

ON GENERALIZED SYMMETRIC SQUARE METRICS

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ABSTRACT. In this paper, we study generalized symmetric Finsler spaces with Z. Shen's square metric and Randers change of square metric. We prove that generalized symmetric Finsler spaces with square metric and Randers change of square metric are Riemannian.

2010 *Mathematics Subject Classification*: 53C60, 53C30.

Keywords: (α, β) -metric, generalized symmetric space, square metric.

1. INTRODUCTION

Generalized symmetric spaces are a larger class of homogeneous spaces which contains the symmetric spaces of É. Cartan. The theory of the generalized symmetric spaces was begun by P. J. Graham and A. J. Ledger in 1967. A systematic study appeared for the first time in the book by O. Kowalski in 1980 [5]. These spaces are Riemannian manifolds (M, g) which admit at each point $x \in M$ an isometry s_x with x as an isolated fixed point.

Symmetric Finsler spaces were first introduced and studied by Z. I. Szabó and S. Deng. The definition of these spaces is a natural generalization of É. Cartan's definition of Riemannian symmetric spaces. A Finsler space (M, F) is called symmetric if for any point $p \in M$ there exist an involutive isometry s_p of (M, F) such that p is an isolated fixed point of s_p . If we drop the involution property in the definition of symmetric Finsler spaces but keep the property that $s_x \circ s_y = s_z \circ s_x$, $z = s_x(y)$, we get presumably a bigger class of Finsler manifolds as symmetric Finsler spaces [4, 9].

The notion of (α, β) -metric was introduced by M. Matsumoto [7] as a generalization of Randers metric introduced by G. Randers [8]. An (α, β) -metric is a Finsler metric of the form $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ where $\alpha = \sqrt{\tilde{a}_{ij}(x)y^i y^j}$ is induced by a Riemannian metric $\tilde{a} = \tilde{a}_{ij} dx^i \otimes dx^j$ on a connected smooth n -dimensional manifold M and $\beta = b_i(x)y^i$ is a 1-form on M . Some important class of (α, β) -metrics are

Randers metric $F = \alpha + \beta$, Kropina metric $F = \frac{\alpha^2}{\beta}$, Matsumoto metric $F = \frac{\alpha^2}{(\alpha - \beta)}$ and Z. Shen's square metric $F = \frac{(\alpha + \beta)^2}{\alpha}$.

In this paper, we study generalized symmetric Finsler spaces with Z. Shen's square metric and Randers change of square metric.

2. BASIC SETTINGS

Let M be a smooth n -dimensional C^∞ manifold and TM be its tangent bundle. A Finsler metric on a manifold M is a non-negative function $F : TM \rightarrow R$ with the following properties [2]:

1. F is smooth on the slit tangent bundle $TM^0 := TM \setminus \{0\}$.
2. $F(x, \lambda y) = \lambda F(x, y)$ for any $x \in M$, $y \in T_x M$ and $\lambda > 0$.
3. The $n \times n$ Hessian matrix $(g_{ij}) = (\frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j})$ is positive definite at every point $(x, y) \in TM^0$.

The following bilinear symmetric form $g_y : T_x M \times T_x M \rightarrow R$ is positive definite

$$g_y(u, v) = \frac{1}{2} \frac{\partial^2}{\partial s \partial t} F^2(x, y + su + tv)|_{s=t=0}.$$

Definition 1. Let $\alpha = \sqrt{\tilde{a}_{ij}(x)y^i y^j}$ be a Riemannian metric and $\beta(x, y) = b_i(x)y^i$ be a 1-form on an n -dimensional manifold M . Let

$$\|\beta(x)\|_\alpha := \sqrt{\tilde{a}^{ij}(x)b_i(x)b_j(x)}. \tag{1}$$

Now, let the function F is defined as follows

$$F := \alpha \phi(s) \quad , \quad s = \frac{\beta}{\alpha}, \tag{2}$$

where $\phi = \phi(s)$ is a positive C^∞ function on $(-b_0, b_0)$ satisfying

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0 \quad , \quad |s| \leq b < b_0. \tag{3}$$

Then by lemma 1.1.2 of [3], F is a Finsler metric if $\|\beta(x)\|_\alpha < b_0$ for any $x \in M$. A Finsler metric in the form (2) is called an (α, β) -metric [1].

