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# NON-ALGORITHMIC PROCEDURES AND ALGORITHM CREATION

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Abstract. Algorithms work blindly, ignoring the meanings of involved symbols and the aims of their actions. They are stated in a fixed language; hence, they are language dependent. In this article, we introduce procedures identifying entities by their attributes, as the scientific method requires, and term them eulerithms. They are language-independent procedures being able to build algorithms.

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## 1. Introduction

Languages being partial free-monoids generated by symbol-sets, can be extended appending words and sentences. A language is extendable when the meaning of its extensions are definable by some of their own sentences. A mathematical construction is finitely-definable when a finite symbol sequence in an extendable language determines it. For example, the number $\pi = 3.141592\ldots$ is finitely definable because the finite English sentence

$$\text{“}\pi \textit{ is the ratio of a circle's circumference to its diameter”} \tag{1}$$

denotes it.

Every Gödel-like numbering function sends each finite symbol sequence into a positive integer. Thus, if there is at least one non-countable set $E$, it must contain some non-finitely definable members in spite of being $E$ definable through some finite expression [7]. As a consequence, if for every member $x$ of any uncountable set $X$, there is a language $\mathcal{L}$ and a finite symbol sequence in it denoting an algorithm calculating $x$, then $X$ is countable. Thus, every uncountable set contains non-computable members [7].

Since algorithms work blindly on symbol-sequences of some language, that is, ignoring their meanings, for some problems, we need non-algorithmic procedures having the capability of assigning meanings to phrases. This fact requires the help of logical methods. For instance, consider the definition above for the number $\pi$. It is not sufficient to understand the meaning of the phrase

$$P = \textit{ the ratio of a circle's circumference to its diameter.}$$

If this ratio were not the same for all circles, expression (1) would define nothing. Accordingly, it is not sufficient to understand $P$ but to know some circle attributes.

Algorithms are written in a fixed symbolic vocabulary, governed by precise instructions, whose execution requires no cleverness, intuition, or intelligence. This is why we introduce procedures working on attributes instead of symbol-sequences; hence they are language independent. We term them *eulerithms*.

Since the word algorithm is a derivative of the Latinized name of the Persian polymath Al-Khwarizmi, our choice is a derivative of Euler's name, which proved the well-known formula

$$e^{ix} = cos(x) + i \cdot sin(x). \tag{2}$$

The original proof of the Euler's formula is based on the properties (attributes) of Taylor series expansions of the functions $e^x$, $sin(x)$, and $cos(x)$. The physicist Richard Feynman called the equation above "*our jewel*" and "*the most remarkable formula in mathematics*" [9].

In any proper scientific method, entities must be identified by their properties (attributes) instead of words assigned by convention. We only consider as attributes those predicates satisfying some conditions. They cannot contain unavoidable disjunctions and self-referential definitions.

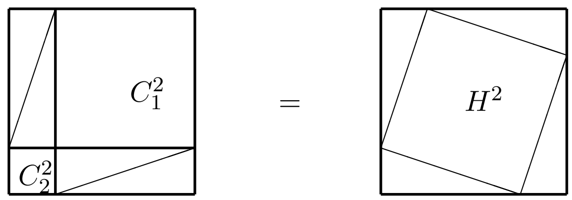
Definition (1) contains a finite symbol sequence, but under the decimal numbering language, we denote $\pi$ by the endless symbol sequence $3.141592\ldots$. Finding the most suitable language for each problem depends on the attributes of the involved concepts [6]. To this end, eulerithms are adequate devices.

We consider as an attribute each predicate that some mathematical construction satisfies. Nevertheless, we cannot consider as attributes those predicates containing unavoidable disjunctions. For instance, the predicate $p(x) =$ "*x is a prime integer*" is an attribute of some members of $\mathbb{N}$. Likewise, the predicate

$$q(x) = \text{“}x \textit{ is a statement with truth-value } 1\text{”}$$

denotes an attribute of every true statement. Both, the integer 5 and the sentence "$x^2+x = 6$ *is a Diophantine equation,*" satisfy their disjunction $p(x) \vee q(x)$; however, only $p(x)$ is an attribute of 5; hence, we cannot consider $p(x) \vee q(x)$ as an attribute of it.

Mainly, we can find object-attributes from analogical representations instead of symbol sequences. Some properties of the equation $x^2 - 4 = 0$ are observable in its formal expression. By contrast, from a phrase defining the triangle concept, it is not possible to find out the existence of barycenter, orthocenter, etc. Even the Pythagorean Theorem is easy to see in analogical representations as follows.




$$C_1^2 + C_2^2 = H^2.$$

By contrast, denoting this theorem in a formal expression, we cannot see the equality $c_1^2 + c_2^2 = h^2$. Finding a suitable language to perform a given procedure is not a task that algorithms can do. They work with a fixed language. In the last section, we state a procedure to build algorithms and suitable languages. This paradigm is a consequence of Theorem 30.

Likewise, we show that eulerithms can build algorithms. By contrast, the procedures building eulerithms must be able to find attributes; hence, they have to create suitable formalisms. We cannot always consider such procedures as eulerithms. This topic requires further research. Since we introduce a new procedure paradigm, references are not essential.

## 2. Preliminaries

Almost every concept is the generic object of an equivalence class. For instance, the word "polygon" denotes an object–class containing triangles, quadrangles, pentagons, etc. Even each of these words denotes an infinite object-set. These classes are homogeneous. Their members are characterized by some common properties. Any property $P$ can be stated by a predicate of the form

*"$x$ satisfies has the property $P$."*

To be homogeneous, not every predicate is suitable as a class definition. For instance, some predicate disjunctions can define the union of heterogeneous classes. Thus, we state the homogeneous class definition as follows.

**Definition 1.** *We say that a class* $\mathbf{K}$ *is homogeneous when can be defined by a disjunction-free predicate.*

To avoid any paradox, the conglomerate $\mathcal{H}$ of all homogeneous classes cannot belong to itself [1]. The predicate

$$\Omega(x) = \text{“}x \textit{ is a non-self-contained mathematical construction, that can be defined by a disjunction-free predicate.”} \quad (3)$$

denotes each member of $\mathcal{H}$; therefore,

$$\Omega(O) \Longleftrightarrow (O \in \mathcal{H}),$$

and $\Omega(x)$ defines $\mathcal{H}$. Statement (3) leads to $\mathcal{H} \notin \mathcal{H}$; hence, the conglomerate $\mathcal{H}$ is homogeneous, but it is not a class. It is a proper conglomerate [1].

Although every homogeneous class is not self-contained, it can contain its generic member. A predicate $p(x)$ defining a class $\mathbf{K}$, determines also its generic member. To avoid any confusion, we write the superscript $^{\curlyvee}$ to denote it. Thus, $\mathbf{K}^{\curlyvee}$ denotes the generic member of $\mathbf{K}$ and $p(x)^{\curlyvee}$ its definition. Likewise, the predicate $\Omega(x)^{\curlyvee}$ defines the generic object $\mathcal{H}^{\curlyvee}$ of $\mathcal{H}$. The generic object of each singleton $\{O\}$ is its member $\{O\}^{\curlyvee} = O$.

