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ALMOST MULTIPLICATIVE LINEAR FUNCTIONALS AND CONDITION SPECTRUM

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Abstract. Let A be a complex commutative unital Fréchet Q-algebra and $\delta > 0$. We prove that if $T : A \longrightarrow \mathbb{C}$ is linear functional and $T(a) \in \sigma_\varepsilon(a)$ for all $a \in A$, where $\sigma_\varepsilon(a)$ is the ε-condition spectrum of a, then T is δ-almost multiplicative for some δ.

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1. Introduction

An algebra A over the complex field $\mathbb{C}$, is called a *Fréchet algebra* if it is a complete metrizable topological linear space. The topology of a Fréchet algebra A can be generated by a sequence (p_k) of separating submultiplicative seminorms, i.e.,

$$p_k(ab) \leqslant p_k(a)p_k(b),$$

for all $k \in \mathbb{N}$ and $a, b \in A$ such that $p_k(a) \leqslant p_{k+1}(a)$, whenever $k \in \mathbb{N}$ and $a \in A$. If A is unital then (p_k) can be chosen such that $p_k(e_A) = 1$ for all $k \in \mathbb{N}$.

A Fréchet algebra A with the above generating sequence of seminorms (p_k) is denoted by $(A, (p_k))$.

For a unital algebra A, the set of all invertible elements and don't invertible elements of A is denoted by $Inv(A)$ and $Sing(A)$, respectively. An unital topological algebra A is called a *Q-algebra* if $Inv(A)$ is open set in A, or equivalently, $Inv(A)$ has an interior point in A by [14, Lemma E2].

The spectrum of an element $a \in A$ is defined as

$$\sigma_A(a) = \{\lambda \in \mathbb{C} : \ \lambda - a \notin Inv(A)\},$$

and the spectral radius $r_A(a)$ of a is $r_A(a) = \sup\{|\lambda| : \ \lambda \in \sigma_A(a)\}$.

Banach algebras are important examples of Fréchet Q-algebras, and Fréchet Q-algebras is the well-known class of Fréchet algebras. By [4, Theorem 6.18], a Fréchet algebra A is a Q-algebra if and only if the spectral radius $r_A(a)$ is finite for all $a \in A$.

Let $\delta > 0$. A linear map $T : A \longrightarrow B$ between Banach algebras A and B is called δ-almost multiplicative if

$$\|T(ab) - T(a)T(b)\| \leqslant \delta\|a\|\|b\|, \quad (a, b \in A).$$

The study of almost multiplicative linear functions introduced by Jarosz [7] with the study of deformation theory of Banach algebras. Such maps have interesting properties and applications. Jarosz investigated the automatic continuity of almost multiplicative linear functionals and proved the following result.

Theorem 1. *[7, Proposition 5.5] Let A be a Banach algebra and $T : A \longrightarrow \mathbb{C}$ be an δ-almost multiplicative. Then $\|T\| \leqslant 1 + \delta$, and hence T is continuous.*

After that, Johnson obtained some results on continuity of almost multiplicative functionals [10]. Recently, the author [20, Theorem 7] proved that Theorem 1 is also true if δ-almost multiplicative is replaced by

$$|T(abc) - T(a)T(b)T(c)| \leqslant \delta\|a\|\|b\|\|b\|, \quad (a, b, c \in A).$$

Let A be a complex unital Banach algebra and $T : A \longrightarrow \mathbb{C}$ be a linear functional such that $T(e_A) = 1$. When is T multiplicative?

The following result, which was obtained independently by Gleason [5], Kahane and Zelazko [11], give the positive answer to this question. Their result, now known as the Gleason-Kahane-Zelazko theorem (see also [2]).

Theorem 2. *Let A be a commutative Banach algebra and $T : A \longrightarrow \mathbb{C}$ be a linear functional such that $T(e_A) = 1$. If for every $a \in A$,*

$$T(a) \in \sigma_A(a), \tag{1}$$

or equivalently, $T(a) \neq 0$ for every $a \in Inv(A)$, then T is multiplicative.

Zelazko in [19] has shown that the conclusion of the above theorem also holds for non commutative A.

