Adv. Oper. Theory 3 (2018), no. 1, 123–136
http://doi.org/10.22034/aot.1703-1145
ISSN: 2538-225X (electronic)
http://aot-math.org

PERMANENCE OF NUCLEAR DIMENSION FOR INCLUSIONS OF UNITAL C^*-ALGEBRAS WITH THE ROKHLIN PROPERTY

HIROYUKI OSAKA[1*] and TAMOTSU TERUYA[2]

Dedicated to the memory of Uffe Haagerup

Communicated by M. Tomforde

ABSTRACT. Let $P \subset A$ be an inclusion of unital C^*-algebras and $E\colon A \to P$ be a faithful conditional expectation of index finite type. Suppose that E has the Rokhlin property. Then $\mathrm{dr}(P) \leq \mathrm{dr}(A)$ and $\dim_{nuc}(P) \leq \dim_{nuc}(A)$. This can be applied to Rokhlin actions of finite groups. We also show that under the same above assumption if A is exact and pure, that is, the Cuntz semigroups $W(A)$ has strict comparison and is almost divisible, then P and the basic contruction $C^*\langle A, e_P\rangle$ are also pure.

1. INTRODUCTION

The nuclear dimension of a C^*-algebra was introduced by Winter and Zacharias [31] as a noncommutative version of topological dimension, which is weaker than the decomposition rank introduced by Kirchberg and Winter [11]. The class of separable, simple, nuclear C^*-algebras with finite nuclear dimension accounts for, however, a large part of separable, simple, nuclear C^*-algebras covered by classification programs, in both stable finite and purely infinite cases [30]. Note that if a C^*-algebra A has finite decomposition rank, then A should be stably finite.

A C^*-algebra A is said to be pure if it has strict comparison of positive elements and an almost divisible Cuntz semigroup $W(A)$. Here, the Cuntz semigroup

Date: Received: Mar. 28, 2017; Accepted: Apr. 27, 2017.
*Corresponding author.
2010 *Mathematics Subject Classification.* Primary 46L55; Secondary 46L35.
Key words and phrases. Rokhlin property, C^*-index, nuclear dimension.

$W(A)$ is said to be almost divisible if, for any positive contraction $a \in M_\infty(A)$ and $0 \neq k \in \mathbb{N}$, there is $x \in W(A)$ such that $k \cdot x \leq \langle a \rangle \leq (k+1) \cdot x$. Winter [30] showed that a separable, simple, unital nonelementary C^*-algebra with finite nuclear dimension is $\mathcal{Z}$-stable, that is, absorbs the Jiang-Su algebra $\mathcal{Z}$ tensorially. Note that the Jiang-Su algebra $\mathcal{Z}$ plays a crucial role in the Elliott conjecture of the classification of separable, simple, unital, nuclear C^*-algebras. Indeed, very recently the calssification theory of separabe, simple , unital, nuclear, $\mathcal{Z}$-absorbing C^*-algebras has been completed by Gong-Lin-Nu [5], Elliott-Gong-Lin-Niu [4], and Tikuisis-White-Winter [26].

In this paper, we first consider the local $\mathcal{C}$-property for separable unital C^*-algebras in the sense of Osaka and Phillips [15] and show that, when A is a local $\mathcal{C}_n$ (respectively, $\mathcal{C}_{\mathrm{nuc}_n}$), separable unital C^*-algebra and α is an action of a finite group G on A, if α has the Rokhlin property in the sense of Izumi [7], then the crossed product algebra $A \rtimes_\alpha G$ belongs to $\mathcal{C}_n$ (respectively, $\mathcal{C}_{\mathrm{nuc}_n}$). This is a partial answer to Problem 9.4 in [31]. We note that the Rokhlin property for an action is essential in the estimate of the nuclear dimension of the crossed product algebra by that of a given C^*-algebra, because there is the symmetry α on the CAR algebra $\mathcal{U}$ without the Rokhlin property such that $\dim_{\mathrm{nuc}}(\mathcal{U} \times_\alpha \mathbb{Z}/2\mathbb{Z}) \neq 0$ (see Remark 3.3).

In Section 3, we extend the above observation for crossed product algebras to inclusions of unital C^*-algebra of index finite type. Let $P \subset A$ be an inclusion of separable unital C^*-algebras of index finite type in the sense of Watatani [28] and let a faithful conditional expectation $E\colon A \to P$ have the Rokhlin property in the sense of Kodaka, Osaka, and Teruya [12]; then P belongs to $\mathcal{C}_n$ (respectively, $\mathcal{C}_{\mathrm{nuc}_n}$) when A is a local $\mathcal{C}_n$ (respectively, $\mathcal{C}_{\mathrm{nuc}_n}$), unital C^*-algebra.

In Section 4, we investigate the permanence property of inclusions with the Rokhlin property with respect to the strict comparison property. We show that under the assumption that an inclusion $P \subset A$ is of index finite type and $E\colon A \to P$ has the Rokhlin property, if A is a unital exact C^*-algebra that has strict comparison, then P and the basic construction $C^*\langle A, e_P\rangle$ have strict comparison. We need the exactness because in this case strict comparison is equivalent to, for x and y in $W(A)$, $x \leq y$ if $d_\tau(x) \leq d_\tau(y)$ for all tracial states τ in A, where the function d_τ is the dimension function on A induced by a trace τ, that is, $d_\tau(a) = \lim_{n\to\infty} \tau(a^{\frac{1}{n}})$ for $a \in M_\infty(A)^+$.