There are predicates containing disjunctions that are equivalent to others being disjunction-free ones. For example, the class that $(x = 0) \vee (x = 1)$ defines is homogeneous because it can be defined by $(x \in \mathbb{N}) \wedge (x^2 - x = 0)$. Similar situations can occur using universal quantifiers. For instance, if a disjunction-free predicate $p(x)$ defines a homogeneous class $\mathbf{K}$, the statement $\forall x \in \mathbf{K} : q(x)$ un many situations is equivalent to $p(x) \wedge q(x)$. By contrast, as a consequence of Morgan's Law, the negation $\neg p(x)$ of a predicate need not be disjunction-free in spite of so being $p(x)$.

**Example 1.** *Let* $\mathbf{K}$ *be the class*

$$\mathbf{K} = \big\{2n \mid n \in \mathbb{N}\big\} \cup \big\{ \textit{ triangle, quadrangle, pentagon}\big\}$$

*The disjunction* $h(x) = p(x) \vee q(x)$ *of the predicates* $p(x) =$ *“*$x \equiv 0 \pmod 2$*” and* $q(x) =$ *“*$x$ *is a polygon of less than* 6 *angles” defines* $\mathbf{K}$*:*

$$h(x) = (p(x) \vee q(x)) \Longleftrightarrow \big(x \in \mathbf{K}\big)$$

*Since* $h(x)$ *contains one disjunction,* $\mathbf{K}$ *is not a homogeneous class. By contrast,* $p(x)$ *defines the homogeneous class* $\mathbf{P} = \big\{2n \mid n \in \mathbb{N}\big\}$ *of even positive integers; hence,* $\mathbf{P}^{\curlyvee} = \big\{2n \mid n \in \mathbb{N}\big\}^{\curlyvee}$ *denotes the even-integer concept.*

**Definition 2.** *We term attribute every finitely-definable disjunction-free predicate defining a nonempty homogeneous class.*

**Lemma 1.** *Every attribute* $p(x)$ *satisfies the relation* $p(x) \Longrightarrow \Omega(x)$*.*

*Proof.* By Definition 2, the mathematical construction that $p(x)$ defines satisfies (3).

**Lemma 2.** *The conjunction of a family of compatible attributes is again an attribute.*

*Proof.* Since each attribute is a disjunction-free predicate, so is the conjunction of every family $\mathbf{A}$ of them. If the members of $\mathbf{A}$ are compatible, $p(x) = \bigwedge_{q(x)\in\mathbf{A}} q(x)$ defines a nonempty class.

**Notation 1.** Let $\mathbf{Attr}$ be the class of all attributes and $\mathfrak{C}^{\ll}$ the thin category [1] such that the object-class $\mathrm{Ob}(\mathfrak{C}^{\ll})$ is $\mathbf{Attr}$. For every pair of attributes $(p(x), q(x))$, the homset $\hom_{\mathfrak{C}^{\ll}}(p(x), q(x))$ either is empty, or it is a singleton $\left\{p(x) \xrightarrow{\ll} q(x)\right\}$ whenever there exists an attribute $h(x)$ satisfying the equivalence.

$$p(x) \Longleftrightarrow q(x) \wedge h(x). \tag{4}$$

As a straightforward consequence of the $\mathfrak{C}^{\ll}$-morphism definition, the following relation holds.

$$\forall\,(p(x), q(x)) \in \mathbf{Attr}\times\mathbf{Attr}: \quad \left(p(x) \xrightarrow{\ll} q(x)\right) \Longrightarrow \left(p(x) \Rightarrow q(x)\right). \tag{5}$$

The converse implication need not hold. If $(p(x) \Rightarrow q(x)$, the existence of the morphism $p(x) \xrightarrow{\ll} q(x)$ requires that some attribute $h(x)$ satisfies the relation $p(x) \Longleftrightarrow (q(x) \wedge h(x))$.

**Definition 3.** *A nonempty subclass $\mathbf{B}$ of an attribute class $\mathbf{K}$ is a basis for it, provided that any attribute $p(x) \in \mathbf{K}$ belongs to $\complement_{\mathbf{K}}\mathbf{B}$ if and only if it is the conjunction of a subset of $\mathbf{B}$ of cardinality greater than* 1.

**Definition 4.** *Let $\mathrm{Ob}(\mathfrak{C}^{\preceq})$ be the class of all entities being definable by attributes. For every $O \in \mathrm{Ob}(\mathfrak{C}^{\preceq})$, let $\mathrm{def}_O(x)$ denote some predicate defining it. The collection*

$$\mathrm{Ob}\left(\mathfrak{C}^{\preceq}\right) = \left\{O \in \mathcal{H} \mid \mathrm{def}_O(x) \in \mathbf{Attr}\right\}$$

*is the object-class of a thin category $\mathfrak{C}^{\preceq}$ such that, for every pair of objects $(O_1, O_2)$, there is the arrow $O_1 \xrightarrow{\preceq} O_2$ if and only if $\mathrm{def}_{O_1}(x) \xrightarrow{\ll} \mathrm{def}_{O_2}(x)$.*

**Lemma 3.** *If for a pair of $\mathfrak{C}^{\preceq}$-objects $(O_1, O_2)$ there is the arrow $O_1 \xrightarrow{\preceq} O_2$, then the relation $\mathrm{def}_{O_1}(x) \Rightarrow \mathrm{def}_{O_2}(x)$ holds.*

*Proof.* By the definition of $\mathfrak{C}^{\preceq}$-morphism, there is the arrow $O_1 \xrightarrow{\preceq} O_2$ when $\mathrm{def}_{O_1}(x) \xrightarrow{\ll} \mathrm{def}_{O_2}(x)$; hence, the lemma is a consequence of equation (5).

**Corollary 4.** *With the same assumptions as in Lemma 3,*

$$\Big((O_1 \xrightarrow{\preceq} O_2) \wedge (O_2 \xrightarrow{\preceq} O_1)\Big) \Longrightarrow \big(O_1 = O_2\big). \tag{6}$$

*Proof.* It is a straightforward consequence of Lemma 3.

**Lemma 5.** *The attribute $\Omega(x)$ (3) and the generic object $\mathcal{H}^{\curlyvee}$ are terminal in $\mathfrak{C}^{\ll}$ and $\mathfrak{C}^{\preceq}$, respectively.*

*Proof.* By Lemma 1, every attribute $p(x)$ satisfies the relation $p(x) \Longrightarrow \Omega(x)$; therefore,

$$p(x) \Longleftrightarrow p(x) \wedge \Omega(x)$$

and $p(x) \xrightarrow{\ll} \Omega(x)$. As a consequence, for every $\mathfrak{C}^{\preceq}$-object $O$, if $\mathrm{def}_O(x)$ is a definition for $O$, then

$$\mathrm{def}_O(x) \xrightarrow{\ll} \Omega(x) = \mathrm{def}_{\mathcal{H}^{\curlyvee}}(x).$$

This equation leads to $O \xrightarrow{\preceq} \mathcal{H}^{\curlyvee}$.