Let M be a smooth manifold. Suppose that \tilde{a} and β are a Riemannian metric and a 1-form on M respectively. In this case we can write the square metric on M as follows:

$$F = \frac{(\alpha + \beta)^2}{\alpha} = \alpha\phi(s),$$

where $\phi(s) = 1 + s^2 + 2s$. The Riemannian metric \tilde{a} induce a linear isomorphism between T_x^*M and T_xM . Then the 1-form β corresponds to a vector field X on M such that

$$\tilde{a}(X_x, y) = \beta(x, y).$$

Therefore we can write the square metric $F = \frac{(\alpha+\beta)^2}{\alpha}$ as follows:

$$F(x, y) = \frac{(\sqrt{\tilde{a}(y, y)} + \tilde{a}(X_x, y))^2}{\sqrt{\tilde{a}(y, y)}}.$$

Symmetric spaces were first defined and studied by É. Cartan, and they have been generalized to generalized Riemannian symmetric spaces by A. J. Ledger [6]. Generalized symmetric Finsler spaces were first defined in [4]. The notion of generalized symmetric Finsler space is a natural generalization of generalized Riemannian symmetric spaces.

Let (M, F) be a connected Finsler space, and $I(M, F)$ the group of all isometries on (M, F) . An isometry s_x on M with an isolated fixed point x will be called a symmetry at x . A symmetry s_x will be called a symmetry of order k at x if there exists a positive integer k such that $s_x^k = Id$.

Definition 2. [4, 9] A family $\{s_x|x \in M\}$ of symmetries on a connected Finsler manifold (M, F) is called an s -structure on (M, F) .

An s -structure $\{s_x|x \in M\}$ is called of order k ($k \geq 2$) if $s_x^k = Id$ for all $x \in M$ and k is the least integer of this property. An s -structure $\{s_x\}$ on (M, F) is called regular if for every points $x, y \in M$

$$s_x \circ s_y = s_z \circ s_x, \quad z = s_x(y).$$

Definition 3. [4, 9] A generalized symmetric Finsler space is a connected Finsler manifold (M, F) admitting a regular s -structure and a Finsler space (M, F) is said to be k -symmetric ($k \geq 2$) if it admits a regular s -structure of order k .

3. THE MAIN RESULT

Theorem 1. Let (M, F) be a generalized symmetric Finsler space with square metric $F = \frac{(\alpha+\beta)^2}{\alpha}$ defined by the Riemannian metric \tilde{a} and the vector field X . Then the

regular s -structure $\{s_x\}$ of (M, F) is also a regular s -structure of the Riemannian manifold (M, \tilde{a}) .

Proof: Let s_x be a symmetry of (M, F) at x and let $p \in M$. Then for any $Y \in T_p M$ we have

$$F(p, Y) = F(s_x(p), ds_x(Y)).$$

Then we have

$$\frac{(\sqrt{\tilde{a}(Y, Y)} + \tilde{a}(X_p, Y))^2}{\sqrt{\tilde{a}(Y, Y)}} = \frac{(\sqrt{\tilde{a}(ds_x Y, ds_x Y)} + \tilde{a}(X_{s_x(p)}, ds_x Y))^2}{\sqrt{\tilde{a}(ds_x Y, ds_x Y)}}. \quad (4)$$

Applying the above equation to $-Y$, we get

$$\frac{(\sqrt{\tilde{a}(Y, Y)} - \tilde{a}(X_p, Y))^2}{\sqrt{\tilde{a}(Y, Y)}} = \frac{(\sqrt{\tilde{a}(ds_x Y, ds_x Y)} - \tilde{a}(X_{s_x(p)}, ds_x Y))^2}{\sqrt{\tilde{a}(ds_x Y, ds_x Y)}}. \quad (5)$$

Subtracting equation (5) from (4) we get

$$\tilde{a}(X_p, Y) = \tilde{a}(X_{s_x(p)}, ds_x Y). \quad (6)$$

On the other hand, adding equation (5) and (4), we get

$$\frac{\tilde{a}(Y, Y) + \tilde{a}(X_p, Y)^2}{\sqrt{\tilde{a}(Y, Y)}} = \frac{\tilde{a}(ds_x Y, ds_x Y) + \tilde{a}(X_{s_x(p)}, ds_x Y)^2}{\sqrt{\tilde{a}(ds_x Y, ds_x Y)}}. \quad (7)$$

By putting (6) in (7), we get

$$\left(\sqrt{\tilde{a}(ds_x Y, ds_x Y)} - \sqrt{\tilde{a}(Y, Y)} \right) \left(\tilde{a}(X_p, Y)^2 - \sqrt{\tilde{a}(ds_x Y, ds_x Y)} \sqrt{\tilde{a}(Y, Y)} \right) = 0. \quad (8)$$

If there exists $Y \neq 0$ such that $\sqrt{\tilde{a}(ds_x Y, ds_x Y)} > \sqrt{\tilde{a}(Y, Y)}$, then by the Cauchy-Schwartz inequality we have

$$\tilde{a}(X, Y)^2 \leq \tilde{a}(X, X) \tilde{a}(Y, Y) < \sqrt{\tilde{a}(Y, Y)} \sqrt{\tilde{a}(ds_x Y, ds_x Y)}.$$