When A has stable rank one, the condition that A has an almost divisible Cuntz semigroup is equivalent to there being a unital $*$-homomorphism from a dimension drop C^*-algebra $Z_{n,n+1} = \{f \in C([0,1], M_n \otimes M_{n+1}) \mid f(0) \in M_n \otimes \mathbb{C} \text{ and} f(1) \in \mathbb{C} \otimes M_{n+1}\}$ by [23, Proposition 2.4]. Note that the Jiang-Su algebra $\mathcal{Z}$ can be constructed as the inductive limit of the sequence of such dimension C^*-algebras [8]. Then we show that, when A is a unital C^*-algebra of stable rank one, if A has an almost divisible Cuntz semigroup, then P and $C^*\langle A, e_P\rangle$ have an almost divisible Cuntz semigroup. Therefore, if A is a separable, unital, exact, pure C^*-algebra of stable rank one, then P and $C^*\langle A, e_P\rangle$ are pure in the sense of Winter [30]. We stress that we do not need the simplicity of A and P.

The first draft of this paper was posted on the arXiv in 2011 (arXiv.1111.1808 v.1). Very recently, Nawata [14] and Santiago [24] studied the general Rokhlin property for actions on projectionless C^*-algebras and Barlak and Szabo [3] defined the sequential split $*$-homomorphism, and they pointed out that the inclusion $*$-homomorphism from P into A is sequentially split using the inclusion map $A \to P^\infty$ in [12, 5.1]. However, the results presented here are of significance.

2. Preliminaries

In this section, we recall the finitely saturated property and local $\mathcal{C}$-property. If we consider the class of unital C^*-algebras with finite decomposable rank (respectively, finite nuclear dimension), we can show that they are finitely saturated. We also recall C^*-index theory and present the relevant basic facts.

2.1. Local $\mathcal{C}$-property and nuclear dimension.

First we recall the definition of the finitely saturated property in [15].

Definition 2.1. Let $\mathcal{C}$ be a class of separable unital C^*-algebras. Then $\mathcal{C}$ is *finitely saturated* if the following closure conditions hold:

(1) If $A \in \mathcal{C}$ and $B \cong A$, then $B \in \mathcal{C}$.
(2) If $A_1, A_2, \dots, A_n \in \mathcal{C}$, then $\bigoplus_{k=1}^n A_k \in \mathcal{C}$.
(3) If $A \in \mathcal{C}$ and $n \in \mathbb{N}$, then $M_n(A) \in \mathcal{C}$.
(4) If $A \in \mathcal{C}$ and $p \in A$ is a nonzero projection, then $pAp \in \mathcal{C}$.

Moreover, the *finite saturation* of a class $\mathcal{C}$ is the smallest finitely saturated class that contains $\mathcal{C}$.

We recall the definition of the local $\mathcal{C}$-property in [15].

Definition 2.2. Let $\mathcal{C}$ be a class of separable unital C^*-algebras. A *unital local $\mathcal{C}$-algebra* is a separable unital C^*-algebra A such that for every finite set $S \subset A$ and every $\varepsilon > 0$ there is a C^*-algebra B in the finite saturation of $\mathcal{C}$ and a unital $*$-homomorphism $\varphi\colon B \to A$ (not necessarily injective) such that $\mathrm{dist}(a, \varphi(B)) < \varepsilon$ for all $a \in S$. If one can always choose $B \in \mathcal{C}$, rather than merely in its finite saturation, we call A a unital strong local $\mathcal{C}$-algebra.

If $\mathcal{C}$ is the set of unital C^*-algebras A with $\mathrm{dr} A < \infty$ (respectively, $\dim_{\mathrm{nuc}} A < \infty$) in the sense of Winter, then any local $\mathcal{C}$-algebra belongs to $\mathcal{C}$. (See Proposition 2.4.)

First, we recall the definition of the covering dimension for nuclear C^*-algebras:

Definition 2.3. [29, 31] Let A be a separable C^*-algebra.

(1) A completely positive map $\varphi\colon \oplus_{i=1}^s M_{r_i} \to A$ has order zero if it preserves orthogonality, that is, $\varphi(e)\varphi(f) = \varphi(f)\varphi(e) = 0$ for $e, f \in \oplus_{i=1}^s M_{r_i}$ with $ef = fe = 0$.
(2) A completely positive map $\varphi\colon \oplus_{i=1}^s M_{r_i} \to A$ is n-decomposable if there is a decomposition $\{1, \dots, s\} = \coprod_{j=0}^n I_j$ such that the restriction $\varphi^{(j)}$ of φ to $\oplus_{i \in I_j} M_{r_i}$ has order zero for each $j \in \{0, \dots, n\}$.

(3) A has decomposition rank n, $\mathrm{dr}A = n$, if n is the least integer such that the following holds: Given $\{a_1, \ldots, a_m\} \subset A$ and $\varepsilon > 0$, there is a completely positive approximation (F, ψ, φ) for $a_1, \ldots, a_m$ within ε, i.e., F is a finite-dimensional C^*-algebra, and $\psi\colon A \to F$ and $\varphi\colon F \to A$ are completely positive contractions such that
  (a) $\|\varphi\psi(a_i) - a_i\| < \varepsilon$ and
  (b) φ is n-decomposable.
  If no such n exists, we write $\mathrm{dr}A = \infty$.

(4) A has nuclear dimension n, $\dim_{\mathrm{nuc}} A = n$, if n is the least integer such that the following holds: Given $\{a_1, \ldots, a_m\} \subset A$ and $\varepsilon > 0$, there is a completely positive approximation (F, ψ, φ) for $a_1, \ldots, a_m$ within ε, i.e., F is a finite-dimensional C^*-algebra, and $\psi\colon A \to F$ and $\varphi\colon F \to A$ are completely positive such that
  (a) $\|\varphi\psi(a_i) - a_i\| < \varepsilon$,
  (b) $\|\psi\| \leq 1$, and
  (c) φ is n-decomposable, and each restriction $\varphi|_{\oplus_{i\in I_j} M_{r_i}}$ is completely positive contractive.
  If no such n exists, we write $\dim_{\mathrm{nuc}} A = \infty$.