If $\mathbf{C}$ is either an abstract or a concrete category, each finitely definable member of $\mathrm{Ob}(\mathbf{C})$ is a $\mathfrak{C}^{\preceq}$-object too. Thus, products and coproducts can be performed as members of $\mathrm{Ob}(\mathfrak{C}^{\preceq})$ or as $\mathbf{C}$-objects. To avoid any confusion, we denote the first case using the symbol $\preceq$ as a superscript in the operators $\prod$ and $\coprod$; therefore, $\prod^{\preceq}$ and $\coprod^{\preceq}$ denote the product and coproduct in the category $\mathfrak{C}^{\preceq}$. Likewise, the symbols $\prod^{\ll}$ and $\coprod^{\ll}$ denote the product and coproduct, respectively, of $\mathfrak{C}^{\ll}$-objects.

**Definition 5.** *For every couple of attributes $p_1(x)$ and $p_2(x)$ that satisfy the relations $p_1(x) \xrightarrow{\ll} p_2(x)$ and $p_1(x) \not\Leftrightarrow p_2(x)$, we say that an attribute $q(x)$ complements $p_2(x)$ to $p_1(x)$ when it satisfies the following conditions.*

$$\begin{cases} p_1(x) \Longleftrightarrow (p_2(x) \wedge q(x)), \\ p_2(x) \not\Rightarrow q(x) \text{ and } q(x) \not\Rightarrow p_2(x) \end{cases} \tag{7}$$

*Likewise, a $\mathfrak{C}^{\preceq}$-object $C$ complements $Q$ to $O$ when an attribute $\mathrm{def}_C(x)$ defining it complements $\mathrm{def}_Q(x)$ to $\mathrm{def}_O(x)$.*

**Definition 6.** *A class $\mathbf{K}$ is discernible provided that it satisfies the following conditions.*

1. *There is an attribute $\mathrm{def}_{\mathbf{K}}(x)$ defining $\mathbf{K}$.*

2. *For every member $O \in \mathbf{K}$, there is an object $Q$ that complements $\mathbf{K}^{\curlyvee}$ to $O$; hence, a predicate $\mathrm{def}_Q(x)$ defining it satisfies the relation below.*

$$\mathrm{def}_O(x) \Longleftrightarrow (\mathrm{def}_{\mathbf{K}}(x) \wedge \mathrm{def}_Q(x)).$$

3. *There is a finite subset $G \subseteq \mathbf{K}$ such that, for every superset $D \supseteq G$, if $D \subseteq \mathbf{K}$, then*

$$\coprod_{O\in D}^{\ll} \mathrm{def}_O(x) \Longleftrightarrow \mathrm{def}_{\mathbf{K}}(x). \tag{8}$$

*We say that $G$ is a determining subset of $\mathbf{K}$.*

**Definition 7.** *Let $\mathbf{K}$ be a discernible class, and for every $O \in \mathbf{K}$, let $Q_O$ be an object that complements $\mathbf{K}^{\curlyvee}$ to $O$. With these assumptions we say that $\{Q_O \mid O \in \mathbf{K}\}$ is a complementing class of $\mathbf{K}$.*

**Example 2.** *Let $\mathbf{K}$ be the solution-set of the equation $x^2 - 4 = 0$; hence, $\mathbf{K} = \{-2, 2\}$. The predicate $\mathrm{def}_{\mathbf{K}}(x) =$ "$x$ is a solution of $x^2 - 4 = 0$" defines $\mathbf{K}$. The attribute conjunctions*

$$\begin{aligned}\mathrm{def}_{-2}(x) &= \mathrm{def}_{\mathbf{K}}(x) \wedge \text{“}x \textit{ is a negative number”}\,,\\ \mathrm{def}_{2}(x) &= \mathrm{def}_{\mathbf{K}}(x) \wedge \text{“}x \textit{ is a positive number”}\,,\end{aligned}$$

*define $-2$ and $2$, respectively. Denoting by $Pos^{\curlyvee}$ and $Neg^{\curlyvee}$ the generic concepts of positive and negative numbers, respectively, $\{Pos^{\curlyvee}, Neg^{\curlyvee}\}$ is the complementing class of $\mathbf{K}$.*

**Notation 2.** If $\mathbf{K}$ is a discernible class, for every $O \in \mathbf{K}$, the expression $\complement^{\curlyvee}_{\preceq O}\mathbf{K}^{\curlyvee}$, denotes the object that complements $\mathbf{K}^{\curlyvee}$ to $O$. Likewise, $\complement^{\curlyvee}_{\preceq \mathbf{K}}\mathbf{K}^{\curlyvee}$ is the complementing class

$$\complement^{\curlyvee}_{\preceq \mathbf{K}}\mathbf{K}^{\curlyvee} = \left\{\complement^{\curlyvee}_{\preceq O}\mathbf{K}^{\curlyvee} \,\middle|\, O \in \mathbf{K}\right\}. \tag{9}$$

**Lemma 6.** *If the members of an attribute set $\mathbf{A} = \{p_i(x) \mid i \in \mathbf{I}\}$ are compatible and $\#(\mathbf{I}) \geq 2$, their conjunction is their $\mathfrak{C}^{\ll}$-product.*

*Proof.* By equation (5), for every $j \in \mathbf{I}$,

$$\bigwedge_{i\in\mathbf{I}} p_i(x) \xrightarrow{\ll} p_j(x). \tag{10}$$

If $h(x)$ is an attribute such that there is a source $\left(h(x) \xrightarrow{\ll} p_i(x)\right)_{i\in\mathbf{I}}$, by definition, for each $i \in \mathbf{i}$, there is an attribute $r_i(x)$ such that

$$\forall i \in \mathbf{I}: \quad h(x) \Longleftrightarrow (p_i(x) \wedge r_i(x))\,; \tag{11}$$

therefore,

$$\bigwedge_{i\in\mathbf{I}} (p_i(x) \wedge r_i(x)) \Longleftrightarrow h(x) \Longrightarrow \bigwedge_{i\in\mathbf{I}} p_i(x)$$

and $h(x) \xrightarrow{\ll} \bigwedge_{i\in\mathbf{I}} p_i(x)$. By Lemma 2, the conjunction $\bigwedge_{i\in\mathbf{I}} p_i(x)$ is an attribute, and the following diagram commutes.

$$\forall i \in \mathbf{I}: \qquad \begin{tikzcd} \bigwedge_{j\in\mathbf{I}} p_j(x) \arrow[dr, "\ll"] & \\ & p_i(x) \\ h(x) \arrow[uu, "\ll"] \arrow[ur, "\ll"'] & \end{tikzcd}$$

