & \bar{a}_1 & \sqrt{2}\bar{a}_2 & \sqrt{6}\bar{a}_3 & \dots \\ \bar{b}_1 & \bar{a}_0 & \sqrt{2}\bar{a}_1 & \sqrt{6}\bar{a}_2 & \dots \\ \sqrt{2}\bar{b}_2 & \sqrt{2}\bar{b}_1 & \bar{a}_0 & \sqrt{3}\bar{a}_1 & \dots \\ \sqrt{6}\bar{b}_3 & \sqrt{6}\bar{b}_2 & \sqrt{3}\bar{b}_1 & \bar{a}_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

which is the matrix of $T_{\hat{\phi}}$ where $\hat{\phi}(z) = \sum_{i=0}^{\infty} \bar{a}_i \bar{z}^i + \sum_{j=1}^{\infty} \bar{b}_j z^j$ which is nothing but $\overline{\phi}(z)$. So we get that $T_\phi^\star = T_{\overline{\phi}}$.

Assume that k is an integer and $k \geq 2$ and now consider an operator W_k on $\mathbb{F}^2$ given by

$$W_k(z^n) = \begin{cases} z^{\frac{n}{k}}, & \text{if } n \text{ is divisible by } k \\ 0 & \text{otherwise} \end{cases}.$$

Note here we denote the operator W_2 by W.

Proposition 2.3. *The operator W_k is a bounded linear operator on $\mathbb{F}^2$ with norm 1.*

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be any arbitrary polynomial, then

$$\|W_k f\|_2^2 = \left\langle \sum_{m=0}^{\infty} a_{km} z^m, \sum_{m=0}^{\infty} a_{km} z^m \right\rangle = \sum_{m=0}^{\infty} |a_{km}|^2 \pi m!$$

$$\leq \pi \sum_{m=0}^{\infty} |a_m|^2 m! = \|f\|_2^2$$

and since the set of all polynomials is dense in $\mathbb{F}^2$, so it follows that $\|W_k\|_2 \leq 1$. Also $\|W_k\|_2 \geq \left\|W_k\left(\frac{1}{\sqrt{\pi}}\right)\right\|_2 = 1$. Hence $\|W_k\|_2 = 1$. $\square$

Proposition 2.4. *For the operator W_k, its adjoint $W_k^\star(z^n) = \frac{n!}{(kn)!} z^{kn}$ for $n = 0, 1, 2, 3 \cdots$ and also $\|W_k^\star f\|_2 \leq \|f\|_2$, for all $f \in \mathbb{F}^2$.*

Proof. Since the set of polynomials is dense in $\mathbb{F}^2$, therefore for any polynomial $f(z) = \sum_{m=0}^{\infty} a_m z^m \in \mathbb{F}^2$, we have

$$\begin{aligned}
\langle W_k^\star z^n, f(z)\rangle &= \left\langle z^n, W_k\left(\sum_{m=0}^{\infty} a_m z^m\right)\right\rangle = \left\langle z^n, \sum_{m=0}^{\infty} a_{km} z^m\right\rangle \\
&= \langle z^n, a_{kn} z^n\rangle = \overline{a_{kn}} \pi n! = \overline{a_{kn}} \frac{\pi n!}{\pi (kn)!} \left\langle z^{kn}, z^{kn}\right\rangle \\
&= \left\langle \frac{n!}{(kn)!} z^{kn}, f(z)\right\rangle.
\end{aligned}$$

This implies that $W_k^\star(z^n) = \frac{n!}{(kn)!} z^{kn}$. Since

$$\begin{aligned}
\|W_k^\star f\|_2^2 = \|W_k^\star \sum_{m=0}^{\infty} a_m z^m\|_2^2 &= \|\sum_{m=0}^{\infty} a_m \frac{m!}{(km)!} z^{km}\|_2^2 \\
&= \sum_{m=0}^{\infty} |a_m|^2 \left(\frac{m!}{(km)!}\right)^2 \left\langle z^{km}, z^{km}\right\rangle = \sum_{m=0}^{\infty} |a_m|^2 (\frac{m!}{(km)!})^2 \pi (km)! \\
&= \pi \sum_{m=0}^{\infty} |a_m|^2 \frac{m!^2}{(km)!} \leq \pi \sum_{m=0}^{\infty} |a_m|^2 m! = \|f\|_2^2,
\end{aligned}$$

We have $\|W_k^\star f\|_2 \leq \|f\|_2$ for all $f \in \mathbb{F}^2$. $\square$

Now we define slant Toeplitz operator and k^{th}-order slant Toeplitz operator on the Fock space.

Definition 2.5. For $\phi \in L^\infty$, we define the *slant Toeplitz operator* on the space $\mathbb{F}^2$ as an operator

$$B_\phi : \mathbb{F}^2 \longrightarrow \mathbb{F}^2 \quad \text{given by} \quad B_\phi(f) = W T_\phi(f) \quad \forall f \in \mathbb{F}^2.$$

Definition 2.6. A k^{th}*-order slant Toeplitz operator* B_ϕ^k for $k \geq 2$ induced by a function $\phi \in L^\infty$ on the space $\mathbb{F}^2$ is defined as

$$B_\phi^k = W_k T_\phi.$$

It is clear that B_ϕ^k is bounded linear operator on $\mathbb{F}^2$ for essentially bounded measurable function ϕ over $\mathbb{C}$ and for $k = 2$, the k^{th}-order slant Toeplitz operator is simply the slant Toeplitz operator B_ϕ.
Following proposition follows easily from the definition of k^{th}-order slant Toeplitz operator B_ϕ^k.