Proposition 2.4. *For each $n \in \mathbb{N} \cup \{0\}$, let $\mathcal{C}_n$ be the set of unital C^*-algebras A with $\mathrm{dr}A \leq n$ and $\mathcal{C}_{\mathrm{nuc}_n}$ be the set of unital C^*-algebras A with $\dim_{\mathrm{nuc}} A \leq n$. Then both $\mathcal{C}_n$ and $\mathcal{C}_{\mathrm{nuc}_n}$ are finitely saturated.*

Proof. By [11, Remark 3.2 (iii): (3.1)–(3.3), Proposition 3.8, and Corollary 3.9], we know that $\mathcal{C}_n$ is finitely saturated.

Similarly, it follows from [31, Propostion 2.3 and Corollary 2.8] that $\mathcal{C}_{\mathrm{nuc}_n}$ is finitely saturated. □

2.2. **C^*-index theory.** We recall an index in terms of a quasi-basis following Watatani [28].

Definition 2.5. Let $A \supset P$ be an inclusion of unital C^*-algebras with a conditional expectation E from A onto P.

(1) A *quasi-basis* for E is a finite set $\{(u_i, v_i)\}_{i=1}^n \subset A \times A$ such that, for every $a \in A$,
$$a = \sum_{i=1}^n u_i E\left(v_i a\right) = \sum_{i=1}^n E\left(a u_i\right) v_i.$$

(2) When $\{(u_i, v_i)\}_{i=1}^n$ is a quasi-basis for E, we define $\mathrm{Index}E$ by
$$\mathrm{Index}E = \sum_{i=1}^n u_i v_i.$$
When there is no quasi-basis, we write $\mathrm{Index}E = \infty$. $\mathrm{Index}E$ is called the Watatani index of E.

Remark 2.6. We give several remarks about the above definitions.

(1) Index$E$ does not depend on the choice of the quasi-basis in the above formula, and it is a central element of A [28, Proposition 1.2.8].
(2) Once we know that there exists a quasi-basis, we can choose one of the form $\{(w_i, w_i^*)\}_{i=1}^m$, which shows that Index$E$ is a positive element [28, Lemma 2.1.6].
(3) By the above statements, if A is a simple C^*-algebra, then Index$E$ is a positive scalar.
(4) If Index$E < \infty$, then E is faithful, i.e., $E(x^*x) = 0$ implies $x = 0$ for $x \in A$.

Remark 2.7. As in the same argument in [25] we have an example of inclusion of C^*-algebras that do not arise as C^*-crossed products. That is, let α be an outer action of a finite group G on a simple C^*-algebra A and let H be a non-normal subgroup of G. Then an inclusion $A^G \subset A^H$ does not arise as a C^*-crossed product.

Remark 2.8. Let $P \subset A$ be an inclusion of unital C^*-algebras and let $E\colon A \to P$ be of index finite type. As shown in [16] and the following sections, we know that several local properties (stable rank one, real rank zero, AF, AI, AT, the order of projections over A determined by traces, and $\mathcal{D}$-absorbing) of A are inherited by P when E has the Rokhlin property. The converse, however, is not true. Indeed, there is an example of an inclusion of C^*-algebras $A^{\mathbb{Z}/2\mathbb{Z}} \subset A$ such that a conditional expectation $E\colon A \to A^{\mathbb{Z}/2Z}$ is of index finite type and has the Rokhlin property, and $A^{\mathbb{Z}/2Z}$ is the CAR algebra, but A is not an AF C^*-algebra.

Let α be the symmetry on the CAR algebra $\mathcal{U}$ constructed by Blackadar [1] such that $\mathcal{U}^{\mathbb{Z}/2\mathbb{Z}}$ is not an AF C^*-algebra. Then α does not have the Rokhlin property. Indeed, this actually has the tracial Rokhlin property. (See the definition in [18].) However, its dual action $\hat{\alpha}$ has the Rokhlin property by [19, Proposition 3.5]. Set $P = \mathcal{U}$ and $A = \mathcal{U} \rtimes_\alpha \mathbb{Z}/2\mathbb{Z}$. Then A is not an AF C^*-algebra, because A and $\mathcal{U}^{\mathbb{Z}/2\mathbb{Z}}$ are stably isomorphic. Since $P = A^{\hat{\alpha}}$ and $\hat{\alpha}$ has the Rokhlin property, the canonical conditional expectation $E\colon A \to P$ is of index finite type and has the Rokhlin property by [12].

Let $\mathcal{C}$ be the set of all finite-dimensional C^*-algebras. Then since P is an AF C^*-algebra, we know that P is a local $\mathcal{C}$-algebra. However, obviously A is not a local $\mathcal{C}$- algebra.

2.3. **Rokhlin property for an inclusion of unital C^*-algebras.** For a C^*-algebra A, we set

$$\begin{aligned} c_0(A) &= \{(a_n) \in l^\infty(\mathbb{N}, A) : \lim_{n\to\infty} \|a_n\| = 0\}, \\ A^\infty &= l^\infty(\mathbb{N}, A)/c_0(A). \end{aligned}$$

We identify A with the C^*-subalgebra of A^∞ consisting of the equivalence classes of constant sequences and set

$$A_\infty = A^\infty \cap A'.$$

For an automorphism $\alpha \in \mathrm{Aut}(A)$, we denote by α^∞ and α_∞ the automorphisms of A^∞ and A_∞ induced by α, respectively.

Izumi defined the Rokhlin property for a finite group action in [7, Definition 3.1] as follows:

Definition 2.9. Let α be an action of a finite group G on a unital C^*-algebra A. α is said to have the *Rokhlin property* if there exists a partition of unity $\{e_g\}_{g\in G} \subset A_\infty$ consisting of projections satisfying

$$(\alpha_g)_\infty(e_h) = e_{gh} \quad \text{for } g, h \in G.$$

We call $\{e_g\}_{g\in G}$ Rokhlin projections.

