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CANAL SURFACE WHOSE CENTER CURVE IS A SPHERICAL CURVE WITH SPHERICAL FRAME

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Abstract. In this paper, we obtain the parametrization of the canal surfaces whose center curves are the spherical curves on the sphere S^2 in $\mathbb{E}^3$. The parametrization of the canal surface is expressed according to the spherical orthonormal frame given in [8]. Then the parallel surface of this surface is studied. Also we define the notion of the associated canal surface. Lastly we give the geometric properties of these surfaces such that Weingarten surface, (X, Y)-Weingarten surface and linear Weingarten surface.

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1. Introduction

Canal surfaces was firstly investigated by Monge in 1850. A canal surface is defined as a surface formed as the envelope of a family of spheres whose centers lie on a space curve $C(t)$ with radius $r(t)$. If the radius $r(t)$ is constant, then the canal surface is called as pipe surface or tubular surface. Canal surfaces play an essential role in descriptive geometry, because in case of an orthographic projection its contour curve can be drawn as the envelope of circles. In technical area canal surfaces can be used for blending surfaces smoothly. Canal surface is useful to represent various objects e.g. pipe, hose, rope or intestine of a body. Moreover, canal surface is an important instrument in surface modelling for CAD/CAM such as tubular surfaces, torus and Dupin cyclides [5].

Canal surfaces and tubular surfaces have been studied by many researchers. In [3], [4], [5], [6], the authors study canal surfaces and tubular surfaces in Euclidean 3-space, Minkowski 3-space, Galilean and Pseudo Galilean spaces. Lately, in [10], the authors consider the new approach to canal surfaces. Also in [2] and [7], the authors study canal surfaces with quaternions.

In [8], the author defines the spherical orthonormal frame of the curves on the sphere S^2.

In this paper, we obtain the parametrization of the canal surfaces whose center curves are the spherical curves on the sphere S^2 in $\mathbb{E}^3$. The parametrization of the canal surface is expressed according to the spherical orthonormal frame given in [8]. Then the parallel surface of this surface is studied. Also we define the notion of the associated canal surface. Lastly we give the geometric properties of these surfaces such that Weingarten surface, (X,Y)-Weingarten surface and linear Weingarten surface.

2. Preliminaries

Let $m \in \mathbb{E}^3$ be a fixed point and $r > 0$ be a constant. Then the sphere is defined by

$$S^2(m,r) = \{u \in \mathbb{E}^3 : \langle u - m, u - m \rangle = r^2\}.$$

We use $S^2(0,1) = S^2$ and $S^2(0,r) = S^2(r)$ throughout this article.

For a unit speed regular curve $x(s) \subset S^2 \subset \mathbb{E}^3$, we choose $\{x(s), \alpha(s), y(s)\}$ forming a standart orthonormal basis of $\mathbb{E}^3$. Then the spherical Frenet formulas of the spherical curve $x(s)$ on S^2 can be written as

$$x'(s) = \alpha(s), \quad \alpha'(s) = -x(s) + \kappa(s) y(s), \quad y'(s) = -\kappa(s) \alpha(s). \tag{1}$$

Here, the function $\kappa(s)$ is called the spherical curvature function (or curvature) of $x(s)$ and the frame $\{x(s), \alpha(s), y(s)\}$ is called the spherical Frenet frame of the spherical curve $x(s)$ ([8]).

We recall some well-known formulas for the surfaces in $\mathbb{E}^3$. Let M be a surface of $\mathbb{E}^3$, the standart connection D on $\mathbb{E}^3$ induces the Levi-Civita connection $\bigtriangledown$ on M. We have the following Gauss formula

$$D_X Y = \nabla_X Y + h(X,Y),$$

and the Weingarten formula

$$D_X \xi = -A_\xi X + {}^{\perp}\nabla_X \, \xi,$$

where $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(TM^{\perp})$. Then ∇ is the Levi-Civita connection of M, h is the second fundamental form, A_ξ is the shape operator, and ${}^{\perp}\nabla$ is the normal connection. We note that

$$\langle h(X,Y), \xi \rangle = \langle A_\xi X, Y \rangle.$$

The mean curvature vector field $\overrightarrow{H}$, the mean curvature H and the Gauss curvature of M are given respectively by

$$\overrightarrow{H} = \frac{1}{2}(h(e_1, e_1) + h(e_2, e_2)), \quad H = \left\| \overrightarrow{H} \right\| \quad \text{and} \quad K = \det A$$

where $\{e_1, e_2\}$ is an orthonormal basis on M ([1]).

Let U be the unit normal vector field on a surface $M(s,t)$ defined by

$$U = \frac{M_s \times M_t}{\|M_s \times M_t\|}.$$

The second fundamental form II of a surface $M(s,t)$ is given as

$$II = eds^2 + 2fdsdt + gdt^2$$

where

$$e = g(M_{ss}, U), \ f = g(M_{st}, U), \ g = g(M_{tt}, U).$$

([11]) Thus the second Gaussian curvature K_{II} of a surface is given as

$$\begin{aligned} K_{II} &= \frac{1}{(eg - f^2)^2} \left\{ \begin{vmatrix} -\frac{1}{2}e_{tt} + f_{st} - \frac{1}{2}g_{ss} & \frac{1}{2}e_s & f_s - \frac{1}{2}e_t \\ f_t - \frac{1}{2}g_s & e & f \\ \frac{1}{2}g_t & f & g \end{vmatrix} \right. \\ &\quad \left. - \begin{vmatrix} 0 & \frac{1}{2}e_t & \frac{1}{2}g_s \\ \frac{1}{2}e_t & e & f \\ \frac{1}{2}g_s & f & g \end{vmatrix} \right\}. \end{aligned}$$

3. Canal surface whose center curve is the spherical curve on S^2

In this section, we consider the canal surfaces whose center curve is the spherical curves on S^2.