Proposition 2.7. *Let $\phi_1, \phi_2 \in L^\infty(\mathbb{C})$ and λ_1, λ_2 be complex numbers. Then*

(1) $B_{\lambda_1\phi_1 + \lambda_2\phi_2}^k = \lambda_1 B_{\phi_1}^k + \lambda_2 B_{\phi_2}^k$.

(2) $\|B^k_{\phi_1}\| \leq \|\phi_1\|_\infty$.

Proposition 2.8. *For $\phi(z) = \sum_{i=0}^{\infty} a_i z^i + \sum_{j=1}^{\infty} b_j \bar{z}^j$, the $(m,n)^{th}$ entry of the matrix of B^k_ϕ with respect to orthonormal basis $\{e_n\}_{n\geq 0}$ of $\mathbb{F}^2$ is given by*

$$\langle B^k_\phi e_n, e_m\rangle = \begin{cases} \sqrt{\frac{m!}{n!}} a_{km-n} & \text{for } km \geq n \\ \frac{\sqrt{(m)!(n)!}}{(km)!} b_{n-km} & \text{for } n > km \end{cases}$$

where m and n are non-negative integers.

Proof. Here the $(m,n)^{th}$ entry of B^k_ϕ with respect to orthonormal basis $\{e_n\}_{n\geq 0}$ of $\mathbb{F}^2$ is

$$\begin{aligned} \langle B^k_\phi e_n, e_m\rangle &= \left\langle W_k T_\phi \frac{z^n}{\sqrt{\pi n!}}, \frac{z^m}{\sqrt{\pi m!}} \right\rangle \\ &= \frac{1}{\pi\sqrt{n!m!}} \langle P(\phi\cdot z^n), W_k^\star z^m\rangle \\ &= \frac{1}{\pi\sqrt{n!m!}} \left\langle \sum_{i=0}^{\infty} a_i z^{i+n} + \sum_{j=1}^{\infty} b_j \bar{z}^j z^n, \frac{m!}{(km)!} z^{km} \right\rangle \\ &= \frac{1}{\pi\sqrt{n!m!}} \left(\sum_{i=0}^{\infty} a_i \left\langle z^{i+n}, \frac{m!}{(km)!} z^{km} \right\rangle + \sum_{j=1}^{\infty} b_j \left\langle \bar{z}^j z^n, \frac{m!}{(km)!} z^{km} \right\rangle \right) \\ &= \frac{1}{\pi(km)!} \sqrt{\frac{m!}{n!}} \left(\sum_{i=0}^{\infty} a_i \left\langle z^{i+n}, z^{km} \right\rangle + \sum_{j=1}^{\infty} b_j \left\langle z^n, z^{km+j} \right\rangle \right). \end{aligned} \tag{2.2}$$

For $km \geq n$, by Lemma 2.1 and by equation (2.2), it follows that

$$\begin{aligned} \langle B^k_\phi e_n, e_m\rangle &= \frac{1}{\pi(km)!} \sqrt{\frac{m!}{n!}} \sum_{i=0}^{\infty} a_i \left\langle z^{i+n}, z^{km} \right\rangle \\ &= \frac{1}{\pi} \frac{1}{(km)!} \sqrt{\frac{m!}{n!}} a_{km-n} \pi (km)! = \sqrt{\frac{m!}{n!}} a_{km-n}. \end{aligned}$$

For $n > km$, by Lemma 2.1 and by equation (2.2), it follows that

$$\begin{aligned} \langle B^k_\phi e_n, e_m\rangle &= \frac{1}{\pi(km)!} \sqrt{\frac{m!}{n!}} \sum_{j=1}^{\infty} b_j \left\langle z^n, z^{km+j} \right\rangle \\ &= \frac{1}{\pi(km)!} \sqrt{\frac{m!}{n!}} b_{n-km} \pi (n)! = \frac{1}{(km)!} \sqrt{m!n!} b_{n-km}. \end{aligned}$$

Thus,

$$\langle B^k_\phi e_n, e_m\rangle = \begin{cases} \sqrt{\frac{m!}{n!}} a_{km-n} & \text{for } km \geq n \\ \frac{\sqrt{(m)!(n)!}}{(km)!} b_{n-km} & \text{for } n > km \end{cases}$$

where m and n are non-negative integers. ∎

Hence explicit form of the matrix of B_ϕ^k is given by

$$\begin{pmatrix} a_0 & b_1 & \sqrt{2}b_2 & \dots \\ a_k & a_{k-1} & \frac{1}{\sqrt{2}}a_{k-2} & \dots \\ \sqrt{2}a_{2k} & \sqrt{2}a_{2k-1} & a_{2k-2} & \dots \\ \sqrt{6}a_{3k} & \sqrt{6}a_{3k-1} & \sqrt{3}a_{3k-2} & \dots \\ \vdots & \vdots & \vdots & \dots \end{pmatrix}$$

For $k = 2$ we get the matrix of slant Toeplitz operator B_ϕ, which is given by

$$\begin{pmatrix} a_0 & b_1 & \sqrt{2}b_2 & \dots \\ a_2 & a_1 & \frac{1}{\sqrt{2}}a_0 & \dots \\ \sqrt{2}a_4 & \sqrt{2}a_3 & a_2 & \dots \\ \sqrt{6}a_6 & \sqrt{6}a_5 & \sqrt{3}a_4 & \dots \\ \vdots & \vdots & \vdots & \dots \end{pmatrix}$$

Following proposition follows from the matrix representation of B_ϕ^k.