Thus

$$\tilde{a}(X, Y)^2 - \sqrt{\tilde{a}(ds_x Y, ds_x Y)} \sqrt{\tilde{a}(Y, Y)} < 0.$$

On the other hand from (8) we have

$$\tilde{a}(X, Y)^2 - \sqrt{\tilde{a}(ds_x Y, ds_x Y)} \sqrt{\tilde{a}(Y, Y)} = 0,$$

which is a contradiction. Therefore for any Y and s_x we have

$$\sqrt{\tilde{a}(ds_x Y, ds_x Y)} \leq \sqrt{\tilde{a}(Y, Y)}. \quad (9)$$

Now we have

$$\sqrt{\tilde{a}(Y, Y)} = \sqrt{\tilde{a}(ds_x^{-1} \circ ds_x Y, ds_x^{-1} \circ ds_x Y)} \leq \sqrt{\tilde{a}(ds_x Y, ds_x Y)}. \quad (10)$$

Therefore from (9) and (10) we have

$$\tilde{a}(ds_x Y, ds_x Y) = \tilde{a}(Y, Y).$$

Therefore s_x is a symmetry with respect to the Riemannian metric \tilde{a} . \square

Theorem 2. *Let (M, \tilde{a}) be a generalized symmetric Riemannian space. Also suppose that F is a square Finsler metric defined by \tilde{a} and a vector field X . Then the regular s -structure $\{s_x\}$ of (M, \tilde{a}) is also a regular s -structure of (M, F) if and only if X is s_x -invariant for all $x \in M$.*

Proof: Let X be s_x -invariant. Therefore for any $p \in M$, we have $X_{s_x(p)} = ds_x X_p$. Then for any $Y \in T_p M$ we have

$$\begin{aligned} F(s_x(p), ds_x Y_p) &= \frac{(\sqrt{\tilde{a}(ds_x Y, ds_x Y)} + \tilde{a}(X_{s_x(p)}, ds_x Y))^2}{\sqrt{\tilde{a}(ds_x Y, ds_x Y)}} \\ &= \frac{(\sqrt{\tilde{a}(ds_x Y, ds_x Y)} + \tilde{a}(ds_x X_p, ds_x Y))^2}{\sqrt{\tilde{a}(ds_x Y, ds_x Y)}} \\ &= \frac{(\sqrt{\tilde{a}(Y, Y)} + \tilde{a}(X_p, Y))^2}{\sqrt{\tilde{a}(Y, Y)}} \\ &= F(p, Y). \end{aligned}$$

Conversely, let s_x be a symmetry of (M, F) at x . Then for any $p \in M$ and $Y \in T_p M$ we have

$$F(p, Y) = F(s_x(p), ds_x Y)$$

which implies

$$\begin{aligned} & \left(\tilde{a}(Y, Y) + \tilde{a}(X_p, Y)^2 + 2\sqrt{\tilde{a}(Y, Y)}\tilde{a}(X_p, Y) \right) \sqrt{\tilde{a}(ds_x Y, ds_x Y)} = \\ & \left(\tilde{a}(ds_x Y, ds_x Y) + \tilde{a}(X_{s_x(p)}, ds_x Y)^2 + 2\sqrt{\tilde{a}(ds_x Y, ds_x Y)}\tilde{a}(X_{s_x(p)}, ds_x Y) \right) \sqrt{\tilde{a}(Y, Y)}. \end{aligned} \quad (11)$$

Substituting Y with $-Y$ in (11), we obtain

$$\begin{aligned} & \left(\tilde{a}(Y, Y) + \tilde{a}(X_p, Y)^2 - 2\sqrt{\tilde{a}(Y, Y)}\tilde{a}(X_p, Y) \right) \sqrt{\tilde{a}(ds_x Y, ds_x Y)} = \\ & \left(\tilde{a}(ds_x Y, ds_x Y) + \tilde{a}(X_{s_x(p)}, ds_x Y)^2 - 2\sqrt{\tilde{a}(ds_x Y, ds_x Y)}\tilde{a}(X_{s_x(p)}, ds_x Y) \right) \sqrt{\tilde{a}(Y, Y)}. \end{aligned} \quad (12)$$