Theorem 1. *Let $x(s)$ be a spherical curve with arc-length parameter s on S^2 and be the center curve of a canal surface obtained from the sphere $S^2(r)$. Then*
(i) the parametrization of the canal surface can be as following

$$\begin{aligned} M(s,t) &= \left(1 + m_1 r(s) \sqrt{1 - r_s^2(s)} \sin t\right) x(s) - r(s) r_s(s) \alpha(s) \\ &\quad + \left(m_2 r(s) \sqrt{1 - r_s^2(s)} \cos t\right) y(s) \end{aligned}$$

(ii) the parametrization of the tubular surface can be as following

$$M(s,t) = (1 + m_1 r \sin t) x(s) + (m_2 r \cos t) y(s)$$

where $m_1, m_2 \in \{-1, 1\}$.

Proof. Let $x(s)$ be a spherical curve with arc-length parameter s on S^2. Assume that M be a parametrization of the envelope of the sphere $S^2(r)$ defining the canal surface and the center curve $x(s)$. Then M can be parametrized as

$$M(s,t) - x(s) = a(s,t)\,x(s) + b(s,t)\,\alpha(s) + c(s,t)\,y(s) \tag{2}$$

where a, b and c are differentiable functions of s and t on the interval I on which x is defined. Moreover, since $M(s,t)$ lies on the sphere $S^2(r)$, we can write

$$\langle M(s,t) - x(s), M(s,t) - x(s)\rangle = r^2. \tag{3}$$

which leads to that

$$a^2 + b^2 + c^2 = r^2 \tag{4}$$

$$aa_s + bb_s + cc_s = rr_s \tag{5}$$

where a_s, b_s, c_s, r_s refer to the derivative of the functions with respect to s.

Differentiating (2) with respect to s and using (1), we get

$$M_s = (a_s - b)x + (1 + a + b_s - c\kappa)\,\alpha + (b\kappa + c_s)\,y \tag{6}$$

where M_s refers to the derivative of M with respect to s. Furthermore, $M(s,t) - x(s)$ is a normal vector to the canal surfaces, which implies that

$$\langle M(s,t) - x(s), M_s\rangle = 0, \tag{7}$$

Then, from (7), (2), (5) and (4), we obtain

$$b = -rr_s, \tag{8}$$

$$a^2 + c^2 = r^2\left(1 - r_s^2\right). \tag{9}$$

which let us take

$$\begin{aligned} a &= \pm r\sqrt{1 - r_s^2}\,\sin t, \\ c &= \pm r\sqrt{1 - r_s^2}\,\cos t. \end{aligned}$$

Then the proof of (i) is complete. If we take r as a constant, we get the proof of (ii).

In the following theorem, we classify all spherical curve on S^2 with constant curvature.

Theorem 2. *Let κ be a real number. Then $x(s)$ is a spherical curve on S^2 with arc-length parameter s and curvature κ if and only if $x(s)$ can be parameterized by*

$$x = \cos\left(\sqrt{1+\kappa^2}s\right)V_1 + \sin\left(\sqrt{1+\kappa^2}s\right)V_2 + V_3$$

where V_1, V_2, V_3 are mutually orthogonal vectors satisfying the following equations

$$\langle V_1, V_1\rangle = \langle V_2, V_2\rangle = \frac{1}{1+\kappa^2} \quad \textit{and} \quad \langle V_3, V_3\rangle = \frac{\kappa^2}{1+\kappa^2}.$$

Proof. Let $x(s)$ be a spherical curve on S^2 with arc-length parameter s and constant curvature κ. By using the spherical Frenet equations, we obtain the following homogeneous differential equation with constant coefficients

$$x''' + \left(1+\kappa^2\right)x' = 0.$$

The characteristic equation of the previous equation is follows

$$r\left(r^2 + \left(1+\kappa^2\right)\right) = 0.$$

Then we get

$$x = \cos\left(\sqrt{1+\kappa^2}s\right)V_1 + \sin\left(\sqrt{1+\kappa^2}s\right)V_2 + V_3. \tag{10}$$

Differentiating (10) with respect to s, we get

$$\alpha = -\sqrt{1+\kappa^2}\sin\left(\sqrt{1+\kappa^2}s\right)V_1 + \sqrt{1+\kappa^2}\cos\left(\sqrt{1+\kappa^2}s\right)V_2.$$

By using $\langle\alpha,\alpha\rangle = 1$, we get V_1, V_2, V_3 are mutually orthogonal vectors satisfying the following equations

$$\langle V_1, V_1\rangle = \langle V_2, V_2\rangle = \frac{1}{1+\kappa^2} \quad \text{and} \quad \langle V_3, V_3\rangle = \frac{\kappa^2}{1+\kappa^2}.$$

Then the proof is complete.