Some famous generalizations of Theorem 2 have been given by many mathematicians (see for example [1, 8, 9, 17]). In addition, there are several possible approximate versions of this theorem which are concerned with identifying the almost multiplicative linear functionals among all linear functionals on Banach algebra A in terms of spectra.

The first result is due to Johnson [10] which says that linear functional T is almost multiplicative if condition (1) is replaced by

$$d(T(a), \sigma_A(a)) < \varepsilon, \quad (a \in A, \ \|a\| = 1).$$

Also, it is shown in [13, Theorem 5] that if the spectrum in condition (1) is replaced by the condition spectrum, then T is almost multiplicative.

In this paper, we prove an approximate version of Gleason-Kahane-Zelazko theorem for commutative Fréchet Q-algebras, which connects the ε-condition spectrum and almost multiplicative linear functionals.

2. Preliminaries

We first introduce the concept of *almost multiplicative* between Fréchet algebras.

Definition 1. *Let $(A, (p_k))$ be a complex Fréchet algebra and $\delta > 0$. A linear functional $T : A \longrightarrow \mathbb{C}$ is called δ-almost multiplicative, if there exists $m \in \mathbb{N}$ such that*

$$|T(ab) - T(a)T(b)| \leqslant \delta p_m(a)p_m(b), \qquad (a, b \in A). \tag{2}$$

Also T is saide to be δ-almost Jordan functional, if there exists $m \in \mathbb{N}$ such that

$$|T(a^2) - T(a)^2| \leqslant \delta p_m(a)^2, \tag{3}$$

for every $a, b \in A$. Since the generating sequence (p_k) in the Fréchet algebra $(A, (p_k))$ is an increasing sequence, both inequality (2) and (3) holds for all $k \geqslant m$.

By Definition 1, there exists smallest $m \in \mathbb{N}$ for which the inequality (2) and (3) holds. In the sequel, we use this fixed m for corresponding definition.

Remark 1. *[6, page 73] Let A and B be Fréchet algebras with generating sequences of seminorms (p_k) and (q_k), respectively. If $\varphi : A \longrightarrow B$ is a linear operator, then φ is continuous if and only if for each $k \in \mathbb{N}$, there exist $n(k) \in \mathbb{N}$ and a constant $c_k > 0$ such that*

$$q_k(\varphi(a)) \leqslant c_k p_{n(k)}(a),$$

for every $a \in A$.

Theorem 3. *[4, Theorem 6.18] Let $(A, (p_k))$ be a Fréchet algebra, then the following statements are equivalent:*

(i) *$(A, (p_k))$ is a Q-algebra.*

(ii) There is $k_0 \in \mathbb{N}$ such that $r_A(a) \leqslant p_{k_0}(a)$, for every $a \in A$.

(iii) $r_A(a) = lim_{n\to\infty} p_{k_0}(a^n)^{\frac{1}{n}}$, for every $a \in A$ and p_{k_0} as in (ii).

If k_0 is the smallest natural number such that p_{k_0} satisfies in the above theorem, we say that p_{k_0} is original seminorm for Fréchet $Q-$algebra $(A, (p_k))$. In the sequel, we use this fixed p_{k_0} as original seminorm for every Fréchet Q-algebra.

Let $(A, (p_k))$ be a unital Fréchet Q-algebra. For $0 < \varepsilon < 1$, the ε-condition spectrum of an element a in A is defined by,

$$\sigma_\varepsilon(a) = \{\lambda \in \mathbb{C} : \ p_{k_0}(\lambda - a)p_{k_0}((\lambda - a)^{-1}) \geqslant \frac{1}{\varepsilon}\},$$

with the convention that $p_{k_0}(\lambda - a)p_{k_0}(\lambda - a)^{-1} = \infty$ when $\lambda - a$ is not invertible. The ε-condition spectral radius $r_\varepsilon(a)$ is define as

$$r_\varepsilon(a) = \sup\{|\lambda| : \ \lambda \in \sigma_\varepsilon(a)\}.$$

Note that $\sigma_A(a) \subseteq \sigma_\varepsilon(a)$ and therefore $r_A(a) \leqslant r_\varepsilon(a)$.

The ε-condition spectrum first introduced by Kulkarni and Sukumar in [12] for complex unital Banach algebras, and some examples and elementary properties of the ε-condition spectrum obtained. Recently, Farajzadeh and Omidi in [3], defined it for unital Fréchet Q-algebras as above and presented the next result.