The arrow uniqueness is a consequence of the thin nature of $\mathfrak{C}^{\ll}$; therefore,

$$\prod_{i\in\mathbf{I}}^{\ll} p_i(x) = \bigwedge_{i\in\mathbf{I}} p_i(x).$$

**Lemma 7.** *Let* $\mathbf{K}$ *be a family of* $\mathfrak{C}^{\preceq}$*-objects of cardinality greater than 1. If for every* $O \in \mathbf{K}$, $\mathrm{def}_O(x)$ *is an attribute defining it and there is the product* $\prod_{O\in\mathbf{K}}^{\preceq} O$, *the following statement holds.*

$$\prod_{O\in\mathbf{K}}^{\ll} \mathrm{def}_O(x) = \mathrm{def}_{\left(\prod_{O\in\mathbf{K}}^{\preceq} O\right)}(x). \tag{12}$$

*Proof.* Let $Q$ be the object that the product $\prod_{O\in\mathbf{K}}^{\ll} \mathrm{def}_O(x)$ defines; hence,

$$\mathrm{def}_Q(x) = \prod_{O\in\mathbf{K}}^{\ll} \mathrm{def}_O(x).$$

As a straightforward consequence of Definition 4, for every source

$$S = \left(W \xrightarrow{\preceq} O\right)_{O\in\mathbf{K}}, \tag{13}$$

there is $\left(\mathrm{def}_W(x) \xrightarrow{\ll} \mathrm{def}_O(x)\right)_{O\in\mathbf{K}}$, and by Lemma 6, the following diagrams commute.

$$
\forall O \in \mathbf{K} : \quad
\begin{tikzcd}
 & \mathrm{def}_Q(x) \arrow[dl, "\gg"'] \\
\mathrm{def}_O(x) & \\
 & \mathrm{def}_W(x) \arrow[ul, "\gg"] \arrow[uu, "\ll"']
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
 & Q \arrow[dl, "\succeq"'] \\
O & \\
 & W \arrow[ul, "\succeq"] \arrow[uu, "\preceq"']
\end{tikzcd}
$$

therefore, $Q = \prod\limits_{O \in \mathbf{K}}^{\preceq} O$ and equation (12) holds.

**Corollary 8.** *For every source* $\left(O \xrightarrow{\preceq} O_i\right)_{i \in \mathbf{I}}$ *the following statement holds.*

$$O \xrightarrow{\preceq} \prod_{i \in \mathbf{I}}^{\preceq} O_i. \tag{14}$$

*Proof.* By definition 4, there is the source $\left(\mathrm{def}_O(x) \xrightarrow{\ll} \mathrm{def}_{O_i}\right)_{i \in \mathbf{I}}$. By equation (4), for each $i \in \mathbf{I}$, there is an attribute $h_i(x)$ such that

$$\forall i \in \mathbf{I} : \quad \mathrm{def}_O(x) \Longleftrightarrow \mathrm{def}_{O_i}(x) \wedge h_i(x). \tag{15}$$

As a consequence, for every $i$, the object $O$ satisfies both attributes $\mathrm{def}_{O_i}(x)$ and $h_i(x)$; therefore, both attribute sets are compatible and

$$\mathrm{def}_O(x) \Longleftrightarrow \left(\left(\bigwedge_{i \in \mathbf{I}} \mathrm{def}_{O_i}(x)\right) \wedge \left(\bigwedge_{i \in \mathbf{I}} h_i(x)\right)\right). \tag{16}$$

By Lemma 2, both conjunctions are attributes; hence,

$$\mathrm{def}_O(x) \xrightarrow{\ll} \bigwedge_{i \in \mathbf{I}} \mathrm{def}_{O_i}(x). \tag{17}$$

The equation above, together with Lemmata 6 and 7, leads to (14).

**Corollary 9.** *For every* $\mathfrak{C}^{\preceq}$*-arrow* $O_1 \xrightarrow{\preceq} O_2$*, an object* $Q$ *complements* $O_2$ *to* $O_1$ *when the following relations hold.*

$$O_1 = \left(O_2 \prod Q\right) \tag{18}$$

$$\hom_{\mathfrak{C}^{\preceq}}(O_2, Q) = \hom_{\mathfrak{C}^{\preceq}}(Q, O_2) = \emptyset. \tag{19}$$

*Proof.* It is a straightforward consequence of Corollary 4, Definition 5, and Lemma 7.

**Lemma 10.** *Let* $\{p_i(x) \mid i \in \mathbf{I}\} \subseteq \mathrm{Ob}(\mathfrak{C}^{\ll})$ *be an attribute set of cardinality greater than* 1. *If* $\mathbf{P}$ *denotes the class*

$$\mathbf{P} = \bigcap_{i \in \mathbf{I}} \Big\{p(x) \in \mathrm{Ob}(\mathfrak{C}^{\ll}) \mid p_i(x) \xrightarrow{\ll} p(x)\Big\},$$

*then the following statement holds.*

$$\coprod_{i \in \mathbf{I}}^{\ll} p_i(x) = \bigwedge_{p(x) \in \mathbf{P}} p(x) = \prod_{p(x) \in \mathbf{P}}^{\ll} p(x). \tag{20}$$

*Proof.* The set $\mathbf{P}$ is nonempty because, by Lemma 5, it contains $\Omega(x)$. It is a straightforward consequence of the definition of $\mathbf{P}$, together with equation (4), that for each $p(x) \in \mathbf{P}$ and every $i$ in $\mathbf{I}$, there is an attribute $h_{p,i}(x)$ such that $p_i(x) \Longleftrightarrow (p(x) \wedge h_{p,i}(x))$; therefore,

$$\forall p(x) \in \mathbf{P} : \quad p_i(x) \Longleftrightarrow (p(x) \wedge h_{p,i}(x)) \,. \tag{21}$$

As a consequence, if for some $i \in \mathbf{I}$ an object $O$ satisfies the attribute $p_i(x)$, then it satisfies every member of $\mathbf{P}$ too. Thus, the members of $\mathbf{P}$ are compatible, and by Lemma 2, their conjunction is again an attribute such that

$$\forall i \in \mathbf{I} : \quad p_i(x) \xrightarrow{\ll} \bigwedge_{p(x) \in \mathbf{P}} p(x).$$

By the definition of $\mathbf{P}$, if an attribute $q(x)$ in **Attr** satisfies the relation

$$\forall i \in \mathbf{I} : \qquad p_i(x) \xrightarrow{\ll} q(x),$$

then it belongs to $\mathbf{P}$. As a consequence, there is the unique morphism

$$\bigwedge_{p(x) \in \mathbf{P}} p(x) \xrightarrow{\ll} q(x)$$

such that, for every $i$ in $\mathbf{I}$, the following diagram commutes.

$$
\begin{tikzcd}
 & \bigwedge\limits_{p(x)\in\mathbf{P}} p(x) \arrow[dd, "\ll"] \\
p_i(x) \arrow[ur, "\ll"] \arrow[dr, "\ll"'] & \\
 & q(x)
\end{tikzcd}
$$

Accordingly, $\bigwedge\limits_{p(x)\in\mathbf{P}} p(x) = \coprod\limits_{i\in\mathbf{I}}^{\ll} p_i(x)$. This equation, together with Lemma 6, leads to (20).