Example 1. *Let us take $\kappa = 1$ in Theorem 2. Then we obtain*

$$\langle V_1, V_1\rangle = \langle V_2, V_2\rangle = \langle V_3, V_3\rangle = \frac{1}{2}.$$

Then we can choose

$$V_1 = \left(\frac{1}{\sqrt{2}}, 0, 0\right), \quad V_2 = \left(0, \frac{1}{\sqrt{2}}, 0\right), \quad V_3 = \left(0, 0, \frac{1}{\sqrt{2}}\right),$$

which implies that

$$\begin{aligned} x &= \left(\frac{\cos\left(\sqrt{2}s\right)}{\sqrt{2}}, \frac{\sin\left(\sqrt{2}s\right)}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \\ \alpha &= \left(-\sin\left(\sqrt{2}s\right), \cos\left(\sqrt{2}s\right), 0\right), \\ y &= \left(-\frac{\cos\left(\sqrt{2}s\right)}{\sqrt{2}}, -\frac{\sin\left(\sqrt{2}s\right)}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right). \end{aligned}$$

Now let us take $m_1 = m_2 = 1$ in Theorem 1 and give the canal surfaces with $r = 2$ and $r = s^2$ (Figure 1).

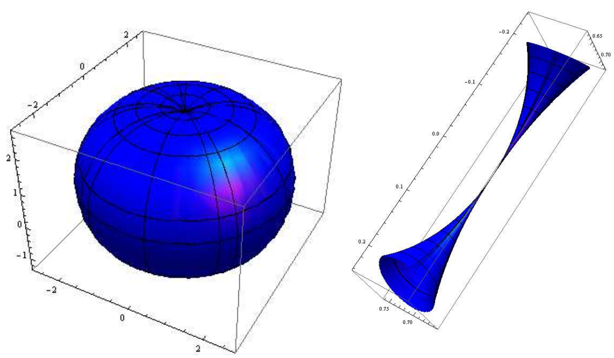

Figure 1: The canal surface for $r = 2$ (left) and $r = s^2$ (right)

4. Tubular surface whose center curve is the spherical curve

In this section we consider the tubular surface whose center curve is the spherical curve in S^2, which is parameterized by

$$M(s,t) = (1 + m_1 r \sin t)\, x(s) + (m_2 r \cos t)\, y(s)$$

where $m_1, m_2 \in \{-1, 1\}$ and $r \in \mathbb{R}$. By taking $m_1 = m_2 = 1$, we have

$$\psi(s,t) = (1 + r \sin t)\, x(s) + (r \cos t)\, y(s). \tag{11}$$

From (11), we find

$$\begin{aligned} \psi_s &= (1 + r \sin t - r\kappa \cos t)\, \alpha, \\ \psi_t &= (r \cos t)\, x - (r \sin t)\, y. \end{aligned}$$

We can find the components of first fundemental form as follows

$$g_{11} = \langle \psi_s, \psi_s \rangle = (1 + r\sin t - r\kappa \cos t)^2, \quad g_{12} = \langle \psi_s, \psi_t \rangle = 0, \quad g_{22} = \langle \psi_t, \psi_t \rangle = r^2.$$

Then $g_{11}g_{22} - (g_{12})^2 = r^2 (1 + r\sin t - r\kappa \cos t)^2$. We assume that $1 + r\sin t - r\kappa \cos t > 0$ for the regularity of the surface ψ.

Now we will give an orthonormal basis on $\psi(s,t)$.

$$\begin{aligned} e_1 &= \frac{1}{\|\psi_s\|}\psi_s = \alpha, \\ e_2 &= \frac{1}{\|\psi_t\|}\psi_t = (\cos t)\, x - (\sin t)\, y, \end{aligned}$$

where $\{e_1, e_2\}$ is an orthonormal frame field on $\psi(s,t)$. Set

$$e_3 = -(\sin t)\, x - (\cos t)\, y,$$

where e_3 is a normal vector field to $\psi(s,t)$. $\{e_1, e_2, e_3\}$ is an orthonormal basis on $\psi(s,t)$. Then we obtain

$$\begin{aligned} D_{e_1}e_1 &= \frac{1}{1 + r\sin t - r\kappa \cos t}(-x + \kappa y), \\ D_{e_1}e_2 &= \frac{\cos t + \kappa \sin t}{1 + r\sin t - r\kappa \cos t}\alpha, \\ D_{e_2}e_2 &= \frac{1}{r}(-(\sin t)\, x - (\cos t)\, y). \end{aligned}$$

The components of the second fundamental form h are calculated as follows

$$\begin{aligned} h_{11} &= \langle D_{e_1}e_1, e_3 \rangle = \frac{\sin t - \kappa \cos t}{1 + r\sin t - r\kappa \cos t}, \\ h_{12} &= \langle D_{e_1}e_2, e_3 \rangle = 0 \quad \text{and} \quad h_{22} = \langle D_{e_2}e_2, e_3 \rangle = \frac{1}{r}. \end{aligned}$$