Lemma 4. *[3, Lemma 3.10] Let $(A, (p_k))$ be a Fréchet Q-algebra, $a \in A$ and let $0 < \varepsilon < 1$. Then*

$$r_\varepsilon(a) \leqslant (\frac{1+\varepsilon}{1-\varepsilon}) p_{k_0}(a).$$

3. Gleason-Kahane-Zelazko theorem

In this section we investigate the Gleason-Kahane-Zelazko theorem for commutative Fréchet Q-algebras.

The following theorem which is essentially [15, Theorem 2.8], gives a sufficient condition for an almost Jordan functional to be almost multiplicative.

Theorem 5. *Let $(A, (p_n))$ be a commutative Fréchet algebras and $T : A \longrightarrow \mathbb{C}$ be an ξ-almost Jordan functional, i.e.,*

$$|T(a^2) - T(a)^2| \leqslant \xi p_m(a)^2, \qquad (a \in A).$$

Then T is an 2ξ-almost multiplicative, that is

$$|T(ab) - T(a)T(b)| \leqslant 2\xi p_m(a)p_m(b), \qquad (a, b \in A).$$

Proposition 1. *Let $(A, (p_k))$ be a unital Fréchet Q-algebra, T be a linear functional on A. If $T(e_A) = 1$ and $kerT \subseteq Sing(A)$, then $T(a) \in \sigma_\varepsilon(a)$ for all $a \in A$.*

Proof. Set $\lambda = T(a)$. Then $T(\lambda - a) = 0$ and hence

$$(\lambda - a) \in kerT \subseteq Sing(A).$$

Therefore $\lambda \in \sigma_A(a)$ and since $\sigma_A(a) \subseteq \sigma_\varepsilon(a)$ for every $0 < \varepsilon < 1$, we get $\lambda \in \sigma_\varepsilon(a)$, for all $a \in A$.

Proposition 2. *Let $(A, (p_k))$ be a unital Fréchet Q-algebra and let $T : A \longrightarrow \mathbb{C}$ be an δ-almost multiplicative. If $T(e_A) \neq 0$, then $T(a) \in \sigma_\varepsilon(a)$ for all $\varepsilon = \frac{\delta}{|T(e_A)|}$.*

Proof. Let $a \in A$ and $T(a) = \lambda$. If $\lambda - a$ is not invertible, then $\lambda \in \sigma_A(a) \subseteq \sigma_\varepsilon(a)$. Thus the conclusion follows. Now assume that $\lambda - a$ is invertible, then

$$|T(e_A) - T(\lambda - a)T((\lambda - a)^{-1})| \leqslant \delta p_{k_0}(\lambda - a)p_{k_0}((\lambda - a)^{-1}).$$

Hence

$$p_{k_0}(\lambda - a)p_{k_0}((\lambda - a)^{-1}) \geqslant \frac{|T(e_A)|}{\delta} = \frac{1}{\frac{\delta}{|T(e_A)|}}.$$

Consequently, $T(a) \in \sigma_\varepsilon(a)$, for $\varepsilon = \frac{\delta}{|T(e_A)|}$.

Corollary 6. *[3, Lemma 3.15] Suppose that $(A, (p_k))$ is a unital Fréchet Q-algebra. If $T : A \longrightarrow \mathbb{C}$ is an ε-almost multiplicative such that $T(e_A) = 1$, then*

$$T(a) \in \sigma_\varepsilon(a),$$

for every $a \in A$.

Theorem 7. *Let $(A, (p_k))$ be a unital Fréchet Q-algebra, and $T : A \longrightarrow \mathbb{C}$ be a linear functional such that $T(e_A) = 1$. Define*

$$\|T\|_p = \sup\{|T(a)| : p_{k_0}(a) = 1\}.$$

Then $\|\cdot\|_p$ is a norm on

$$\mathfrak{M}(A) = \{T : A \longrightarrow \mathbb{C} : \quad T \textit{ is linear and } T(e_A) = 1\}.$$