Subtracting (12) from (11) we get

$$\tilde{a}(X_p, Y) = \tilde{a}(X_{s_x(p)}, ds_x Y).$$

Therefore $(ds_x)_p X_p = X_{s_x(p)}$. \square

Theorem 3. *A generalized symmetric Finsler square metric must be Riemannian*

Proof: Let (M, F) be a generalized symmetric Finsler space with square metric $F = \frac{(\alpha+\beta)^2}{\alpha}$ defined by the Riemannian metric \tilde{a} and the vector field X and let $\{s_x\}$ be the regular s -structure of (M, F) . Let s_x be a symmetry of (M, F) . Then by the theorem 1, s_x is also a symmetry of (M, \tilde{a}) . Thus we have

$$\begin{aligned} F(x, ds_x Y) &= \frac{(\sqrt{\tilde{a}(ds_x Y, ds_x Y)} + \tilde{a}(X_x, ds_x Y))^2}{\sqrt{\tilde{a}(ds_x Y, ds_x Y)}} \\ &= \frac{(\sqrt{\tilde{a}(Y, Y)} + \tilde{a}(X_x, ds_x Y))^2}{\sqrt{\tilde{a}(Y, Y)}} \\ &= F(x, Y). \end{aligned}$$

Therefore $\tilde{a}(X_x, ds_x Y) = \tilde{a}(X_x, Y)$, $\forall Y \in T_x M$. Since x is an isolated fixed point of the symmetry s_x , the tangent map $S_x = (ds_x)_x$ is an orthogonal transformation of $T_x M$ having no nonzero fixed vectors. So we have

$$\tilde{a}(X_x, (S - id)_x(Y)) = 0, \quad \forall Y \in T_x M.$$

Since $(S - id)_x$ is an invertible linear transformation, we have $X_x = 0$, $\forall x \in M$. Hence F is Riemannian. \square

If $F(\alpha, \beta)$ is a Finsler metric, then $F(\alpha, \beta) \rightarrow \tilde{F}(\alpha, \beta)$ is called a Randers change if

$$\tilde{F}(\alpha, \beta) = F(\alpha, \beta) + \beta.$$

Randers change of a Finsler metric has been introduced by M. Matsumoto [7]. In the following, we deal with the Randers changed square metric

$$F = \frac{(\alpha + \beta)^2}{\alpha} + \beta = \alpha\phi(s), \tag{13}$$

where $\phi(s) = 1 + s^2 + 3s$.

Theorem 4. *Let (M, F) be a generalized symmetric Finsler space with Randers changed square Finsler metric $F = \frac{(\alpha+\beta)^2}{\alpha} + \beta$ defined by the Riemannian metric \tilde{a} and the vector field X . Then the regular s -structure s_x of (M, F) is also a regular s -structure of the Riemannian metric (M, \tilde{a}) .*

Proof: Let s_x be a symmetry of (M, F) and let $p \in M$. Therefore for every $Y \in T_p M$ we have $F(p, Y) = F(s_x(p), ds_x Y)$. Applying the equation (13) we get

$$\frac{(\sqrt{\tilde{a}(Y, Y)} + \tilde{a}(X_p, Y))^2}{\sqrt{\tilde{a}(Y, Y)}} + \tilde{a}(X_p, Y) = \frac{(\sqrt{\tilde{a}(ds_x Y, ds_x Y)} + \tilde{a}(X_{s_x(p)}, ds_x Y))^2}{\sqrt{\tilde{a}(ds_x Y, ds_x Y)}} + \tilde{a}(X_{s_x(p)}, ds_x Y). \quad (14)$$

Substituting Y with $-Y$ in (13), we obtain

$$\frac{(\sqrt{\tilde{a}(Y, Y)} - \tilde{a}(X_p, Y))^2}{\sqrt{\tilde{a}(Y, Y)}} - \tilde{a}(X_p, Y) = \frac{(\sqrt{\tilde{a}(ds_x Y, ds_x Y)} - \tilde{a}(X_{s_x(p)}, ds_x Y))^2}{\sqrt{\tilde{a}(ds_x Y, ds_x Y)}} - \tilde{a}(X_{s_x(p)}, ds_x Y). \quad (15)$$

Subtracting equation (15) from equation (14), we get

$$\tilde{a}(X_p, Y) = \tilde{a}(X_{s_x(p)}, ds_x Y). \quad (16)$$

Adding equation (15) and (14) and using (16) we get

$$(\sqrt{\tilde{a}(ds_x Y, ds_x Y)} - \sqrt{\tilde{a}(Y, Y)}) \left(\frac{\tilde{a}(X, Y)^2}{\sqrt{\tilde{a}(ds_x Y, ds_x Y)} \sqrt{\tilde{a}(Y, Y)}} - 1 \right) = 0, \quad (17)$$

which leads to $\tilde{a}(Y, Y) = \tilde{a}(ds_x Y, ds_x Y)$. Therefore s_x is a symmetry with respect to the Riemannian metric \tilde{a} . \square

Theorem 5. *A generalized symmetric Randers changed square metric must be Riemannian.*

Proof: The proof is similar to the proof of Theorem 3. \square

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