Proposition 2.9. *Let $\phi \in L^\infty$ be analytic function then $B_\phi^k = 0$ if and only if $\phi = 0$ almost everywhere.*

Remark 2.10. For an analytic function $\phi \in L^\infty$ the correspondence $\phi \longrightarrow B_\phi^k$ is one-one.

It is evident that if T_ϕ is bounded on $\mathbb{F}^2$ then B_ϕ is always bounded. However reverse statement need not be true which is shown in the following example.

Example 2.11. B_z and $B_{\bar{z}}$ are bounded linear operators on $\mathbb{F}^2$ while T_z and $T_{\bar{z}}$ are not bounded on $\mathbb{F}^2$.

For any non-negtive integer p, using the lemma 2.1 we have

$$\begin{aligned} \|B_z \cdot z^p\|^2 &= \|WT_z \cdot z^p\|^2 = \|Wz^{p+1}\|^2 \\ &= \begin{cases} \|z^{\frac{p+1}{2}}\|^2, & \text{if } p \text{ is odd} \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} \dfrac{(\frac{p+1}{2})!}{p!}\|z^p\|^2, & \text{if } p \text{ is odd} \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

which shows B_z is a bounded linear operator.

Now for $p \in \mathbb{N}$, consider

$$\begin{aligned}
\|B_{\bar{z}} \cdot z^p\|^2 &= \|WT_{\bar{z}} \cdot z^p\|^2 = \|WP(\bar{z} \cdot z^p\|^2 \\
&= \left\| W \frac{p!}{(p-1)!} z^{p-1} \right\|^2 = \begin{cases} p^2 \|z^{\frac{p-1}{2}}\|^2, & \text{if } p \text{ is odd} \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} p \dfrac{(\frac{p-1}{2})!}{(p-1)!} \|z^p\|^2, & \text{if } p \text{ is odd} \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

which implies that $B_{\bar{z}}$ is bounded. Now consider

$$\|T_z z^p\|^2 = \left\langle z^{p+1}, z^{p+1} \right\rangle = \pi(p+1)! = (p+1) \left\langle z^p, z^p \right\rangle = (p+1)\|z^p\|^2.$$

This means $\|T_z z^p\| = \sqrt{p+1}\|z^p\|$, thus T_z is not bounded. Similarly we have $\|T_{\bar{z}} z^p\| = \sqrt{p}\|z^p\|$ and hence $T_{\bar{z}}$ is also not bounded.

Now we conclude this section by giving a result about the point spectra of kth-order slant Toeplitz operators.

Theorem 2.12. *Let ϕ be an invertible, analytic function in L^∞, then $\sigma_p(B_\phi^k) = \sigma_p(B_{\phi(z^k)}^k)$, where $\sigma_p(B_\phi^k)$ denotes the point spectrum of B_ϕ^k.*

Proof. Suppose that $\lambda \in \sigma_p(B_\phi^k)$. Then there exists a non-zero function f in $\mathbb{F}^2$ such that $B_\phi^k f = \lambda f$. Let $M = \phi f$, then by using the definition of W_k it follows that

$$\begin{aligned}
B_\phi^k(z^k)M &= W_k T_{\phi(z^k)} M \\
&= W_k P[\phi(z^k)(\phi f)] = W_k[\phi(z^k)(\phi f)] \\
&= \phi(z) W_k(\phi f) = \phi(z) \cdot B_\phi^k(f) \\
&= \phi(z) \cdot \lambda f = \lambda \phi f \\
&= \lambda M.
\end{aligned}$$

Since ϕ is invertible and f is non-zero, so $M \neq 0$ and therefore $\lambda \in \sigma_p(B_\phi^k)$. Conversely, let $\mu \in \sigma_p(B_{\phi(z^k)}^k)$. Then there exist a non-zero function g in $\mathbb{F}^2$ satisfying $B_\phi^k(z^k)g = \mu g$. Let $N = \phi^{-1} g$. Then $N \in \mathbb{F}^2$ satisfies

$$\begin{aligned}
B_\phi^k N &= W_k T_\phi(\phi^{-1} g) \\
&= W_k P(\phi \cdot (\phi^{-1} g)) = W_k g \\
&= \phi^{-1} \phi W_k g \\
&= \phi^{-1} W_k(\phi(z^k) g) \\
&= \phi^{-1} B_{\phi(z^k)}^k g \\
&= \phi^{-1} \mu g = \mu \phi^{-1} g \\
&= \mu N
\end{aligned}$$

Since ϕ is invertible and g is non-zero, so $N \neq 0$ and therefore, $\mu \in \sigma_p(B_\phi^k)$.
Thus $\sigma_p(B_\phi^k) = \sigma_p(B_{\phi(z^k)}^k)$.