Let $A \supset P$ be an inclusion of unital C^*-algebras. For a conditional expectation E from A onto P, we denote by E^∞ the natural conditional expectation from A^∞ onto P^∞ induced by E. If E has a finite index with a quasi-basis $\{(u_i, v_i)\}_{i=1}^n$, then E^∞ also has a finite index with a quasi-basis $\{(u_i, v_i)\}_{i=1}^n$ and $\mathrm{Index}(E^\infty) = \mathrm{Index}E$.

Motivated by Definition 2.9, Kodaka, Osaka, and Teruya introduced the Rokhlin property for an inclusion of unital C^*-algebras with a finite index [12].

Definition 2.10. A conditional expectation E of a unital C^*-algebra A with a finite index is said to have the *Rokhlin property* if there exists a projection $e \in A_\infty$ satisfying

$$E^\infty(e) = (\mathrm{Index}E)^{-1} \cdot 1$$

and a map $A \ni x \mapsto xe$ is injective. We call e a Rokhlin projection.

The following result states that the Rokhlin property of an action in the sense of Izumi implies that the canonical conditional expectation from a given simple C^*-algebra to its fixed-point algebra has the Rokhlin property in the sense of Definition 2.10.

Proposition 2.11. [12] *Let α be an action of a finite group G on a simple unital C^*-algebra A and E be the canonical conditional expectation from A onto the fixed-point algebra $P = A^\alpha$ defined by*

$$E(x) = \frac{1}{\#G}\sum_{g\in G}\alpha_g(x) \quad \text{for } x \in A,$$

where $\#G$ is the order of G. Then α has the Rokhlin property if and only if there is a projection $e \in A_\infty$ such that $E^\infty(e) = \frac{1}{\#G}\cdot 1$, where E^∞ is the conditional expectation from A^∞ onto P^∞ induced by E.

Remark 2.12. In Proposition 2.11 we need the simplicity of a given C^*-algebra A so that the canonical condition expectation $E\colon A \to A^\alpha$ has the Rokhlin property, because we do not know whether a map $A \ni a \mapsto ae \in A^\infty$ is injective. When A is simple, there is no difference between the Rokhlin property of an action α and that of a conditional expectation E.

3. Permanence properties for decomposition rank and nuclear dimension

In this section, we give a partial answer to Problem 9.4 in [31]. More generally, we give formulas for decomposable rank and nuclear dimension for an inclusion of unital C^*-algebras of finite index under the assumption of the Rokhlin property for the inclusion.

The following should be well known.

Theorem 3.1. *Let $n \in \mathbb{N} \cup \{0\}$ and $\mathcal{C}_n$ be the set of separable unital C^*-algebras D with $\mathrm{dr} D \leq n$ and let $\mathcal{C}_{\mathrm{nuc}_n}$ be the set of separable unital C^*-algebras D with $\dim_{\mathrm{nuc}} D \leq n$.*

1. *If A is a separable, unital, local $\mathcal{C}_n$, C^*-algebra, then A belongs to $\mathcal{C}_n$, i.e., $\mathrm{dr} A \leq n$.*
2. *If A is a separable, unital, local $\mathcal{C}_{\mathrm{nuc}_n}$, C^*-algebra, then A belongs to $\mathcal{C}_{\mathrm{nuc}_n}$, i.e., $\dim_{\mathrm{nuc}} A \leq n$.*

Corollary 3.2(2) is a partial answer to Problem 9.4 in [31].

Corollary 3.2. *For $n \in \mathbb{N} \cup \{0\}$, let $\mathcal{C}_n$ (respectively, $\mathcal{C}_{\mathrm{nuc}_n}$ or $\mathcal{C}_{\mathrm{lnuc}} = \cup_{n \in \mathbb{N}} \mathcal{C}_{\mathrm{nuc}_n}$) be the set of separable unital C^*-algebras D with $\mathrm{dr} D \leq n$ (respectively, $\dim_{\mathrm{nuc}} D \leq n$, or locally finite nuclear dimension). Let A be a separable unital C^*-algebra and α be an action of a finite group G on A. Suppose that α has the Rokhlin property. Then we have the following:*

1. *If A is a local $\mathcal{C}_n$, then $\mathrm{dr}(A^\alpha) \leq n$ and $\mathrm{dr}(A \rtimes_\alpha G) \leq n$.*
2. *If A is a local $\mathcal{C}_{\mathrm{nuc}_n}$, then $\dim_{\mathrm{nuc}}(A^\alpha) \leq n$ and $\dim_{\mathrm{nuc}}(A \rtimes_\alpha G) \leq n$.*
3. *If A has locally finite nuclear dimension, then A^α and $A \rtimes_\alpha G$ have locally finite nuclear dimension.*

Proof. We will show that, if A is a local $\mathcal{C}_n$, C^*-algebra, $A \rtimes_\alpha G$ is a local $\mathcal{C}_n$-algebra.

Since α has the Rokhlin property, for any finite set $\mathcal{S} \subset A \rtimes_\alpha G$ and $\varepsilon > 0$, there are n, projection $f \in A$, and a unital $*$-homomorphism $\varphi\colon M_n \otimes fAf \to A \rtimes_\alpha G$ such that $\mathrm{dist}(a, \varphi(M_n \otimes fAf)) < \varepsilon$ by [18, Theorem 2.2]. Since $A \in \mathcal{C}_n$ by Theorem 3.1, $M_n \otimes fAf \in \mathcal{C}_n$. Hence, $A \rtimes_\alpha G$ is a local $\mathcal{C}_n$-algebra. Again from Theorem 3.1, $\mathrm{dr}(A \rtimes_\alpha G) \leq n$. Since A^α is isomorphic to a corner C^*-subalgebra $q(A \rtimes_\alpha G)q$ for some projection $q \in A \rtimes_\alpha G$, we have $\mathrm{dr}(A^\alpha) \leq n$ by Proposition 2.4.