Theorem 3. *The mean curvature H of $\psi(s,t)$ is obtained as*

$$H = \frac{1}{2}(h_{11} + h_{22}) = \frac{1 - 2r\kappa \cos t + 2r\sin t}{2r(1 + r\sin t - r\kappa \cos t)}. \tag{12}$$

Theorem 4. *The Gauss curvature K of $\psi(s,t)$ is obtained as*

$$K = h_{11}h_{22} - (h_{12})^2 = \frac{\sin t - \kappa \cos t}{r(1 + r\sin t - r\kappa \cos t)}. \tag{13}$$

A surface is called Weingarten surface if there exist a non-trivial function $\Psi(K,H)$ such that $\Psi(K,H) = K_sH_t - K_tH_s = 0$ for the Gauss curvature K and mean curvature H of the surface. Here subscripts denote partial derivatives. Also we a surface is called as a linear Weingarten surface if there exist real numbers a, b, $c \in \mathbb{R}\backslash\{0\}$ such that the linear combination $aK + bH = c$ is satisfied. For $(X,Y) \in \{(K, K_{II}), (H, K_{II})\}$, the surface is called as (X,Y)-Weingarten surface if $\Psi(X,Y) = 0$ ([9]).

From (12) and (13), we have

$$K_s = \frac{-\kappa' \cos t}{r(1 + r\sin t - r\kappa \cos t)^2}, \quad K_t = \frac{\cos t + \kappa \sin t}{r(1 + r\sin t - r\kappa \cos t)^2}$$

and

$$H_s = \frac{-\kappa' \cos t}{2(1 + r\sin t - r\kappa \cos t)^2}, \quad H_t = \frac{\cos t + \kappa \sin t}{2(1 + r\sin t - r\kappa \cos t)^2}.$$

Thus it can be easily seen that $\Psi(K,H) = K_sH_t - K_tH_s = 0$. So we can give the following theorem.

Theorem 5. *The surface $\psi(s,t)$ is a Weingarten surface.*

Now assume that there exist real numbers a, b, $c \in \mathbb{R}\backslash\{0\}$ such that the linear combination $aK + bH = c$ is satisfied.

$$aK + bH - c = \frac{b - 2cr + 2\left(a - cr^2 + br\right)\sin t - 2\left(a - cr^2 + br\right)\kappa \cos t}{2r(1 + r\sin t - r\kappa \cos t)} = 0$$

which implies that $b = 2cr$ and $a + cr^2 = 0$. So we can give the following theorem.

Theorem 6. *Let K and H be the Gauss curvature and mean curvature of the surface $\psi(s,t)$. Then there exists the following relation between K and H :*

$$-r^2K + 2rH = 1$$

where r is a positive real number.

From above theorem, we get the following corollary.

Corollary 7. *The surface $\psi(s,t)$ is a linear Weingarten surface.*

Definition 1. *The parallel surface of the surface $X(s,t)$ defined by*

$$X^*(s,t) = X(s,t) + \mu U(s,t)$$

where

$$U(s,t) = \frac{X_s \times X_t}{\|X_s \times X_t\|}$$

is the unit normal vector of the surface $X(s,t)$ and $\mu \in \mathbb{R}$.

Now we will define the parallel surface $\psi^*(s,t)$ of the surface $\psi(s,t)$ as follows

$$\begin{aligned}\psi^*(s,t) &= \psi(s,t)+\mu e_3 \\ &= (1+(r-\mu)\sin t)\,x(s)+((r-\mu)\cos t)\,y(s)\end{aligned} \tag{14}$$

From (14), we find

$$\begin{aligned}\psi_s^* &= (1+(r-\mu)\sin t-(r-\mu)\,\kappa\cos t)\,\alpha,\\ \psi_t^* &= ((r-\mu)\cos t)\,x-((r-\mu)\sin t)\,y.\end{aligned}$$

We can find the components of first fundemental form as follows

$$\begin{aligned}g_{11}^* &= \langle\psi_s^*,\psi_s^*\rangle=(1+(r-\mu)\sin t-(r-\mu)\,\kappa\cos t)^2\,,\\ g_{12}^* &= \langle\psi_s^*,\psi_t^*\rangle=0,\quad g_{22}=\langle\psi_t,\psi_t\rangle=(r-\mu)^2\,.\end{aligned}$$

Then $g_{11}^*g_{22}^*-(g_{12}^*)^2=(r-\mu)^2\,(1+(r-\mu)\sin t-(r-\mu)\,\kappa\cos t)^2$. We assume that $r-\mu>0$ and $1+(r-\mu)\sin t-(r-\mu)\,\kappa\cos t>0$ for the regularity of the surface $\psi^*(s,t)$.