□

Corollary 2.13. *For an invertible, analytic function ϕ in L^∞, $\sigma_p(B_\phi) = \sigma_p(B_{\phi(z^2)})$, where $\sigma_p(B_\phi)$ denotes the point spectrum of B_ϕ.*

3. Berezin Transform of B_ϕ

Let $\mathbb{H}$ be any reproducing kernel Hilbert space on an open subset Ω of $\mathbb{C}$. For a bounded operator S on $\mathbb{H}$, the *Berezin transform* [15] denoted by $\tilde{S}$, is the complex valued function on Ω

$$\tilde{S}(z) = \langle Sk_z, k_z\rangle \qquad \text{for} \quad z \in \Omega.$$

For every bounded operator S on $\mathbb{H}$, the Berezin transform $\tilde{S}$ is a bounded function on Ω.
The normalized reproducing kernel in the Fock space is given by

$$k_z(w) = \frac{1}{\sqrt{\pi}} e^{w\bar{z} - \frac{|z|^2}{2}} = \frac{1}{\sqrt{\pi}} \left(\sum_{n=0}^{\infty} \frac{(w\bar{z})^n}{n!} \right) e^{-\frac{|z|^2}{2}}.$$

The following proposition gives the Berezin transform of the operator W_k:

Proposition 3.1. *For the operator W_k, we have $W_k\left(k_z(w)\right) = \frac{1}{\sqrt{\pi}} e^{-\frac{|z|^2}{2}} \sum_{m=0}^{\infty} \frac{(w\bar{z})^m}{(km)!}$ and its Berezin transform is given by*

$$\tilde{W_k}(z) = \frac{1}{\sqrt{\pi}} e^{-\frac{|z|^2}{2}} \sum_{m=0}^{\infty} \frac{|z|^{2m}}{(km)!}.$$

Proof. By the definitions of operator W_k and normalized reproducing kernel k_z, it follows that

$$\begin{aligned} W_k\left(k_z(w)\right) &= W_k\left(\frac{1}{\sqrt{\pi}} e^{-\frac{|z|^2}{2} + w\bar{z}} \right) = \frac{1}{\sqrt{\pi}} e^{-\frac{|z|^2}{2}} W_k \left(\sum_{n=0}^{\infty} \frac{(w\bar{z})^n}{n!} \right) \\ &= \frac{1}{\sqrt{\pi}} e^{-\frac{|z|^2}{2}} \sum_{m=0}^{\infty} \frac{(w\bar{z})^m}{(km)!} \end{aligned}$$

and

$$\begin{aligned} \tilde{W_k}(z) = \langle W_k k_z, k_z \rangle &= \left\langle \frac{1}{\sqrt{\pi}} e^{-\frac{|z|^2}{2}} \sum_{m=0}^{\infty} \frac{(w\bar{z})^m}{(km)!}, k_z(w) \right\rangle \\ &= \frac{1}{\sqrt{\pi}} e^{-\frac{|z|^2}{2}} \sum_{m=0}^{\infty} \left\langle \frac{(w\bar{z})^m}{(km)!}, k_z(w) \right\rangle \\ &= \frac{1}{\sqrt{\pi}} e^{-\frac{|z|^2}{2}} \sum_{m=0}^{\infty} \frac{(z\bar{z})^m}{(km)!} = \frac{1}{\sqrt{\pi}} e^{-\frac{|z|^2}{2}} \sum_{m=0}^{\infty} \frac{|z|^{2m}}{(km)!}. \end{aligned}$$

□

The explicit expression for the Berezin transform of the slant Toeplitz operator is given in the next result:

Theorem 3.2. *For an operator W its adjoint $W^\star(k_z(w)) = \frac{1}{\sqrt{\pi}}e^{-\frac{|z|^2}{2}}\cosh(w\bar{z})$ and the Berezin transform of slant Toeplitz operator B_ϕ is*

$$\tilde{B_\phi}(z) = \frac{1}{2\pi}\int_{\mathbb{C}} \phi(w)e^{-|z-w|^2}dA(w) + \frac{1}{2\pi}\int_{\mathbb{C}} \phi(w)e^{-|z-w|^2}e^{-2(z\bar{w})}dA(w).$$

Proof. By the proposition 2.4 and the definition of normalized reproducing kernel k_z, it follows that

$$\begin{aligned}
W^\star(k_z(w)) &= \frac{1}{\sqrt{\pi}}e^{-\frac{|z|^2}{2}}W^\star\left(\sum_{n=0}^{\infty}\frac{(w\bar{z})^n}{n!}\right)\\
&= \frac{1}{\sqrt{\pi}}e^{-\frac{|z|^2}{2}}\sum_{n=0}^{\infty}\frac{(w\bar{z})^{2n}}{(2n!)}\\
&= \frac{1}{\sqrt{\pi}}e^{-\frac{|z|^2}{2}}\cosh(w\bar{z}).
\end{aligned}$$

Now the Berezin transform of B_ϕ is given by

$$\begin{aligned}
\tilde{B_\phi}(z) &= \langle B_\phi k_z, k_z\rangle\\
&= \langle W T_\phi k_z, k_z\rangle\\
&= \langle T_\phi k_z, W^\star k_z\rangle\\
&= \langle P(\phi k_z), W^\star k_z\rangle\\
&= \frac{1}{\sqrt{\pi}}\int_{\mathbb{C}} \phi(w)k_z(w)e^{-\frac{|z|^2}{2}}\overline{\cosh(w\bar{z})}e^{-|w|^2}dA(w)\\
&= \frac{1}{\pi}\int_{\mathbb{C}} \phi(w)e^{w\bar{z}-|z|^2-|w|^2}\cosh(z\bar{w})dA(w)\\
&= \frac{1}{\pi}\int_{\mathbb{C}} \phi(w)e^{w\bar{z}-|z|^2-|w|^2}\left(\frac{e^{z\bar{w}}+e^{-(z\bar{w})}}{2}\right)dA(w)\\
&= \frac{1}{2\pi}\int_{\mathbb{C}} \phi(w)e^{w\bar{z}+z\bar{w}-(|z|^2+|w|^2)}dA(w) + \frac{1}{2\pi}\int_{\mathbb{C}} \phi(w)e^{w\bar{z}-z\bar{w}-(|z|^2+|w|^2)}dA(w)\\
&= \frac{1}{2\pi}\int_{\mathbb{C}} \phi(w)e^{-|z-w|^2}dA(w) + \frac{1}{2\pi}\int_{\mathbb{C}} \phi(w)e^{-|z-w|^2}e^{-2(z\bar{w})}dA(w). \qquad (3.1)
\end{aligned}$$

□

In the expression (3.1), the term $\frac{1}{\pi}\int_{\mathbb{C}}\phi(w)e^{-|z-w|^2}dA(w)$ is basically the Berezin transform of T_ϕ or simply the Berezin transform of ϕ. Also from (3.1) it is clear that $\tilde{B_\phi}(z) \to 0$ as $|z| \to \infty$.