**Theorem 11.** *Let $\{O_i \mid i \in \mathbf{I}\} \subseteq \mathrm{Ob}(\mathfrak{C}^{\preceq})$ be an object-set of cardinality greater than 1 and $\mathbf{K}$ the class*

$$\mathbf{K} = \bigcap_{i\in\mathbf{I}} \left\{ O \in \mathrm{Ob}(\mathfrak{C}^{\preceq}) \mid O_i \xrightarrow{\preceq} O \right\}.$$

*With these assumptions the following statements hold.*

1. *There is the coproduct $\coprod\limits_{i\in\mathbf{I}}^{\preceq} O$ and satisfies the relation below.*

$$\coprod_{i\in\mathbf{I}}^{\preceq} O = \prod_{O\in\mathbf{K}}^{\preceq} O. \tag{22}$$

2. *The attribute-coproduct $\coprod\limits_{i\in\mathbf{I}}^{\ll} \mathrm{def}_{O_i}(x)$ defines $\coprod\limits_{i\in\mathbf{I}}^{\preceq} O_i$; hence,*

$$\coprod_{i\in\mathbf{I}}^{\ll} \mathrm{def}_{O_i}(x) \Longleftrightarrow \mathrm{def}_{\left(\coprod_{i\in\mathbf{I}}^{\preceq} O_i\right)}(x). \tag{23}$$

*Proof.*

1. By the definition of $\mathbf{K}$, for every $i \in \mathbf{I}$, there is the source $\left(O_i \xrightarrow{\preceq} O\right)_{O \in \mathbf{K}}$; hence, by Corollary 8,

$$\forall i \in \mathbf{I}: \quad O_i \xrightarrow{\preceq} \prod_{O \in \mathbf{K}}^{\preceq} O. \tag{24}$$

Thus, for every $i \in \mathbf{I}$, the diagram below commutes.

$$\begin{tikzcd} & \prod\limits_{O\in\mathbf{K}}^{\preceq} O \\ O_i \arrow[ur, "\preceq"] \arrow[dr, "\preceq"'] & \\ & \coprod\limits_{j\in\mathbf{I}}^{\preceq} O_i \arrow[uu, "\preceq"'] \end{tikzcd} \tag{25}$$

By the definition of $\mathfrak{C}^{\preceq}$-morphism, there is an attribute $h(x)$ such that

$$\operatorname{def}_{\left(\coprod_{i\in\mathbf{I}}^{\preceq} O_i\right)}(x) \Longleftrightarrow \operatorname{def}_{\left(\prod_{O\in\mathbf{K}} O\right)}(x) \wedge h(x). \tag{26}$$

The relation above, (25), and Lemma 3 lead to

$$\forall i \in \mathbf{I}: \quad \operatorname{def}_{O_i}(x) \Longrightarrow \operatorname{def}_{\left(\prod_{O\in\mathbf{K}} O\right)}(x) \wedge h(x) \Longrightarrow h(x).$$

If $Q$ is the generic object of the class that $h(x)$ defines, then the relation above leads to

$$\forall i \in \mathbf{I}: \quad \operatorname{def}_{O_i}(x) \xrightarrow{\ll} \operatorname{def}_Q(x) \ \textit{and}\ O_i \xrightarrow{\preceq} Q;$$

therefore, $Q \in \mathbf{K}$, and by (26),

$$\operatorname{def}_{\left(\coprod_{i\in\mathbf{I}}^{\preceq} O_i\right)}(x) \Longleftrightarrow \operatorname{def}_{\left(\prod_{O\in\mathbf{K}} O\right)}(x);$$

therefore, $\coprod\limits_{i\in\mathbf{I}}^{\preceq} O_i = \prod\limits_{O\in\mathbf{K}}^{\preceq} O$.

2. By the definition of $\mathbf{K}$, the equation below holds.

$$\mathbf{P} = \bigcap_{i\in\mathbf{I}} \Big\{ \mathrm{def}_O(x) \in \mathrm{Ob}(\mathfrak{C}^{\ll}) \mid \mathrm{def}_{O_i}(x) \xrightarrow{\ll} \mathrm{def}_O(x) \Big\} = \Big\{ \mathrm{def}_O(x) \mid O \in \mathbf{K} \Big\}. \quad (27)$$

The equation above, together with Lemma 10, leads to

$$\coprod_{i\in\mathbf{I}}^{\ll} \mathrm{def}_{O_i}(x) = \prod_{O\in\mathbf{K}}^{\ll} \mathrm{def}_O(x). \quad (28)$$

Taking into account (22) and Lemma 7,

$$\mathrm{def}_{\left(\coprod_{i\in\mathbf{I}}^{\preceq} O_i\right)}(x) = \mathrm{def}_{\left(\prod_{O\in\mathbf{K}}^{\preceq} O\right)}(x) = \prod_{O\in\mathbf{K}}^{\ll} \mathrm{def}_O(x) = \coprod_{i\in\mathbf{I}}^{\ll} \mathrm{def}_{O_i}(x)$$

and equation (23) holds.

**Example 3.** *Let $T$ and $Q$ denote the sets of all triangles and quadrangles, respectively. If we consider both sets as members of* $\mathrm{Ob}(\mathbf{Set})$*, the coproduct is* $T \coprod Q = (T \times \{0\}) \bigcup (Q \times \{1\})$*. However, as $\mathfrak{C}^{\preceq}$-objects,* $T \coprod^{\preceq} Q$ *is the generic object of the set of polygons with less than five sides. To see this fact, consider the following attributes.*

$$\begin{aligned} p(x) = &\ \text{“}x \text{ is a polygon with less than five sides,”} \\ q_1(x) = &\ \text{“}x \text{ is a shape with three angles,”} \\ q_2(x) = &\ \text{“}x \text{ is a shape with four angles.”} \end{aligned}$$

*The conjunctions $p(x) \wedge q_1(x)$ and $p(x) \wedge q_2(x)$ define $T$ and $Q$, respectively. If $\mathbf{K} \subseteq \mathbf{Attr}$ is the attribute class*

$$\mathbf{K} = \Big\{ q(x) \in \mathbf{Attr} \mid (p(x) \wedge q_1(x)) \xrightarrow{\ll} q(x) \Big\} \bigcap \Big\{ q(x) \in \mathbf{Attr} \mid (p(x) \wedge q_2(x)) \xrightarrow{\ll} q(x) \Big\}.$$

*then, by Lemma 10,*

$$\big(p(x) \wedge q_1(x)\big) \coprod^{\ll} \big(p(x) \wedge q_2(x)\big) = \bigwedge_{q(x)\in\mathbf{K}} q(x) \Longrightarrow p(x).$$

*If* $h(x) = \bigwedge_{q(x)\in\mathbf{K}} q(x)$*, then the equation above, together with Lemma 3, leads to*

$$T \coprod^{\preceq} Q = \left\{ X \in \mathrm{Ob}(\mathfrak{C}^{\preceq}) \,\middle|\, h(X) \right\}^{\curlyvee}.$$

*Thus, the coproduct is the object with more common attributes with $T$ and $Q$.*

**Notation 3.** For every nonempty subset $\mathbf{M}$ of $\mathbf{Attr}$, let $\mathcal{F}_{\mathbf{M},Ob} : \mathrm{Ob}(\mathfrak{C}^{\preceq}) \longrightarrow \wp(\mathbf{M})$ be the map sending each $O \in \mathrm{Ob}(\mathfrak{C}^{\preceq})$ into

$$\mathcal{F}_{\mathbf{M},Ob}(O) = \left\{ p(x) \in \mathbf{M} \,\middle|\, \mathrm{def}_O(x) \xrightarrow{\ll} p(x) \right\}. \tag{29}$$

As a consequence of Lemma 5, if the class $\mathbf{M}$ contains the attribute $\Omega(x)$, so does $\mathcal{F}_{\mathbf{M},Ob}(O)$, for every $\mathrm{Ob}(\mathfrak{C}^{\preceq})$-object $O$.