Now we will give an orthonormal basis on $\psi^*(s,t)$.

$$\begin{aligned}e_1^* &= \frac{1}{\|\psi_s^*\|}\psi_s^*=\alpha,\\ e_2^* &= \frac{1}{\|\psi_t^*\|}\psi_t^*=(\cos t)\,x-(\sin t)\,y,\end{aligned}$$

where $\{e_1^*,e_2^*\}$ is an orthonormal frame field on $\psi^*(s,t)$. Set

$$e_3^*=-(\sin t)\,x-(\cos t)\,y,$$

where e_3^* is a normal vector field to $\psi^*(s,t)$. $\{e_1^*,e_2^*,e_3^*\}$ is an orthonormal basis on $\psi^*(s,t)$. Then we obtain

$$\begin{aligned}D_{e_1^*}e_1^* &= \frac{1}{1+(r-\mu)\sin t-(r-\mu)\,\kappa\cos t}\,(-x+\kappa y)\,,\\ D_{e_1^*}e_2^* &= \frac{\cos t+\kappa\sin t}{1+(r-\mu)\sin t-(r-\mu)\,\kappa\cos t}\alpha,\\ D_{e_2^*}e_2^* &= \frac{1}{(r-\mu)}\,(-(\sin t)\,x-(\cos t)\,y)\,.\end{aligned}$$

The components of the second fundamental form h^* are calculated as follows

$$\begin{aligned}h_{11}^* &= \left\langle D_{e_1^*}e_1^*,e_3^*\right\rangle=\frac{\sin t-\kappa\cos t}{1+(r-\mu)\sin t-(r-\mu)\,\kappa\cos t},\\ h_{12}^* &= \left\langle D_{e_1^*}e_2^*,e_3^*\right\rangle=0\quad\text{and}\quad h_{22}=\left\langle D_{e_2^*}e_2^*,e_3^*\right\rangle=\frac{1}{(r-\mu)}.\end{aligned}$$

Similarly we can find the following results.

Theorem 8. *The mean curvature H^* of $\psi^*(s,t)$ is obtained as*

$$H^* = \frac{1 - 2(r-\mu)\kappa\cos t + 2(r-\mu)\sin t}{2(r-\mu)(1 + (r-\mu)\sin t - (r-\mu)\kappa\cos t)}.$$

Theorem 9. *The Gauss curvature K^* of $\psi^*(s,t)$ is obtained as*

$$K^* = \frac{\sin t - \kappa\cos t}{(r-\mu)(1 + (r-\mu)\sin t - (r-\mu)\kappa\cos t)}.$$

Theorem 10. *The surface $\psi^*(s,t)$ is a Weingarten surface.*

Now assume that there exist real numbers a, b, $c \in \mathbb{R}\backslash\{0\}$ such that the linear combination $aK^* + bH^* = c$ is satisfied.

$$\begin{aligned} & aK^* + bH^* - c \\ = & \frac{b - 2c(r-\mu) + 2\left(a - c(r-\mu)^2 + b(r-\mu)\right)(\sin t - \kappa\cos t)}{2r(1 + r\sin t - r\kappa\cos t)} \\ = & \ 0 \end{aligned}$$

which implies that $b = 2c(r-\mu)$ and $a + c(r-\mu)^2 = 0$. So we can give the following theorem.

Theorem 11. *Let K^* and H^* be the Gauss curvature and mean curvature of the surface $\psi^*(s,t)$. Then there exists the following relation between K^* and H^* :*

$$-(r-\mu)^2 K^* + 2(r-\mu)H^* = 1$$

where r is a positive real number and μ is a real number.

From above theorem, we get the following corollary.

Corollary 12. *The surface $\psi^*(s,t)$ is a linear Weingarten surface.*

5. Associated canal surfaces

In this section, we will give the definition of the associated canal surfaces.

In [8], the author defines the associated curve of the spherical curve $x(s)$ in S^2 with the spherical frame $\{x(s), \alpha(s), y(s)\}$. Let $x_1(\overline{s})$ be the associated curve of $x(s)$ such that $x_1(\overline{s}) = y(s)$ where there exists a diffeomorfism $\overline{s} = f_1(s)$. In this paper, we will call $x_1(\overline{s}) = y(s)$ as **the first associated curve** of the spherical curve $x(s)$.

Let $x_2(s^*) = \alpha(s)$ where there exists a diffeomorfism $s^* = f_2(s)$ Then we will call $x_2(s^*) = \alpha(s)$ as **the second associated curve** of the spherical curve $x(s)$.

So we can give the following corollaries.

Corollary 13. *Let $x_1(\overline{s})$ be the first associated curve of the spherical curve $x(s)$ in S^2 with the spherical frame $\{x(s), \alpha(s), y(s)\}$ such that $x_1(\overline{s}) = y(s)$ where there exists a diffeomorfism $\overline{s} = f_1(s)$. Then we have*

$$x_1 = y, \quad \alpha_1 = -\alpha, \quad y_1 = x, \quad \kappa_1 = \frac{1}{\kappa}, \quad \frac{df_1}{ds} = \kappa,$$

where $\{x_1(\overline{s}), \alpha_1(\overline{s}), y_1(\overline{s})\}$ is the spherical frame of $x_1(\overline{s})$ and $\kappa_1(\overline{s})$ is the spherical curvature of $x_1(\overline{s})$.