For $\phi \in L^\infty$, we observe that

(1) $\tilde{B_\phi}(z) \in L^\infty(\mathbb{C}, d\lambda)$.
(2) $\|\tilde{B_\phi}(z)\|_\infty \le \|\phi\|_\infty$.
(3) $\tilde{B}_{\lambda_1\phi_1+\lambda_2\phi_2} = \lambda_1\tilde{B}_{\phi_1} + \lambda_2\tilde{B}_{\phi_2}$.

Corollary 3.3. *Let ϕ be essentially bounded measurable function on $\mathbb{C}$. Then B_ϕ = 0 if and only if $\tilde{B_\phi} = 0$.*

Proposition 3.4. *If $\phi(z)$ is a non-negative bounded measurable function on $\mathbb{C}$ and if $|z| \to \infty$ then B_ϕ is compact.*

Proof. If $|z| \to \infty$ then $\tilde{B}_\phi \to 0$ and so by the expression (3.1) it follows that $\int_{\mathbb{C}} \phi(w)e^{-|z-w|^2}dA(w) \to 0$ as $z \to \infty$, that is, $\tilde{T}_\phi(z) \to 0$ as $z \to \infty$. But we know that for a given ϕ, T_ϕ is compact on $\mathbb{F}^2$ if and only if $\tilde{T}_\phi \to 0$ as $z \to \infty$, ([15], Proposition 5.3). So it follows that T_ϕ is compact. Hence B_ϕ is compact on $\mathbb{F}^2$. $\square$

4. Commutativity of kth-order slant Toeplitz operators

In this section the commutativity of kth-order slant Toeplitz operators with coanalytic symbols and harmonic symbols has been studied. The following lemma follows from [12].

Lemma 4.1. *Let f and g be analytic functions in $L^\infty(\mathbb{C})$, both of which are not identically zero and let $q \geq 0$ be an integer. If $fW_k^\star g = gW_k^\star f$, then the following are equivalent;*

1. *$f^{(i)}(0) = 0$ for any integers i with $0 \leq i \leq q$ and $f^{(q+1)}(0) \neq 0$*
2. *$g^{(i)}(0) = 0$ for any integers i with $0 \leq i \leq q$ and $g^{(q+1)}(0) \neq 0$.*

Now in the next theorem we obtain the necessary and sufficient condition for the commutativity of kth-order slant Toeplitz operator with co-analytic symbols.

Theorem 4.2. *Let $\phi, \psi \in L^\infty(\mathbb{C})$ be such that $\bar{\phi}, \bar{\psi}$ are analytic functions then the following statements are equivalent:*

1. *B_ϕ^k and B_ψ^k commute ;*
2. *there exist scalers α and β, not both zero, such that $\alpha\phi + \beta\psi = 0$.*

Proof. Firstly suppose that (2) holds. Without loss of generality assume that $\alpha \neq 0$, then $\phi = \gamma\psi$ where $\gamma = -\beta/\alpha$. Then

$$B_\phi^k B_\psi^k = W_k T_\phi W_k T_\psi = W_k T_{\gamma\psi} W_k T_\psi = \gamma W_k T_\psi W_k T_\psi = W_k T_\psi W T_{\gamma\psi} = B_\psi^k B_\phi^k.$$

Conversely suppose that B_ϕ^k and B_ψ^k commutes. Therefore we get that $B_\phi^{k\star} B_\psi^{k\star}(1) = B_\psi^{k\star} B_\phi^{k\star}(1)$, or equivalently, $\bar{\phi} W_k^\star \bar{\psi} = \bar{\psi} W_k^\star \bar{\phi}$.
Now we have the following three cases.
Case I. If $\phi \equiv 0$ or $\psi \equiv 0$, then the result is obvious.
Case II. If $\overline{\phi}(0) \neq 0$ and $\overline{\psi}(0) \neq 0$. Since $\overline{\phi}, \overline{\psi}$ are analytic functions in $L^\infty(\mathbb{C})$, so let $\overline{\phi}(z) = \sum_{r=0}^\infty a_r z^r$ and $\overline{\psi}(z) = \sum_{s=0}^\infty b_s z^s$, then $a_0 \neq 0, b_0 \neq 0$ and $W_k^\star(\overline{\phi}(z)) = \sum_{r=0}^\infty a_r \dfrac{r!}{(kr)!} z^{kr}$ and $W_k^\star(\overline{\psi}(z)) = \sum_{s=0}^\infty b_s \dfrac{s!}{(ks)!} z^{ks}$. Also

$$\overline{\psi} W_k^\star(\overline{\phi}(z)) = \left(\sum_{s=0}^\infty b_s z^s\right)\left(\sum_{r=0}^\infty a_r \frac{r!}{(kr)!} z^{kr}\right) = \sum_{s=0}^\infty \sum_{r=0}^\infty \frac{r!}{(kr)!} b_s a_r z^{s+kr}$$

and

$$\overline{\phi} W_k^\star(\overline{\psi}(z)) = \left(\sum_{r=0}^\infty a_r z^r\right)\left(\sum_{s=0}^\infty b_s \frac{s!}{(ks)!} z^{ks}\right) = \sum_{r=0}^\infty \sum_{s=0}^\infty \frac{s!}{(ks)!} a_r b_s z^{r+ks}.$$