**Lemma 12.** *If a subset $\mathbf{M}$ of $\mathbf{Attr}$ contains an attribute $\mathrm{def}_O(x)$ defining a $\mathfrak{C}^{\preceq}$-object $O$, then the following relation holds.*

$$\mathrm{def}_O(x) \in \mathcal{F}_{\mathbf{M},\mathrm{Ob}}(O). \tag{30}$$

*Proof.* Since $\mathrm{def}_O(x) \xrightarrow{\ll} \mathrm{def}_O(x)$, then equation (29) leads to (30).

**Theorem 13.** *For every nonempty subset $\mathbf{M}$ of $\mathbf{Attr}$, there is a faithful contravariant functor*

$$\mathcal{F}_{\mathbf{M}} : \mathrm{Ob}(\mathfrak{C}^{\preceq}) \longrightarrow \mathbf{Set}$$

*with the object-map $\mathcal{F}_{\mathbf{M},Ob}$.*

*Proof.* Let $O_1 \xrightarrow{\preceq} O_2$ a $\mathfrak{C}^{\preceq}$-morphism. By definition, this morphism leads to $\mathrm{def}_{O_1}(x) \xrightarrow{\ll} \mathrm{def}_{O_2}(x)$. As a consequence of equation (29),

$$\mathcal{F}_{\mathbf{M},Ob}(O_2) \subseteq \mathcal{F}_{\mathbf{M},Ob}(O_1);$$

therefore, the inclusion map

$$m : \mathcal{F}_{\mathbf{M},Ob}(O_2) \longrightarrow \mathcal{F}_{\mathbf{M},Ob}(O_1)$$

is a **Set**-morphism. If $O_1 = O_2$, then the relation $\mathcal{F}_{\mathbf{M}}(O_1) = \mathcal{F}_{\mathbf{M}}(O_2)$ holds and $\mathcal{F}_{\mathbf{M}}\left(O_1 \xrightarrow{\preceq} O_2\right)$ is the identity-map. Since $\mathfrak{C}^{\preceq}$ is a thin category, $\mathcal{F}_{\mathbf{M}}$ is faithful.

**Remark 1.** *By equation (29), the image $\mathcal{F}_{\mathbf{M}}(O)$ of each member of $\mathrm{Ob}(\mathfrak{C}^{\preceq})$ is intrinsic because it consists of attributes of $O$.*

**Corollary 14.** *With the same assumption as in Theorem 13, for every couple of objects $O_1$ and $O_2$, if $\mathbf{M}$ contains their definitions, then the relation*

$$\mathcal{F}_{\mathbf{M}}(O_1) \subseteq \mathcal{F}_{\mathbf{M}}(O_2)$$

*leads to* $O_2 \xrightarrow{\preceq} O_1$.

*Proof.* It is a straightforward consequence of Lemma 3 and Theorem 13, together with equation (29) (see Definition 4).

**Theorem 15.** *If* $\mathbf{K}$ *be a subset of* $\mathrm{Ob}(\mathfrak{C}^{\ll}))$ *of cardinality greater than* 1, *and* $\mathbf{M} \subseteq$ **Attr** *a nonempty attribute set, then the following statements hold.*

1. *If there is the product* $\overset{\preceq}{\prod\limits_{O\in\mathbf{K}}} O$, *then*

$$\bigcup_{O\in\mathbf{K}} \mathcal{F}_{\mathbf{M}}(O) \subseteq \mathcal{F}_{\mathbf{M}}\left(\overset{\preceq}{\prod_{O\in\mathbf{K}}} O\right). \tag{31}$$

2. *The coproduct* $\overset{\preceq}{\coprod\limits_{O\in\mathbf{K}}} O$ *satisfies the following equation.*

$$\mathcal{F}_{\mathbf{M}}\left(\overset{\preceq}{\coprod_{O\in\mathbf{K}}} O\right) = \bigcap_{O\in\mathbf{K}} \mathcal{F}_{\mathbf{M}}(O). \tag{32}$$

*Proof.*

1. It is a property of the product in $\mathfrak{C}^{\preceq}$, that

$$\forall Q \in \mathbf{K} : \quad \overset{\preceq}{\prod_{O\in\mathbf{K}}} O \xrightarrow{\preceq} Q.$$

   The equation above and Theorem 13 lead to (31).

2. By Lemma 10, there is always the coproduct; therefore,

$$\forall O \in \mathbf{K} : \quad O \xrightarrow{\preceq} \overset{\preceq}{\coprod_{U\in\mathbf{K}}} U.$$

As a consequence of Theorem 13,

$$\forall O \in \mathbf{K}: \quad \mathcal{F}_{\mathbf{M}}\left(\coprod_{U \in \mathbf{K}}^{\preceq} U\right) \xrightarrow{m_O} \mathcal{F}_{\mathbf{M}}(O)$$

where $m_O$ is the inclusion map; hence,

$$\mathcal{F}_{\mathbf{M}}\left(\coprod_{O \in \mathbf{K}}^{\preceq} O\right) \subseteq \bigcap_{O \in \mathbf{K}} \mathcal{F}_{\mathbf{M}}(O). \tag{33}$$

Every object $O \in \mathbf{K}$ satisfies each member of $\mathbf{H} = \bigcap_{Q \in \mathbf{K}} \mathcal{F}_{\mathbf{M}}(Q)$. As a consequence, for every $p(x) \in \mathbf{H}$, the class $\left\{O \in \mathrm{Ob}(\mathfrak{C}^{\preceq}) \mid p(O)\right\}$ that it defines and its generic member

$$Q = \left\{O \in \mathrm{Ob}(\mathfrak{C}^{\preceq}) \mid p(O)\right\}^{\curlyvee}$$

the following relation holds.

$$\forall O \in \mathbf{K}: \quad \mathrm{def}_O(x) \Longleftrightarrow p(x)^{\curlyvee} \wedge \mathrm{def}_O(x) = \mathrm{def}_Q(x) \wedge \mathrm{def}_O(x) \tag{34}$$

because $p(x)^{\curlyvee}$ defines $Q$. Thus, the equation above leads to

$$\forall O \in \mathbf{K}: \quad O \xrightarrow{\preceq} Q.$$

As a consequence, the diagram below commutes.