Corollary 14. *Let $x_2(s^*)$ be the second associated curve of the spherical curve $x(s)$ in S^2 with the spherical frame $\{x(s), \alpha(s), y(s)\}$ such that $x_2(s^*) = y(s)$ where there exists a diffeomorfism $s^* = f_2(s)$. Then we have*

$$\begin{aligned} x_2 &= \alpha, \quad \alpha_2 = \frac{1}{\sqrt{1+\kappa^2}}(-x + \kappa y), \quad y_2 = \frac{1}{\sqrt{1+\kappa^2}}(\kappa x + y), \\ \kappa_2 &= \frac{\kappa'}{(1+\kappa^2)^{3/2}}, \quad \frac{df_2}{ds} = \sqrt{1+\kappa^2}, \end{aligned}$$

where $\{x_2(s^*), \alpha_2(s^*), y_2(s^*)\}$ is the spherical frame of $x_2(s^*)$ and $\kappa_2(s^*)$ is the spherical curvature of $x_2(s^*)$.

Definition 2. *Let $x_1(\overline{s})$ be the first associated curve of the spherical curve $x(s)$ in S^2, $\psi(s,t)$ and $\psi_1(\overline{s},t)$ be canal surfaces (or tubular surfaces) whose center curves are $x(s)$ and $x_1(\overline{s})$, respectively. Then $\psi_1(\overline{s},t)$ is called as* ***"the first associated canal surface (or the first associated tubular surface)"*** *of $\psi(s,t)$.*
Similarly, let $x_2(s^*)$ be the second associated curve of the spherical curve $x(s)$ in S^2, $\psi(s,t)$ and $\psi_2(s^*,t)$ be canal surfaces (or tubular surfaces) whose center curves are $x(s)$ and $x_2(s^*)$, respectively. Then $\psi_2(s^*,t)$ is called as ***"the second associated canal surface (or the second associated tubular surface)"*** *of $\psi(s,t)$.*

Firstly, we consider the first associated tubular surface of $\psi(s,t)$. Let $\psi_1(\overline{s},t)$ be the first associated tubular surface of $\psi(s,t)$. Then we can write

$$\begin{aligned} \psi_1(\overline{s},t) &= (1 + r\sin t)\, x_1(\overline{s}) + (r\cos t)\, y_1(\overline{s}) \\ &= (r\cos t)\, x(s) + (1 + r\sin t)\, y(s). \end{aligned} \tag{15}$$

From (15), we have

$$\begin{aligned} (\psi_1)_{\overline{s}} &= \frac{1}{\kappa}(r\cos t - (1 + r\sin t))\, \alpha, \\ (\psi_1)_t &= -(r\sin t)\, x + (r\cos t)\, y, \end{aligned}$$

which implies that

$$\begin{aligned}\langle(\psi_1)_{\overline{s}},(\psi_1)_{\overline{s}}\rangle &= \frac{(r\cos t-(1+r\sin t)\,\kappa)^2}{\kappa^2},\\ \langle(\psi_1)_{\overline{s}},(\psi_1)_t\rangle &= 0,\quad \langle(\psi_1)_t,(\psi_1)_t\rangle = r^2.\end{aligned}$$

Then

$$\langle(\psi_1)_{\overline{s}},(\psi_1)_{\overline{s}}\rangle\,\langle(\psi_1)_t,(\psi_1)_t\rangle-\langle(\psi_1)_{\overline{s}},(\psi_1)_t\rangle^2 = r^2\frac{(r\cos t-(1+r\sin t)\,\kappa)^2}{\kappa^2}.$$

Theorem 15. *Let $\psi_1(\overline{s},t)$ be the first associated tubular surfaces of $\psi(s,t)$. Then $\psi_1(\overline{s},t)$ has a singular point at $\psi(s_0,t_0)$ if and only if*

$$r\cos t_0-(1+r\sin t_0)\,\kappa(s_0)=0.$$

Now we assume that $r\cos t-(1+r\sin t)\,\kappa\neq 0$ for all (t,s). Then we will give an orthonormal basis on $\psi_1(\overline{s},t)$.

$$\begin{aligned}\overline{e}_1 &= \frac{1}{\|(\psi_1)_{\overline{s}}\|}(\psi_1)_{\overline{s}}=\varepsilon_1\alpha,\\ \overline{e}_2 &= \frac{1}{\|(\psi_1)_t\|}(\psi_1)_t=-(\sin t)\,x+(\cos t)\,y,\end{aligned}$$

where $\varepsilon_1=sgn\,(r\cos t-(1+r\sin t)\,\kappa)$ and $\{\overline{e}_1,\overline{e}_2\}$ is an orthonormal frame field on $\psi_1(\overline{s},t)$. Set

$$\overline{e}_3=-(\cos t)\,x-(\sin t)\,y,$$

where $\overline{e}_3$ is a normal vector field to $\psi_1(\overline{s},t)$. $\{\overline{e}_1,\overline{e}_2,\overline{e}_3\}$ is an orthonormal basis on $\psi_1(\overline{s},t)$. Then we obtain

$$\begin{aligned}D_{\overline{e}_1}\overline{e}_1 &= \frac{1}{r\cos t-(1+r\sin t)\,\kappa}(-x+\kappa y),\\ D_{\overline{e}_1}\overline{e}_2 &= \frac{-\varepsilon_1(\kappa\cos t+\sin t)}{r\cos t-(1+r\sin t)\,\kappa}\alpha,\\ D_{\overline{e}_2}\overline{e}_2 &= \frac{1}{r}(-(\cos t)\,x-(\sin t)\,y).\end{aligned}$$