Since $\overline{\phi}W_k^\star\overline{\psi} = \overline{\psi}W_k^\star\overline{\phi}$, therefore it follows that

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{s!}{(ks)!}a_r b_s z^{r+ks} = \sum_{s=0}^{\infty}\sum_{r=0}^{\infty}\frac{r!}{(kr)!}b_s a_r z^{s+kr}$$

or equivalently,

$$\sum_{s=0}^{\infty}\sum_{p=ks}^{\infty}\frac{s!}{(ks)!}a_{p-ks}b_s z^{p} = \sum_{r=0}^{\infty}\sum_{p=kr}^{\infty}\frac{r!}{(kr)!}a_r b_{p-kr} z^{p}. \tag{4.1}$$

For $p = 0$, equation (4.1) gives that $a_0b_0 = b_0a_0$, so $b_0 = (b_0/a_0)a_0$, since $a_0 \neq 0$. Now take $\lambda = b_0/a_0$, so then $b_0 = \lambda a_0$.
For $1 \leq p \leq k-1$, then by equation (4.1) it follows that $a_pb_0 = b_pa_0$. This means $b_p = (b_0/a_0)a_p$, since $a_0 \neq 0$, that is $b_p = \lambda a_p$.
For $k \leq p \leq 2k-1$, then by equation (4.1) it follows that $a_pb_0 + \frac{1}{(k)!}a_{p-k}b_1 = b_pa_0 + \frac{1}{k!}b_{p-k}a_1$, but $b_m = \lambda a_m$ for each m such that $0 \leq m \leq k-1$, therefore $b_p = \lambda a_p$, since $a_0 \neq 0$. So continuing in this manner it follows that $b_i = \lambda a_i$ for each non-negative integer i, where $\lambda = b_0/a_0$. Therefore in this case we get $\overline{\psi}(z) = \sum_{s=0}^{\infty} b_s z^s = \sum_{s=0}^{\infty}\lambda a_s z^s = \lambda\overline{\phi}(z)$.
Case III. If ϕ and ψ are both not zero identically and $\overline{\phi}(0) = 0$ or $\overline{\psi}(0) = 0$. Without loss of generality, suppose that $\overline{\phi}^{(m)}(0) = 0$ for any integer m such that $0 \leq m \leq m_1$ and $\overline{\phi}^{(m+1)}(0) \neq 0$, where m_1 is a non-negative integer. Therefore by Lemma 4.1, it follows that $\overline{\psi}^{(m)}(0) = 0$ for any integer m such that $0 \leq m \leq m_1$ and $\overline{\psi}^{(m+1)}(0) \neq 0$. Now we can write $\overline{\phi}, \overline{\psi}$ as

$$\overline{\phi}(z) = z^{m_1+1}\phi_1(z), \qquad \overline{\psi}(z) = z^{m_1+1}\psi_1(z) \tag{4.2}$$

where ϕ_1, ψ_1 are analytic functions in $L^\infty(\mathbb{C})$ and $\phi_1(0) \neq 0,\ \psi_1(0) \neq 0$. Therefore by equation (4.2) it follows that,

$$\overline{\psi}W_k^\star(\overline{\phi}(z)) = z^{(k+1)(m_1+1)}\psi_1 W_k^\star(\phi_1(z))$$

and

$$\overline{\phi}W_k^\star(\overline{\psi}(z)) = z^{(k+1)(m_1+1)}\phi_1 W_k^\star(\psi_1(z)).$$

Now as $\overline{\psi}W_k^\star(\overline{\phi}) = \overline{\phi}W_k^\star(\overline{\psi})$, so it gives $\psi_1 W_k^\star(\phi_1) = \phi_1 W_k^\star(\psi_1)$. Since $\phi_1(0) \neq 0$ and $\psi_1(0) \neq 0$, so by the second case it follows that $\psi_1 = \lambda_1\phi_1$, where $\lambda_1 = \psi_1(0)/\phi_1(0)$. Therefore we have

$$\overline{\psi}(z) = z^{m_1+1}\psi_1(z) = \lambda_1 z^{m_1+1}\phi_1(z) = \lambda_1\overline{\phi}(z).$$

Thus the symbol functions ϕ and ψ are linearly dependent. □

In ([12]), C. Liu and Yufeng Lu discussed the commutativity of kth-order slant Toeplitz operators on Bergman space and gave the necessary and sufficient conditions for commutativity of kth-order slant Toeplitz operators. In the next theorem, we show that the two kth-order slant Toeplitz operators having harmonic symbols commute if and only if their symbol functions are linearly dependent.