Proof. If $p_{k_0}(a) \neq 0$, then $p_{k_0}(\frac{a}{p_{k_0}(a)}) = 1$. Therefore $|T(\frac{a}{p_{k_0}(a)})| \leqslant \|T\|_p$, and so

$$|T(a)| \leqslant p_{k_0}(a)\|T\|_p. \tag{4}$$

Since $T(e_A) = 1$, it follows from (4) that $\|T\|_p \neq 0$. This means that $\|\cdot\|_p$ is a norm on $\mathfrak{M}(A)$. Indeed, if $\|T\|_p = 0$, then $T(a) = 0$ for each $a \in A$, where $p_{k_0}(a) = 1$. Suppose that $T \neq 0$ and set $a = e_A$. Then $T(a) = p_{k_0}(a) = 1$, but this is not possible by (4). Thus, we conclude that $T = 0$ and so $\|\cdot\|_p$ is a norm on $\mathfrak{M}(A)$.

Our main theorem is the following.

Theorem 8. *Let $(A, (p_k))$ be a commutative unital Fréchet Q-algebra,* $0 < \varepsilon < \frac{1}{3}$, *and let T be a linear functional on A such that*

$$T(a) \in \sigma_\varepsilon(a), \qquad (a \in A).$$

Then T is δ-almost multiplicative.

Proof. Suppose that T is a linear functional on A with

$$T(a) \in \sigma_\varepsilon(a), \quad (a \in A).$$

Then $T(e_A) \in \sigma_\varepsilon(e_A) = 1$ and hence $T(e_A) = 1$. It follows from Lemma 4 that

$$|T(a)| \leqslant \big(\frac{1+\varepsilon}{1-\varepsilon}\big) p_{k_0}(a),$$

for all $a \in A$. Therefore T is continuous and $\|T\|_p \leqslant \frac{1+\varepsilon}{1-\varepsilon}$.

Let $a \in A$ with $p_{k_0}(a) = 1$. Since T is continuous and linear, we have

$$T(\exp(za)) = \sum_{n=0}^{\infty} \frac{T(a^n)}{n!} z^n,$$

for all $z \in \mathbb{C}$. Hence the function $\varphi : \mathbb{C} \longrightarrow \mathbb{C}$ defined by $\varphi(z) = T(\exp(za))$ is an entire function. Also, since for all $z \in \mathbb{C}$,

$$|\varphi(z)| \leqslant \|T\|_p p_{k_0}(\exp(za)) \leqslant \frac{1+\varepsilon}{1-\varepsilon} \exp(|z| p_{k_0}(a)) \leqslant \frac{1+\varepsilon}{1-\varepsilon} e^{|z|}. \tag{5}$$

Therefore, φ has growth order less than or equal to one. Suppose that $\alpha_1, \alpha_2, ...$ are the zeros of φ indexed with

$$|\alpha_1| \leqslant |\alpha_2| \leqslant ...$$

Using the Hadamard's factorization theorem [18, Page 250] and by the same method as in the proof of [13, Theorem 5], we get

$$T(a^2) - T(a)^2 = -\sum_j \frac{1}{\alpha_j}.$$

Now let α_j be a zero of φ. Using Jensen's formula [16] and the inequality (5), by the same reasoning as in the proof of [13, Theorem 5], we obtain

$$j \ln 2 \leqslant \ln \big(\frac{1+\varepsilon}{1-\varepsilon}\big) + 2|\alpha_j|.$$

On the other hand, since

$$0 = \varphi(\alpha_j) = T(\exp(\alpha_j a)) \in \sigma_\varepsilon(\exp(\alpha_j a)),$$

we get

$$\frac{1}{\varepsilon} \leqslant p_{k_0}(\exp(\alpha_j a))p_{k_0}(\exp(\alpha_j a)^{-1}) \leqslant \exp(2|\alpha_j|p_{k_0}(a))$$

As $p_{k_0}(a) = 1$,

$$|\alpha_j| \geqslant \frac{1}{2}\ln\big(\frac{1}{\varepsilon}\big).$$

Now similar to [13, Theorem 5] for all $a \in A$ with $p_{k_0}(a) = 1$, we conclude that

$$|T(a^2) - T(a)^2| \leqslant \xi := \frac{2}{\ln(1/\varepsilon)}\big(1 + \frac{2}{(\ln(2)/3)^2}\big).$$

Suppose that $a \in A$ is arbitrary. By replacing a by $\frac{a}{p_{k_0}(a)}$ in the above inequality, we have

$$|T(a^2) - T(a)^2| \leqslant \xi p_{k_0}(a)^2,$$

for all $a \in A$. Now it follows from Theorem 5 that T is δ-almost multiplicative with $\delta := 2\xi$. This completes the proof.