The components of the second fundamental form $\overline{h}$ are calculated as follows

$$\begin{aligned}\overline{h}_{11} &= \langle D_{\overline{e}_1}\overline{e}_1,\overline{e}_3\rangle=\frac{\cos t-\kappa\sin t}{r\cos t-(1+r\sin t)\,\kappa},\\ \overline{h}_{12} &= \langle D_{\overline{e}_1}\overline{e}_2,\overline{e}_3\rangle=0\quad\text{and}\quad \overline{h}_{22}=\langle D_{\overline{e}_2}\overline{e}_2,\overline{e}_3\rangle=\frac{1}{r}.\end{aligned}$$

Theorem 16. *The mean curvature H_1 of $\psi_1(\overline{s},t)$ is obtained as*

$$H_1 = \frac{1}{2}\left(\overline{h}_{11} + \overline{h}_{22}\right) = \frac{2r\cos t - \kappa\left(1 + 2r\sin t\right)}{2r\left(r\cos t - \left(1 + r\sin t\right)\kappa\right)}. \tag{16}$$

Theorem 17. *The Gauss curvature K_1 of $\psi_1(\overline{s},t)$ is obtained as*

$$K_1 = \overline{h}_{11}\overline{h}_{22} - \left(\overline{h}_{12}\right)^2 = \frac{\cos t - \kappa\sin t}{r\left(r\cos t - \left(1 + r\sin t\right)\kappa\right)}. \tag{17}$$

From (16) and (17), we have

$$(K_1)_{\overline{s}} = \frac{\kappa'\cos t}{r\kappa\left(r\cos t - \left(1 + r\sin t\right)\kappa\right)^2}, \quad (K_1)_t = \frac{\kappa\left(\sin t + \kappa\cos t\right)}{r\left(r\cos t - \left(1 + r\sin t\right)\kappa\right)^2}$$

and

$$(H_1)_{\overline{s}} = \frac{\kappa'\cos t}{2\kappa\left(r\cos t - \left(1 + r\sin t\right)\kappa\right)^2}, \quad (H_1)_t = \frac{\kappa\left(\sin t + \kappa\cos t\right)}{2\left(r\cos t - \left(1 + r\sin t\right)\kappa\right)^2}.$$

Thus it can be easily seen that $\Psi(K_1, H_1) = (K_1)_{\overline{s}}(H_1)_t - (K_1)_t(H_1)_{\overline{s}} = 0$. So we can give the following theorem.

Theorem 18. *The surface $\psi_1(\overline{s},t)$ is a Weingarten surface.*

Now assume that there exist real numbers a, b, $c \in \mathbb{R}\backslash\{0\}$ such that the linear combination $aK_1 + bH_1 = c$ is satisfied.

$$aK_1 + bH_1 - c = \frac{2\left(a - cr^2 + br\right)\cos t - \left(b - 2cr + 2\left(a - cr^2 + br\right)\right)\kappa\sin t}{2r\left(r\cos t - \left(1 + r\sin t\right)\kappa\right)} = 0$$

which implies that $b = 2cr$ and $a + cr^2 = 0$. So we can give the following theorem.

Theorem 19. *Let K_1 and H_1 be the Gauss curvature and mean curvature of the surface $\psi_1(\overline{s},t)$. Then there exists the following relation between K_1 and H_1:*

$$-r^2K_1 + 2rH_1 = 1$$

where r is a positive real number.

From above theorem, we get the following corollary.

Corollary 20. *The surface $\psi_1(\overline{s},t)$ is a linear Weingarten surface.*

Now, we consider the second associated tubular surface of $\psi(s,t)$. Assume that $\kappa(s)=\kappa$ (constant). Let $\psi_2(s^*,t)$ be the second associated tubular surface of $\psi(s,t)$. Then we can write

$$\begin{aligned}\psi_2(s^*,t) &= (1+r\sin t)\,x_2(s^*)+(r\cos t)\,y_2(s^*)\\ &= \frac{\kappa r\cos t}{\sqrt{1+\kappa^2}}x(s)+(1+r\sin t)\,\alpha(s)+\frac{r\cos t}{\sqrt{1+\kappa^2}}y(s).\end{aligned} \tag{18}$$

From (18), we have

$$\begin{aligned}(\psi_2)_{s^*} &= \frac{1+r\sin t}{\sqrt{1+\kappa^2}}(-x+\kappa y),\\ (\psi_2)_t &= \frac{-\kappa r\sin t}{\sqrt{1+\kappa^2}}x+(r\cos t)\,\alpha-\frac{r\sin t}{\sqrt{1+\kappa^2}}y,\end{aligned}$$

which implies that

$$\langle(\psi_2)_{s^*},(\psi_2)_{s^*}\rangle=(1+r\sin t)^2,\quad \langle(\psi_2)_{s^*},(\psi_2)_t\rangle=0,\quad \langle(\psi_2)_t,(\psi_2)_t\rangle=r^2.$$

Then

$$\langle(\psi_2)_{s^*},(\psi_2)_{s^*}\rangle\langle(\psi_2)_t,(\psi_2)_t\rangle-\langle(\psi_2)_{s^*},(\psi_2)_t\rangle^2=r^2(1+r\sin t)^2.$$