Similarly, if A is a local $\mathcal{C}_{\mathrm{nuc}_n}$ (respectively, a local $\mathcal{C}_{\mathrm{lnuc}}$), we have $\dim_{\mathrm{nuc}}(P^\alpha) \leq n$ and $\dim_{\mathrm{nuc}}(A \rtimes_\alpha G) \leq n$ (respectively, A^α and $A \rtimes_\alpha G$ are local $\mathcal{C}_{\mathrm{lnuc}}$, that is, A^α and $A \rtimes_\alpha G$ have locally finite nuclear dimension). □

Remark 3.3. When α does not have the Rokhlin property, generally the estimate in Corollary 3.2 is not correct. Indeed, let α be the symmetry action on the CAR algebra $\mathcal{U}$ constructed by Blackadar in [1] such that $\mathcal{U}^{\mathbb{Z}/2\mathbb{Z}}$ is not an AF C^*-algebra. Then α does not have the Rokhlin property by [19, Proposition 3.5], and $\dim_{\mathrm{nuc}}(\mathcal{U}^{\mathbb{Z}/2\mathbb{Z}}) \neq 0$, but $\dim_{\mathrm{nuc}}(\mathcal{U}) = 0$.

In Corollary 3.2, since α is outer by [18, Lemm 1.5], $A \subset A \rtimes_\alpha G$ is of finite index in the sense of Watatani by [28, Proposition 2.8.6]. Therefore, we shall extend Corollary 3.2 for a pair of unital C^*-algebras $P \subset A$ of index finite type.

Theorem 3.4. *For $n \in \mathbb{N} \cup \{0\}$, let $\mathcal{C}_n$ be the set of separable unital C^*-algebras D with $\mathrm{dr} D \leq n$ and let $\mathcal{C}_{\mathrm{nuc}_n}$ be the set of separable unital C^*-algebras D with $\dim_{\mathrm{nuc}} D \leq n$. Further, let $P \subset A$ be an inclusion of unital C^*-algebras and $E\colon A \to P$ be a faithful conditional expectation of index finite type. Suppose that E has the Rokhlin property.*

(1) *If A is a unital, local $\mathcal{C}_n$, C^*-algebra, then*

$$\mathrm{dr} P \leq n.$$

(2) *If A is a unital, local $\mathcal{C}_{\mathrm{nuc}_n}$, C^*-algebra, then*

$$\dim_{\mathrm{nuc}} P \leq n.$$

Proof. (1) For any finite set $\mathcal{F} = \{a_1, a_2, \ldots, a_l\} \subset P$ and $\varepsilon > 0$, since $\mathrm{dr} A \leq n$, there are $B \in \mathcal{C}_n$, a $*$-homomorphism $\rho\colon B \to A$, a finite set $\{b_1, b_2, \ldots, b_l\}$ in $\rho(B)$, and a completely positive approximation (F, ψ, φ) such that

(1) ψ and φ are completely positive contractive,
(2) there are n-central projections $q^{(m)}$ of F such that $F = \oplus q^{(m)} F q^{(m)}$ and $\varphi_{|q^{(m)}Fq^{(m)}}$ is order zero,
(3) $\mathcal{F} \subset_\varepsilon \rho(B)$, i.e., $\|a_i - b_i\| < \varepsilon$ for $1 \leq i \leq l$, and
(4) $\|\varphi \circ \psi(b_i) - b_i\| < \varepsilon$ for $1 \leq i \leq l$.

For $x \in \rho(B)$, we have

$$\begin{aligned}\varphi \circ \psi(x) &= \varphi(\sum_m q^{(m)} \psi(x) q^{(m)}) \\ &= \sum_m (\varphi_{|q^{(m)}Fq^{(m)}} \circ q^{(m)} \psi q^{(m)})(x).\end{aligned}$$

Then each $\varphi_{|q^{(m)}Fq^{(m)}}$ is an order-zero map.
From applying the same argument to each $q^{(m)}\psi q^{(m)}\colon \rho(B) \to q^{(m)}Fq^{(m)}$ and $\varphi_{|q^{(m)}Fq^{(m)}}\colon q^{(m)}Fq^{(m)} \to C^*(\varphi_{|q^{(m)}Fq^{(m)}}(q^{(m)}Fq^{(m)}))$, we have completely positive contractions $\psi_m\colon A \to q^{(m)}Fq^{(m)}$ and
$\varphi_m\colon C^*(\varphi_{|q^{(m)}Fq^{(m)}}(q^{(m)}Fq^{(m)})) \to P$ such that

(1) $(\psi_m)_{|\rho(B)} = q^{(m)}\psi q^{(m)}$ and
(2) $\|\sum_m (\varphi_m \circ \psi_m)(b_i) - a_i\| < 2n\varepsilon$ for $1 \leq i \leq l$.

Set $\hat{\varphi} = \sum_m \varphi_m$ and $\hat{\psi} = \sum_m \psi_m$. Then $\hat{\varphi}$ is n-decomposable. We can show that $(F, \hat{\varphi}, \hat{\psi})$ is the completely positive approximation for $a_1, a_2, \ldots, a_l$ within

$(2n+1)\varepsilon$. Indeed,

$$\begin{aligned}\|(\hat{\varphi}\circ\hat{\psi})(a_i)-a_i\| &\le \|(\hat{\varphi}\circ\hat{\psi})(a_i-b_i)\|+\|(\hat{\varphi}\circ\hat{\psi})(b_i)-a_i\| \\ &\le \|a_i-b_i\|+\|\sum_m(\varphi_m\circ\psi_m)(b_i)-a_i\| \\ &\le \varepsilon+2n\varepsilon \\ &=(2n+1)\varepsilon\end{aligned}$$

for $1\le i\le l$. Therefore, we conclude that $\mathrm{dr}P\le n$.