The theorem as it stands, is false for real Fréchet Q-algebras. The following example illustrating this fact.

Example 1. *Let $A = C_{\mathbb{R}}[0,1]$, the space of all real valued continuous functions defined on $[0,1]$, and define $T : A \longrightarrow \mathbb{C}$ by*

$$T(f) = \int_0^1 f(t)dt, \qquad (f \in A).$$

Then $T(e_A) = 1$ and for all $f \in A$, $\sigma_A(f) = Range f$. Hence

$$T(f) \in \sigma_A(f) \subseteq \sigma_\varepsilon(f).$$

Let $f(t) = g(t) = t$, then $|T(fg) - T(f)T(g)| = \frac{1}{12}$. On the other hand,

$$\delta = 2\xi := \frac{4}{\ln(1/\varepsilon)}\big(1 + \frac{2}{(\ln(2)/3)^2}\big) < \frac{1}{12},$$

for all ε which satisfies

$$\varepsilon < e^{-48\left(1+\frac{2}{(\ln(2)/3)^2}\right)}. \tag{6}$$

Thus, with condition (6), T is not δ-almost multiplicative. However, $T(f) \in \sigma_\varepsilon(f)$.

Using Theorem 8, we can deduce the classical Gleason-Kahane-Zelazko Theorem for commutative Fréchet Q-algebras.

Corollary 9. *Let $(A,(p_k))$ be a commutative unital Fréchet Q-algebra, $T : A \longrightarrow \mathbb{C}$ be a linear functional. If $T(a) \in \sigma_A(a)$ for every $a \in A$, then T is multiplicative.*

Proof. Since $\sigma_A(a) \subseteq \sigma_\varepsilon(a)$ for every $0 < \varepsilon < 1$, hence $T(a) \in \sigma_\varepsilon(a)$ for all $a \in A$ with $0 < \varepsilon < \frac{1}{3}$. By preceding theorem, T is δ-almost multiplicative. As $\varepsilon \longrightarrow 0^+$ implies that $\delta \longrightarrow 0^+$, we conclude that T is multiplicative.

Theorem 10. *Suppose that $(A,(p_k))$ is a Fréchet Q-algebra, $0 < \varepsilon < 1$, and T is a linear functional on A such that $T(a) \in \sigma_\varepsilon(a)$ for all $a \in A$. Then T is δ-almost multiplicative, where*

$$\delta = \Big(\frac{1+\varepsilon}{1-\varepsilon}\Big) + \Big(\frac{1+\varepsilon}{1-\varepsilon}\Big)^2.$$

Proof. Let T be a linear functional on A such that $T(a) \in \sigma_\varepsilon(a)$, for all $a \in A$. By Lemma 4, we have

$$|T(a)| \leqslant \Big(\frac{1+\varepsilon}{1-\varepsilon}\Big) p_{k_0}(a),$$

for all $a \in A$. Let $\xi = \big(\frac{1+\varepsilon}{1-\varepsilon}\big)$. Then

$$\begin{aligned} |T(ab) - T(a)T(b)| &\leqslant |T(ab)| + |T(a)||T(b)| \\ &\leqslant \xi p_{k_0}(ab) + \xi^2 p_{k_0}(a) p_{k_0}(b) \\ &\leqslant \xi p_{k_0}(a) p_{k_0}(b) + \xi^2 p_{k_0}(a) p_{k_0}(b) \\ &\leqslant (\xi + \xi^2) p_{k_0}(a) p_{k_0}(b). \end{aligned}$$

Therefore, T is δ-almost multiplicative for $\delta = (\xi + \xi^2)$.

We mention that δ in above theorem is not sharp. Because if $\varepsilon \longrightarrow 0^+$ then $\delta \longrightarrow 2$, so we cannot conclude that T is multiplicative. However, in Theorem 8, δ is sharp.

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