$$\forall O \in \mathbf{K}: \qquad \begin{tikzcd} & \coprod\limits_{O \in \mathbf{K}}^{\preceq} O \arrow[d, "\preceq"] \\ O \arrow[ur, "\preceq"] \arrow[r, "\preceq"'] & Q; \end{tikzcd}$$

hence, there is the arrow

$$\coprod_{O \in \mathbf{K}}^{\preceq} O \xrightarrow{\preceq} Q. \tag{35}$$

This morphism, together with Lemma 12 and Theorem 13, leads to

$$p(x) \in \mathcal{F}_{\mathbf{M}}(Q) \subseteq \mathcal{F}_{\mathbf{M}}\left(\coprod_{O \in \mathbf{M}}^{\preceq} O\right);$$

therefore,

$$\forall p(x) \in \bigcap_{O\in\mathbf{K}} \mathcal{F}_{\mathbf{M}}(O): \quad p(x) \in \mathcal{F}_{\mathbf{M}}\left(\coprod_{O\in\mathbf{K}}^{\preceq} O\right).$$

The equation above and (33) lead to (32).

**Remark 2.** *Equation 32 (Theorem 15) means that the coproduct* $\coprod_{O\in\mathbf{K}}^{\preceq} O$ *possesses all attributes in* $\mathbf{M}$ *that are common to all members of* $\mathbf{K}$*.*

**Lemma 16.** *Let* $O_1$*,* $O_2$*, and* $Q$ *be three* $\mathfrak{C}^{\preceq}$*-objects. If there is the morphism*

$$O_1 \xrightarrow{\preceq} O_2, \tag{36}$$

*the following statements hold.*

1. $\left(O_1 \coprod^{\preceq} O_2\right) = O_2$.

2. $\left(O_1 \coprod^{\preceq} Q\right) \xrightarrow{\preceq} \left(O_2 \coprod^{\preceq} Q\right)$.

3. *If there are the products* $O_1 \prod^{\preceq} Q$ *and* $O_2 \prod^{\preceq} Q$*, then*

$$\left(O_1 \prod^{\preceq} Q\right) \xrightarrow{\preceq} \left(O_2 \prod^{\preceq} Q\right).$$

4. *If* $\left(O_1 \coprod^{\preceq} Q\right) = O_1$*, then* $Q \xrightarrow{\preceq} O_2$*.*

*Proof.* 1. By equation (36), together with coproduct definition, the following diagram commutes.

$$\begin{tikzcd}
& O_1 \coprod^{\preceq} O_2 \arrow[dd, "\preceq"] & \\
O_1 \arrow[ur, "\preceq"] \arrow[dr, "\preceq"'] & & O_2 \arrow[ul, "\preceq"'] \arrow[dl, "\preceq"] \\
& O_2 &
\end{tikzcd}$$

By Definition 4, the diagram above leads to

$$\mathrm{def}_{(O_1 \coprod O_2)}(x) \Longleftrightarrow \mathrm{def}_{O_2}(x);$$

hence, $O_2 = (O_1 \coprod O_2)$.

2. As a consequence of (36), the following diagram commutes

$$\begin{tikzcd}
& O_1 \overset{\preceq}{\coprod} Q \arrow[ddd, "\preceq"] & \\
O_1 \arrow[ur, "\preceq"] \arrow[d, "\preceq"'] & & Q \arrow[ul, "\succeq"'] \arrow[ddl, "\succeq"] \\
O_2 \arrow[dr, "\preceq"'] & & \\
& O_2 \overset{\preceq}{\coprod} Q &
\end{tikzcd}$$

Thus, there is the morphism $\left(O_1 \overset{\preceq}{\coprod} Q\right) \xrightarrow{\preceq} \left(O_2 \overset{\preceq}{\coprod} Q\right)$.

3. The proof for Statement (3) is the dual of preceding one.

4. The relation $\left(O_1 \overset{\preceq}{\coprod} Q\right) = O_1$ leads to

$$Q \xrightarrow{\preceq} \left(O_1 \overset{\preceq}{\coprod} Q\right) = O_1 \xrightarrow{\preceq} O_2.$$

**Theorem 17.** *Let* $\mathbf{M}$ *be a nonempty attribute class and* $\mathbf{K} = \{O_i \mid i \in \mathbf{I}\}$ *a set of* $\mathfrak{C}^{\preceq}$*-objects such that* $\#(\mathbf{I}) \geq 2$*. For every attribute set*

$$\Big\{p_i(x) \in \mathcal{F}_{\mathbf{M}}(O_i) \mid i \in \mathbf{I}\Big\},$$

*the following statements hold.*

1. *If there are the products* $\prod^{\preceq}_{i\in\mathbf{I}} O_i$ *and* $\prod^{\ll}_{i\in\mathbf{I}} p_i(x)$, *then the relation* $\prod^{\ll}_{i\in\mathbf{I}} p_i(x) \in \mathbf{M}$ *leads to*

$$\prod^{\ll}_{i\in\mathbf{I}} p_i(x) \in \mathcal{F}_{\mathbf{M}}\left(\prod^{\preceq}_{i\in\mathbf{I}} O_i\right) \tag{37}$$

2. *If* $\coprod^{\ll}_{i\in\mathbf{I}} p_i(x) \in \mathbf{M}$, *then*

$$\coprod^{\ll}_{i\in\mathbf{I}} p_i(x) \in \mathcal{F}_{\mathbf{M}}\left(\coprod^{\preceq}_{i\in\mathbf{I}} O_i\right) \tag{38}$$

*Proof.*

1. By equation (29), the statement $\forall i \in \mathbf{I} :\ p_i(x) \in \mathcal{F}_{\mathbf{M}}(O_i)$ leads to

$$\forall i \in \mathbf{I} : \quad \mathrm{def}_{O_i}(x) \xrightarrow{\ll} p_i(x); \tag{39}$$

therefore,

$$\bigwedge_{i\in\mathbf{I}} \mathrm{def}_{O_i}(x) \xrightarrow{\ll} \bigwedge_{i\in\mathbf{I}} p_i(x).$$

By Lemma 6, the morphism above leads to

$$\prod^{\ll}_{i\in\mathbf{I}} \mathrm{def}_{O_i}(x) \xrightarrow{\ll} \prod^{\ll}_{i\in\mathbf{I}} p_i(x).$$

This morphism, together with equation (29) and Lemma 7, leads to statement (1).

2. As in the proof of Statement (1), by (39), the diagram below commutes.

$$\forall i \in \mathbf{I} : \qquad \begin{tikzcd} & \coprod^{\ll}_{j\in\mathbf{I}} p_j(x) \\ \mathrm{def}_{O_i}(x) \arrow[ur, "\ll"] \arrow[dr, "\ll"'] & \\ & \coprod^{\ll}_{j\in\mathbf{I}} \mathrm{def}_{O_j}(x) \arrow[uu, "\ll"'] \end{tikzcd}$$

Thus, $\coprod\limits_{i\in\mathbf{I}}^{\ll} \mathrm{def}_{O_i}(x) \xrightarrow{\ll} \coprod\limits_{i\in\mathbf{I}}^{\ll} p_i(x)$. This arrow, together with equation (29), leads to (38).