Theorem 4.3. *Let $\phi(z) = \sum_{i=0}^{n} a_i z^i + \sum_{i=1}^{n} a_{-i}\bar{z}^i$ and $\psi(z) = \sum_{j=0}^{n} b_j z^j + \sum_{j=1}^{n} b_{-j}\bar{z}^j$, where $b_{-n} \neq 0$ such that the ratio $\frac{\overline{a_{-n}}}{\overline{b_{-n}}}$ is real and $n \geq 1$ be an integer, then the following statements are equivalent:*

(1) *B_ϕ^k and B_ψ^k commute;*
(2) *there exist scalers α and β, not both zero, such that $\alpha\phi + \beta\psi = 0$.*

Proof. Suppose that (2) holds then B_ϕ^k and B_ψ^k commute.
Now suppose that (1) holds. Let $\phi_1(z) = \sum_{i=0}^{n} a_i z^i$, $\overline{\phi_2}(z) = \sum_{i=1}^{n} a_{-i}\bar{z}^i$, $\psi_1(z) = \sum_{j=0}^{n} b_j z^j$ and $\overline{\psi_2}(z) = \sum_{j=1}^{n} b_{-j}\bar{z}^j$. Then $\phi = \phi_1 + \overline{\phi_2}$ and $\psi = \psi_1 + \overline{\psi_2}$. Since B_ϕ^k and B_ψ^k commute, therefore it gives $T_{\overline{\phi}} W_k^\star T_{\overline{\psi}} W_k^\star(1) = T_{\overline{\psi}} W_k^\star T_{\overline{\phi}} W_k^\star(1)$, that is

$$\phi_2 W_k^\star \psi_2 + \overline{\psi_1}(0)\phi_2 + P(\overline{\phi_1} W_k^\star \psi_2) = \psi_2 W_k^\star \phi_2 + \overline{\phi_1}(0)\psi_2 + P(\overline{\psi_1} W_k^\star \phi_2) \tag{4.3}$$

or, equivalently,

$$\begin{aligned}
&(\sum_{i=1}^{n} \overline{a_{-i}} z^i)(\sum_{j=1}^{n} \overline{b_{-j}} \frac{j!}{(kj)!} z^{kj}) + \overline{b_0} \sum_{i=1}^{n} \overline{a_{-i}} z^i + P\left((\sum_{i=0}^{n} \overline{a_i}\bar{z}^i)(\sum_{j=1}^{n} \overline{b_{-j}} \frac{j!}{(kj)!} z^{kj})\right)\\
&= (\sum_{j=1}^{n} \overline{b_{-j}} z^j)(\sum_{i=1}^{n} \overline{a_{-i}} \frac{i!}{(ki)!} z^{ki}) + \overline{a_0} \sum_{j=1}^{n} \overline{b_{-j}} z^j + P\left((\sum_{j=0}^{n} \overline{b_j}\bar{z}^j)(\sum_{i=1}^{n} \overline{a_{-i}} \frac{i!}{(ki)!} z^{ki})\right).
\end{aligned}$$

Now for any integer r such that $kn + 1 \leq r \leq kn + n$, it follows that

$$\sum_{i=1}^{n} \sum_{\substack{j=1\\ i+kj=r}}^{n} \overline{a_{-i}}\,\overline{b_{-j}} \frac{j!}{(kj)!} z^r = \sum_{j=1}^{n} \sum_{\substack{i=1\\ j+ki=r}}^{n} \overline{b_{-j}}\,\overline{a_{-i}} \frac{i!}{(ki)!} z^r \tag{4.4}$$

where i and j are positive integers not greater than n. Now we apply induction to complete the proof.
When $r = kn + n$, then by equation (4.4) it follows that

$$\overline{a_{-n}}\,\overline{b_{-n}} \frac{n!}{(kn)!} = \overline{b_{-n}}\,\overline{a_{-n}} \frac{n!}{(kn)!},$$

so $\overline{a_{-n}} = \overline{b_{-n}}\left(\overline{a_{-n}}/\overline{b_{-n}}\right)$, since $\overline{b_{-n}} \neq 0$. Let $\gamma = \overline{a_{-n}}/\overline{b_{-n}}$, so we get $\overline{a_{-n}} = \gamma\overline{b_{-n}}$.
When $r = kn + n - 1$, then by equation (4.4) it follows that

$$\overline{a_{-n+1}}\,\overline{b_{-n}} \frac{n!}{(kn)!} = \overline{b_{-n+1}}\,\overline{a_{-n}} \frac{n!}{(kn)!},$$

so $\overline{a_{-n+1}} = \gamma\overline{b_{-n+1}}$, where $\gamma = \overline{a_{-n}}/\overline{b_{-n}}$, since $\overline{b_{-n}} \neq 0$. Suppose that $\overline{a_{-n+s}} = \gamma\overline{b_{-n+s}}$ for any integer s with $0 \leq s \leq l < n - 1$. Now we consider the connection between a_{-n+l+1} and b_{-n+l+1}. When $r = kn + n - l - 1$, then by equation (4.4) we get that

$$\begin{aligned}
&\overline{a_{-n+l+1}}\,\overline{b_{-n}} \frac{n!}{(kn)!} + \cdots + \overline{a_{-n+l+1-k\lambda}}\,\overline{b_{-n}} \frac{(n+\lambda)!}{(k(n+\lambda))!}\\
&= \overline{b_{-n+l+1}}\,\overline{a_{-n}} \frac{n!}{(kn)!} + \cdots + \overline{b_{-n+l+1-k\lambda}}\,\overline{a_{-n+\lambda}} \frac{(n+\lambda)!}{(k(n+\lambda))!}
\end{aligned}$$

where $\lambda = \left[\frac{(l+1)}{k}\right]$ and $[x]$ denotes the greatest integer function not greater than x. Now from the assumption it follows that

$$\overline{a_{-n+l+1}}\overline{b_{-n}}\frac{n!}{(kn)!} = \overline{b_{-n+l+1}}\overline{a_{-n}}\frac{n!}{(kn)!},$$