(2) By a similar argument to that for (1), we can conclude that $\dim_{\mathrm{nuc}}P\le n$. □

4. Pureness for C^*-algebras

In this section, we consider the pureness for a pair $P\subset A$ of unital C^*-algebras, which is defined in [30], and show that, if the inclusion $P\subset A$ has the Rokhlin property and A is pure, then P is pure.

Definition 4.1. [10, 20] Let $M_\infty(A)$ denote the algebraic limit of the direct system $(M_n(A),\phi_n)$, where $\phi_n\colon M_n(A)\to M_{n+1}(A)$ is given by

$$a\mapsto\begin{pmatrix}a&0\\0&0\end{pmatrix}.$$

Let $M_\infty(A)_+$ (respectively, $M_n(A)_+$) denote the positive elements in $M_\infty(A)$ (respectively, $M_n(A)_+$). Given $a,b\in M_\infty(A)_+$, we say that a is *Cuntz subequivalent* to b (written $a\precsim b$) if there is a sequence $(v_n)_{n=1}^\infty$ of elements in some $M_k(A)$ such that

$$\|v_nbv_n^*-a\|\to 0\ (n\to\infty).$$

We say that a and b are *Cuntz equivalent* if $a\precsim b$ and $b\precsim a$. This relation is an equivalence relation, and we write $\langle a\rangle$ for the equivalence class of a. The set $W(A):=M_\infty(A)_+/\sim$ becomes a positive ordered Abelian semigroup when equipped with the operation

$$\langle a\rangle+\langle b\rangle=\langle a\oplus b\rangle$$

and the partial order

$$\langle a\rangle\le\langle b\rangle\Longleftrightarrow a\precsim b.$$

Let $T(A)$ and $QT(A)$ denote the tracial state space and the space of the normalized 2-quasitraces on A [2, Definition II. 1. 1], respectively. Note that $T(A)\subset QT(A)$ and equality holds when A is exact [6]. Let $S(W(A))$ denote the set of additive and order-preserving maps d from $W(A)$ to $\mathbb{R}^+$ having the property $d(\langle 1_A\rangle)=1$. Such maps are called states. When $d\colon M_\infty(A)_+\to\mathbb{R}_+$ is a dimension function, that is, $d(a\oplus b)=d(a)+d(b)$, and $d(a)\le d(b)$ if $a\precsim b$ for all $a,b\in M_\infty(A)_+$, this gives an additive order-preserving map $\tilde{d}\colon W(A)\to\mathbb{R}^+$ given by $\tilde{d}(\langle a\rangle)=d(a)$ for all $a\in M_\infty(A)_+$.

Given τ in $QT(A)$, one may define a map $d_\tau\colon M_\infty(A)_+ \to \mathbb{R}^+$ by

$$d_\tau(a) = \lim_{n\to\infty} \tau(a^{\frac{1}{n}}).$$

This map is lower semicontinuous and depends only on the Cuntz equivalence class of a. Then $d_\tau \in S(W(A))$. Such states are called *lower semicontinuous dimension functions* and the set of all such functions is denoted by $LDF(A)$. It was proved in [2, Theorem II. 4. 4] that $QT(A)$ is a simplex and the map from $QT(A)$ to $LDF(A)$ by $\tau \mapsto d_\tau$ in the above is bijective and affine by [2, Theorem II. 2. 2].

Definition 4.2. A C^*-algebra A is said to have *strict comparison of positive elements* or simply *strict comparison* if, for all $a, b \in M_\infty(A)_+$, A has the property that $a \preceq b$ whenever $s(a) < s(b)$ for every $s \in LDF(A)$.

Remark 4.3. When A is a simple, unital, C^*-algebra, A has strict comparison if and only if $W(A)$ is *almost unperforated* by [22, Corollary 4.6]. Recall that $W(A)$ is almost unperforated if, for $x, y \in W(A)$ and for all natural numbers n, $(n+1)x \leq ny$ implies that $x \leq y$.

The following should be well known, so we omit its proof.

Lemma 4.4. *Let A be a unital C^*-algebra and suppose that $W(A)$ has strict comparison. Then we have the following:*

1. *For $n \in \mathbb{N}$, $M_n(A)$ has strict comparison.*
2. *For a nonzero hereditary C^*-subalgebra B of A, B has strict comparison.*

Theorem 4.5. *Let A be a unital exact C^*-algebra that has strict comparison. Let $E\colon A \to P$ be of index finite type. Suppose that E has the Rokhlin property. Then we have the following:*

1. *P has strict comparison.*
2. *The basic construction $C^*\langle A, e_P\rangle$ has strict comparison.*

Proof. Note that P is also exact [27, 9]. Hence, we know that $QT(A) = T(A)$ and $QT(P) = T(P)$ by [6].

Since $E \otimes id\colon A \otimes M_n \to P \otimes M_n$ is of index finite type and has the Rokhlin property, it suffices to verify the condition that, whenever $a, b \in P$ are positive elements such that $d_\tau(a) < d_\tau(b)$ for all $\tau \in T(P)$, then $a \preceq b$.

Let $a, b \in P$ be projections such that $d_\tau(a) < d_\tau(b)$ for all $\tau \in T(P)$. Since the restriction of a tracial state on A is a tracial state on P, $d_\tau(a) < d_\tau(b)$ for all $\tau \in T(A)$. Since A has strict comparison, $a \preceq b$ in A.

Since $E\colon A \to P$ has the Rokhlin property, there is an injective $*$-homomorphism $\beta : A \to P^\infty$ such that $\beta(x) = x$ for $x \in P$ by [12] [16, Lemma 2.5]. Then $\beta(a) \preceq \beta(b)$ in P^∞, that is, $a \preceq b$ in P^∞. Hence, $a \preceq b$ in P. Therefore, P has strict comparison.