Theorem 21. *Let $\psi_2(s^*,t)$ be the second associated tubular surfaces of $\psi(s,t)$. Then $\psi_2(s^*,t)$ has a singular point at $\psi(s,t_0)$ if and only if $1+r\sin t_0=0$.*

Now we assume that $1+r\sin t\neq 0$ for all (t,s). Then we will give an orthonormal basis on $\psi_2(s^*,t)$.

$$\begin{aligned}e_1^* &= \frac{1}{\|(\psi_2)_{s^*}\|}(\psi_2)_{s^*}=\frac{\varepsilon_2}{\sqrt{1+\kappa^2}}(-x+\kappa y),\\ e_2^* &= \frac{1}{\|(\psi_2)_t\|}(\psi_2)_t=\frac{-\kappa\sin t}{\sqrt{1+\kappa^2}}x+(\cos t)\,\alpha-\frac{\sin t}{\sqrt{1+\kappa^2}}y,\end{aligned}$$

where $\varepsilon_2=sgn(1+r\sin t)$ and $\{e_1^*,e_2^*\}$ is an orthonormal frame field on $\psi_1(\overline{s},t)$. Set

$$e_3^*=-\frac{\varepsilon_2\kappa\cos t}{\sqrt{1+\kappa^2}}x-(\varepsilon_2\sin t)\,\alpha-\frac{\varepsilon_2\cos t}{\sqrt{1+\kappa^2}}y,$$

where e_3^* is a normal vector field to $\psi_2(s^*,t)$. $\{e_1^*,e_2^*,e_3^*\}$ is an orthonormal basis on $\psi_2(s^*,t)$. Then we obtain

$$\begin{aligned}D_{e_1^*}e_1^* &= \frac{1}{1+r\sin t}\alpha\\ D_{e_1^*}e_2^* &= \frac{\varepsilon_2\cos t}{(1+r\sin t)\sqrt{1+\kappa^2}}(-x+\kappa y),\\ D_{e_2^*}e_2^* &= -\frac{\kappa\cos t}{r\sqrt{1+\kappa^2}}x-\frac{\sin t}{r}\alpha-\frac{\cos t}{r\sqrt{1+\kappa^2}}y.\end{aligned}$$

The components of the second fundamental form h^* are calculated as follows

$$\begin{aligned} h^*_{11} &= \left\langle D_{e^*_1} e^*_1, e^*_3 \right\rangle = \frac{-\varepsilon_2 \sin t}{1 + r \sin t}, \\ h^*_{12} &= \left\langle D_{e^*_1} e^*_2, e^*_3 \right\rangle = 0 \quad \text{and} \quad h^*_{22} = \left\langle D_{e^*_2} e^*_2, e^*_3 \right\rangle = \frac{\varepsilon_2}{r}. \end{aligned}$$

Theorem 22. *The mean curvature H_2 of $\psi_2(s^*, t)$ is obtained as*

$$H_2 = \frac{\varepsilon_2}{2r(1 + r \sin t)}. \tag{19}$$

Theorem 23. *The Gauss curvature K_2 of $\psi_2(s^*, t)$ is obtained as*

$$K_2 = \frac{-\sin t}{r(1 + r \sin t)}. \tag{20}$$

From (19) and (20), we have

$$(K_2)_{s^*} = 0, \quad (K_2)_t = \frac{-\cos t}{r(1 + r \sin t)^2}$$

and

$$(H_2)_{s^*} = 0, \quad (H_2)_t = \frac{-\varepsilon_2 \cos t}{2(1 + r \sin t)^2}.$$

Thus it can be easily seen that $\Psi(K_2, H_2) = 0$. So we can give the following theorem.

Theorem 24. *The surface $\psi_2(s^*, t)$ is a Weingarten surface.*

Now assume that there exist real numbers a, b, $c \in \mathbb{R} \backslash \{0\}$ such that the linear combination $aK_2 + bH_2 = c$ is satisfied.

$$aK_2 + bH_2 - c = \frac{b\varepsilon_2 - 2cr - 2\left(cr^2 + a\right) \sin t}{2r(1 + r \sin t)} = 0$$

which implies that $b = 2\varepsilon_2 cr$ and $a + cr^2 = 0$. So we can give the following theorem.

Theorem 25. *Let K_2 and H_2 be the Gauss curvature and mean curvature of the surface $\psi_2(s^*, t)$. Then there exists the following relation between K_2 and H_2 :*

$$-r^2 K_2 + 2\varepsilon_2 r H_2 = 1$$

where r is a positive real number.

From above theorem, we get the following corollary.

Corollary 26. *The surface $\psi_2(s^*, t)$ is a linear Weingarten surface.*

The second Gaussian curvature K_{II} of the surface $\psi_2(s^*, t)$ is obtained that

$$K_{II} = \frac{\cot^2 t + 2(1 + r\sin t)(1 + 2r\sin t)}{4\varepsilon_2 r (1 + r\sin t)^2}.$$

Then it can be easily seen that $\Psi(K_{II}, H_2) = 0$ and $\Psi(K_{II}, K_2) = 0$. So we can give the following theorem.

Theorem 27. *The surface $\psi_2(s^*, t)$ is a (X, Y)-Weingarten surface where $(X, Y) \in \{(K_2, K_{II}), (H_2, K_{II})\}$.*

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