which implies that $\overline{a_{-n+l+1}} = \gamma\overline{b_{-n+l+1}}$, where $\gamma = \overline{a_{-n}}/\overline{b_{-n}}$, since $\overline{b_{-n}} \neq 0$. Thus, from the induction we obtain that $\overline{a_{-n+s}} = \gamma\overline{b_{-n+s}}$ for any integer s such that $0 \leq s \leq n-1$. Hence $\phi_2(z) = \sum_{r=1}^{n}\overline{a_{-r}}z^r = \sum_{r=1}^{n}\gamma\overline{b_{-r}}z^r = \gamma\psi_2(z)$. Now since $\phi_2(z) = \gamma\psi_2(z)$, so by the equation (4.3) it follows that

$$\overline{\psi_1}(0)\phi_2 + P(\overline{\phi_1}W_k^\star\psi_2) = \overline{\phi_1}(0)\psi_2 + P(\overline{\psi_1}W_k^\star\phi_2). \tag{4.5}$$

Then

$$\left\langle\overline{\psi_1}(0)\phi_2 + P(\overline{\phi_1}W_k^\star\psi_2), z^{kn}\right\rangle = \left\langle\overline{\phi_1}(0)\psi_2 + P(\overline{\psi_1}W_k^\star\phi_2), z^{kn}\right\rangle,$$

or equivalently,

$$\left\langle\overline{\phi_1}W_k^\star\psi_2, z^{kn}\right\rangle = \left\langle\overline{\psi_1}W_k^\star\phi_2, z^{kn}\right\rangle,$$

which further implies that $\overline{b_{-n}}\overline{a_0}\ \pi(kn)! = \overline{a_{-n}}\overline{b_0}\ \pi(kn)!$ and hence $\overline{a_0} = \gamma\overline{b_0}$, where $\gamma = \overline{a_{-n}}/\overline{b_{-n}}$, since $\overline{b_{-n}} \neq 0$. Now as $a_0 = \gamma b_0$, so from (4.5) it follows that

$$\overline{\psi_1}(0)\phi_2 = \overline{\phi_1}(0)\psi_2 \text{ and } P(\overline{\phi_1}W_k^\star\psi_2) = P(\overline{\psi_1}W_k^\star\phi_2),$$

that is, $P\left(\overline{(\gamma\psi_1 - \phi_1)}\cdot W_k^\star\psi_2\right) = 0$. Hence for any integer u such that $kn - n \leq u \leq kn - 1$, we have

$$\left\langle z^u, P\left(\overline{(\gamma\psi_1 - \phi_1)}\cdot W_k^\star\psi_2\right)\right\rangle = 0,$$

or equivalently,

$$\langle(\gamma\psi_1 - \phi_1)z^u, W_k^\star\psi_2\rangle = 0,$$

which on putting the values of ϕ_1, ψ_1 and ψ_2 gives that

$$\left\langle\sum_{r=0}^{n}(\gamma b_r - a_r)z^{r+u}, \sum_{r=1}^{n}\frac{r!}{(kr)!}\overline{b_{-r}}z^{kr}\right\rangle = 0.$$

Since $a_0 = \gamma b_0$, so we have

$$\left\langle\sum_{r=1}^{n}(\gamma b_r - a_r)z^{r+u}, \sum_{r=1}^{n}\frac{r!}{(kr)!}\overline{b_{-r}}z^{kr}\right\rangle = 0. \tag{4.6}$$

When $u = kn-1$, then equation (4.6) gives that

$$(\gamma b_1 - a_1)\, b_{-n}\ \pi(kn)! = 0,$$

so this gives $a_1 = \gamma b_1$, since $b_{-n} \neq 0$. So, then it follows

$$\left\langle\sum_{r=2}^{n}(\gamma b_r - a_r)z^{r+u}, \sum_{r=1}^{n}\frac{r!}{(kr)!}\overline{b_{-r}}z^{kr}\right\rangle = 0.$$

Now suppose that $a_j = \gamma b_j$ for any integer j such that $0 \le j \le s$, where $0 \le s \le n-1$.Then by equation (4.6), it follows that

$$\left\langle \sum_{r=s+1}^{n} (\gamma b_r - a_r) z^{r+u}, \sum_{r=1}^{n} \frac{r!}{(kr)!} \overline{b_{-r}} z^{kr} \right\rangle = 0. \tag{4.7}$$

Now consider the connection between a_{s+1} and b_{s+1}. When $u = kn - s - 1$, then by equation (4.7), it follows that

$$(\gamma b_{s+1} - a_{s+1})\, b_{-n} \pi (kn)! = 0,$$

which gives $a_{s+1} = \gamma b_{s+1}$, since $b_{-n} \neq 0$. Thus from the induction we obtain that $a_j = \gamma b_j$ for any integers j with $0 \le j \le n$. Therefore we get $\phi_1(z) = \sum_{r=1}^{n} a_r z^r = \sum_{r=1}^{n} \gamma b_r z^r = \gamma \psi_1(z)$. Also from above we have $\phi_2 = \gamma \psi_2$, so it follows that $\phi = \phi_1 + \overline{\phi_2} = \gamma \psi_1 + \gamma \overline{\psi_2} = \gamma \psi$. Thus the symbol functions ϕ and ψ are linearly dependent. □

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[1]Department of Mathematics, University of Delhi, Delhi-110007, India.
E-mail address: shivamkumarsingh14@gmail.com

[2]Department of Mathematics, Delhi College of Arts and Commerce, University of Delhi, Delhi-110023, India.
E-mail address: dishna2@yahoo.in