Since $C^*\langle A, e_P\rangle$ is isomorphic to the corner C^*-algebra $qM_n(P)q$ for some $n \in \mathbb{N}$ and projection $q \in M_n(P)$, from Lemma 4.4 we conclude that $C^*\langle A, e_P\rangle$ has strict comparison. □

Definition 4.6. We say that the order on projections over a unital C^*-algebra A is determined by traces if, whenever $p, q \in M_\infty(A)$ are projections such that $\tau(p) < \tau(q)$ for all $\tau \in T(A)$, then p is the Murray–von Neumann equivalent to a subprojection of q.

Theorem 4.7. *Let A be a unital C^*-algebra such that the order on projections over A is determined by traces. Let $E\colon A \to P$ be of index finite type. Suppose that E has the Rokhlin property. Then the order on projections over P is determined by traces.*

Proof. Note that when p and q are projections, $p \preceq q$ is equivalent to p being the Murray–von Neumann equivalent to a subprojection of q

As in the proof of Theorem 4.5 it suffices to verify the condition that, whenenver $p, q \in P$ are projections such that $d_\tau(p) < d_\tau(q)$ for $\tau \in T(P)$, then $p \preceq q$ in P.

As in the proof of Theorem 4.5, there is an injective $*$-homomorphism $\beta : A \to P^\infty$ sych that $\beta(x) = x$ for $x \in P$. Then $\beta(p) \preceq \beta(q)$ in P^∞, that is, $p \preceq q$ in P^∞. Hence $p \preceq q$ in P. Therefore, we get the conclusion. □

Definition 4.8. Let A be a unital C^*-algebra. A is said to have an *almost divisible* Cuntz semigroup if, for any positive contraction $a \in M_\infty(A)$ and $0 \neq k \in \mathbb{N}$, there is $x \in W(A)$ such that

$$k \cdot x \leq \langle a\rangle \leq (k+1) \cdot x.$$

Proposition 4.9. *Let A be a unital C^*-algebra of stable rank one that has an almost divisible Cuntz semigroup $W(A)$. Let $E\colon A \to P$ be of index finite type. Suppose that E has the Rokhlin property. Then we have the following:*

(1) *P has an almost divisible Cuntz semigroup $W(P)$.*
(2) *The basic construction $C^*\langle A, e_P\rangle$ has an almost divisible Cuntz semigroup $W(C^*\langle A, e_P\rangle)$.*

Proof. (1) Let $a \in M_\infty(P)$ be a positive contraction and $0 \neq k \in \mathbb{N}$. Since A has an almost divisible Cuntz semigroup, there is $x \in W(A)$ such that

$$k \cdot x \leq \langle a\rangle \leq (k+1) \cdot x.$$

It follows from [23, Proposition 5.1] that it is equivalent to there being a unital $*$-homomorphism from the C^*-algebra $Z_{n,n+1}$ into A, where $Z_{k,k+1} = \{f \in C([0,1], M_k \otimes M_{k+1}) \mid f(0) \in M_k \otimes \mathbb{C}, f(1) \in \mathbb{C} \otimes M_{k+1}\}$.

Since the inclusion $P \subset A$ has the Rokhlin property, there is an injective $*$-homomorphism $\beta\colon A \to P^\infty$ such that $\beta(a) = a$ for all $a \in P$. Hence, there is a unital $*$-homomorphism h from $Z_{k,k+1}$ into P^∞. Since $Z_{k,k+1}$ is weakly semiprojective by [8], there are an $m \in \mathbb{N}$ and unital $*$-homomorphism $\tilde{h}$ from $Z_{k,k+1}$ into $\Pi_{n=m}^\infty P$ [13]. Therefore, there is a unital $*$-homomorphism from $Z_{k,k+1}$ into P.

Again from [23, Proposition 5.1], there is $y \in W(P)$ such that

$$k \cdot y \leq \langle a \rangle \leq (k+1) \cdot y.$$

(2) Since $W(A)$ is almost divisible, for any $k \in \mathbb{N}$, there exists a unital $*$-homomorphism $h\colon Z_{k,k+1} \to A$. Hence, there is a unital $*$-homomorphism $\iota \circ h\colon Z \to C^*\langle A, e_P \rangle$. Then we conclude that $W(C^*\langle A, e_P\rangle)$ is almost divisible by [22, Lemma 4.2]. □

Definition 4.10. [30] Let A be a separable unital C^*-algebra. We say that A is pure if $W(A)$ has strict comparison and is almost divisible.

We note that any separable simple unital Jiang-Su absorbing C^*-algebra is pure. It is not yet known whether the converse is true.

Theorem 4.11. *Let A be a separable, unital, exact, pure C^*-algebra of stable rank one; that is, A has strict comparison and $W(A)$ is an almost divisible Cuntz semigroup. Let $E\colon A \to P$ be of index finite type. Suppose that E has the Rokhlin property. Then we have the following:*

1. *P is pure.*
2. *The basic construction $C^*\langle A, e_p \rangle$ is pure.*

Proof. These results follow from Theorem 4.5 and Proposition 4.9. □

Corollary 4.12. *Let A be a separable, simple, unital, exact, pure C^*-algebra of stable rank one and let α be an action of a finite group G on A. Suppose that α has the Rokhlin property. Then A^G and $A \rtimes_\alpha G$ are pure.*

Proof. Since α has the Rokhlin property, the canonical conditional expectation $E\colon A \to A^\alpha$ has the Rokhlin projection e by Proposition 2.11. Since A is simple, a map $A \ni x \mapsto xe$ is injective. This means that E has the Rokhlin property. Hence, the conclusion follows from Theorem 4.11. □

Remark 4.13. From [30, Corollary 6.2], if A is a separable, simple, unital, pure C^*-algebra with locally finite nuclear dimension, then A is $\mathcal{Z}$-absorbing. Hence, it seems that the $\mathcal{Z}$-absorbing property is stable under the condition that a pair of unital C^*-algebras is of index finite type and has the Rokhlin property. Indeed, under this assumption, if A is $\mathcal{D}$-absorbing (i.e., $A \otimes \mathcal{D} \cong A$) for a strongly self-absorbing C^*-algebra $\mathcal{D}$, then P is also $\mathcal{D}$-absorbing [16].

References

1. B. Blackadar, *Symmetries of the CAR algebras*, Ann. Math. **131**(1990), 589–623.
2. B. Blackadar and D. Handelman, *Dimension functions and traces on C^*-algebras*, J. Funct. Anal. **45** (1982), 297–340.
3. S. Barlak and G. Szabo, *Sequentially split $*$-homomorphoisms between C^*-algebras*, Internat. J. Math. **27** (2016), no. 13, 1650105, 48 pp.
4. G. A. Elliott, G. Gong, H. Lin, and Z. Niu, *On the classification of simple amenable C^*-algebras with finite decomposition rank,II*, arXiv:1507.03437.

5. G. Gong, H. Lin, and Z. Niu, *Classification of finite simple amenable $\mathcal{Z}$-stable C^*-algebras*, arXiv:1501.00135.
6. U. Haagerup, *Quasi-traces on exact C^*-algebras are traces*, preprint, 1991.
7. M. Izumi, *Finite group actions on C^*-algebras with the Rohlin property–I*, Duke Math. J. **122** (2004), 233–280.
8. X. Jiang and H. Su, *On a simple unital projectionless C^*-algebra*, Amer. J. Math. **121** (1999), 359–413.
9. E. Kirchberg, *On subalgebras of the CAR-algebra*, J. Funct. Anal. **129** (1995), no. 1, 35–63.
10. E. Kirchberg and M. Rørdam, *Non-simply purely infinite C^*-algebras*, Amer. J. Math. **122** (2000), 637–666.
11. E. Kirchberg and W. Winter, *Covering dimension and quasidiagonality*, Internat. J. Math. **15** (2004), 63–85.
12. K. Kodaka, H. Osaka, and T. Teruya, *The Rohlin property for inclusions of C^*-algebras with a finite Watatani index*, Contemp. Math. **503** (2009), 177–195.
13. T. A. Loring, *Lifting Solutions to Perturbing Problems in C^*-algebras*, Fields Institute Monographs no. 8, American Mathematical Society, Providence RI, 1997.
14. N. Nawata, *Finite group actions on certain stably projectionless C^* -algebras with the Rohlin property*, Trans. Amer. Math. Soc. **368** (2016), no. 1, 471–493.
15. H. Osaka and N. C. Phillips, *Crossed products by finite group actions with the Rokhlin property*, Math. Z. **270** (2012), 19–42.
16. H. Osaka and T. Teruya, *Strongly self-absorbing property for inclusions of C^*-algebras with a finite Watatani index*, Trans. Amer. Math. Soc. **366** (2014), no. 3, 1685–1702.
17. V. I. Paulsen, *Completely bounded maps and dilations*, Pitman Research Notes in Mathematics Series, 146. Longman Scientific & Technical, Harlow; John Wiley & Sons, Inc., New York, 1986.
18. N. C. Phillips, *The tracial Rokhlin property for actions of finite groups on C^*-algebras*, Amer. J. Math. **133** (2011), no. 3, 581–636 (arXiv:math.OA/0609782).
19. N. C. Phillips, *Finite cyclic actions with the tracial Rokhlin property*, Trans. Amer. Math. Soc. **367** (2015), no. 8, 5271–5300. (arXiv:mathOA/0609785).
20. M. Rørdam, *On the structure of simple C^*-algebras tensored with a UHF algebra II*, J. Funct. Anal. **107** (1992), 255–269.
21. M. Rørdam, *Classification of nuclear, simple C^*-algebras*, Encycl. Math. Sci. **126**, Springer, Berlin, 2002.
22. M. Rørdam, *The stable and the real rank of $\mathcal{Z}$-absorbing C^*-algebras*, Int. J. Math. **10** (2004), 1065–1084.
23. M. Rørdam and W. Winter, *The Jiang-Su algebra revised*, J. Reine Angew. Math. **642** (2010), 129–155.
24. L. Santiago, *Crossed products by actions of finite groups with the Rokhlin property*, Int. J. Math. **26** (2015), no. 7, 1550042, 31 pp.
25. T. Teruya, *A characterization of normal extensions for subfactors*, Proc. Amer. Math. Soc. **120** (1994), 781–783.
26. A. Tikusis, S. White, and W. Winter, *Quasidiagonality of nuclear C^*-algebras*, arXiv:1509.08318.
27. S. Wassermann, *Slice map problem for C^*-algebras*, Proc. London Math. Soc. (3), **32** (1976), no. 3, 537–559.
28. Y. Watatani, *Index for C^*-subalgebras*, Mem. Amer. Math. Soc. **83** (1990), no. 424, vi+117 pp.
29. W. Winter, *Covering dimension for nuclear C^*-algebras I*, J. Funct. Anal. **199** (2003), 535–556.
30. W. Winter, *Nuclear dimension and $\mathcal{Z}$-stability of pure C^*-algebras*, Invent. Math. **187** (2012), 259–342 (arXiv:mathOA/1006.2731v2.).
31. W. Winter and J. Zacharias, *The nuclear dimension of C^*-algebras*, Adv. Math. **224** (2010), 461–498.

[1]Department of Mathematical Sciences, Ritsumeikan University, Kusatsu, Shiga 525-8577, Japan.
E-mail address: osaka@se.ritsumei.ac.jp

[2]Faculty of Education, Gunma University, Maebashi, Gunma 371-8510, Japan.
E-mail address: teruya@gunma-u.ac.jp