## 3. Morphisms among heterogeneous structures

Let $S_1$, $S_2$, and $S_3$ be three systems evaluating the map

$$\forall n \in \mathbb{N}: \quad f(n) = n^2, \tag{40}$$

and working at each $n \in \mathbb{N}$ as follows.

1. The first one, $S_1$, consists of an infinite table

| $n$ | $n^2$ |
|---|---|
| 1 | 1 |
| 2 | 4 |
| 3 | 9 |
| ...... | |

Table 1.

   together with an algorithm searching each positive integer $n$ in the first column, and returning $n^2$ lying in the same row, but in the second column.

2. The second system, $S_2$, consists of an algorithm calculating the square of each positive integer $n$ and returning $n \times n = n^2$.

3. The last system, $S_3$, from a finite sub-table of Table 1, can find the law $n \mapsto n^2$ and then works as $S_2$.

The three systems evaluate the map $f(n)$. Nevertheless, $S_2$ is more efficient than $S_1$, and $S_3$ is cleaver than $S_2$. This system extends $f$ from a sample of its values. To this end, it finds out the underlying law defining it. The law is a common attribute of all pairs $(n, n^2)$ in $f$; hence, a predicate that we can denote as follows.

$$p(x) = \text{ “}x \textit{ is a pair } (n, m) \textit{ of positive integers such that } m = n^2.\text{”} \tag{41}$$

By Theorem 15, if $\mathbf{M}$ contains $p(x)$, then

$$\forall (n,m) \in \mathbb{N} \times \mathbb{N} : \quad p(x) \in \mathcal{F}_{\mathbf{M}} \left( (n, n^2) \overset{\preceq}{\coprod} (m, m^2) \right).$$

Thus, unlike algorithms, $S_3$ handles attributes instead of symbol sequences. As in (41), we can obtain the existence of common attributes through coproducts in the category $\mathfrak{C}^{\preceq}$.

By definition, the class $\mathrm{Ob}(\mathfrak{C}^{\preceq})$ contains every finitely-definable entity. Nevertheless, there are sets in $\mathrm{Ob}(\mathfrak{C}^{\preceq})$ such that not every member is a $\mathfrak{C}^{\preceq}$-object. For example, consider the real number set $\mathbb{R}$. It is finitely-definable, but it contains some members that are not [7]. If for every $\alpha \in \mathbb{R}$ there is a finite symbol sequence defining it, any Gödel-like numbering function sends $\mathbb{R}$ into $\mathbb{N}$ and $\mathbb{R}$ would be countable [7].

As a consequence, $\mathrm{Ob}(\mathfrak{C}^{\preceq})$ contains every finitely-definable category together with its objects and morphisms. Unlike classical universal categories [8], if $\mathbf{C}$ is a finitely-definable category, $\mathbf{C}$-morphisms are $\mathfrak{C}^{\preceq}$-objects.

**Notation 4.** To simplify expressions, we denote by $\overset{\star}{\prod}$ and $\overset{\star}{\coprod}$ the operators defined as follows. For every subset $\mathbf{K}$ of $\mathrm{Ob}(\mathfrak{C}^{\preceq})$,

$$\#(\mathbf{K}) \geq 2 : \quad \begin{cases} \overset{\star}{\prod\limits_{O \in \mathbf{K}}} O = \overset{\preceq}{\prod\limits_{O \in \mathbf{K}}} O \\ \overset{\star}{\coprod\limits_{O \in \mathbf{K}}} O = \overset{\preceq}{\coprod\limits_{O \in \mathbf{K}}} O \end{cases} \tag{42}$$

$$\#(\mathbf{K}) = 1 : \quad \begin{cases} \overset{\star}{\prod\limits_{O \in \mathbf{K}}} O = O \\ \overset{\star}{\coprod\limits_{O \in \mathbf{K}}} O = O \end{cases} \tag{43}$$

$$\mathbf{K} = \emptyset : \quad \overset{\star}{\prod\limits_{O \in \mathbf{K}}} O = \overset{\star}{\coprod\limits_{O \in \mathbf{K}}} O = \emptyset. \tag{44}$$

Again to simplify expressions, for every subset $\mathbf{K}$ of $\mathrm{Ob}(\mathfrak{C}^{\ll})$, we write the following notations.

$$\#(\mathbf{K}) \geq 2: \quad \begin{cases} \overset{\lozenge}{\prod_{O\in\mathbf{K}}} O = \overset{\ll}{\prod_{O\in\mathbf{K}}} O \\ \overset{\lozenge}{\coprod_{O\in\mathbf{K}}} O = \overset{\ll}{\coprod_{O\in\mathbf{K}}} O \end{cases} \tag{45}$$

$$\#(\mathbf{K}) = 1: \quad \begin{cases} \overset{\lozenge}{\prod_{O\in\mathbf{K}}} O = O \\ \overset{\lozenge}{\coprod_{O\in\mathbf{K}}} O = O \end{cases} \tag{46}$$

$$\mathbf{K} = \emptyset: \quad \overset{\lozenge}{\prod_{O\in\mathbf{K}}} O = \overset{\lozenge}{\coprod_{O\in\mathbf{K}}} O = \emptyset. \tag{47}$$

When there is no confusion, we denote products and coproducts by the generic superscript $^{\odot}$. Thus, $\overset{\odot}{\prod_{O\in\mathbf{K}}} O$, denotes either the product $\overset{\lozenge}{\prod_{O\in\mathbf{K}}} O$ when $\mathbf{K} \subset \mathrm{Ob}(\mathfrak{C}^{\ll})$, or $\overset{\preceq}{\prod_{O\in\mathbf{K}}} O$ when $\mathbf{K} \subseteq \mathrm{Ob}(\mathfrak{C}^{\preceq})$.

**Notation 5.** By the expression $\mathrm{Ob}\,(\Omega[\mathbf{DCat}])$ we denote the collection of all subclasses of $\mathrm{Ob}(\mathfrak{C}^{\ll}) \cup \mathrm{Ob}(\mathfrak{C}^{\preceq})$.

**Notation 6.** The symbol $\wp_F$, denotes the map sending each set into the class of its finite subsets; hence,

$$\forall \mathbf{K} \in \mathrm{Ob}(\mathbf{Set}): \quad \wp_F(\mathbf{K}) = \Big\{ X \subseteq \mathbf{K} \mid \#(X) < \infty \Big\}.$$

**Definition 8.** *For every member $O$ of $\mathrm{Ob}(\Omega[\mathbf{DCat}])$, we say that a subset $\mathbf{K}_O$ of $\wp_F\left(\mathrm{Ob}\left(\Omega[\mathbf{DCat}]\right)\right)$ is a generator for $O$ when the following equation holds.*

$$O = \overset{\odot}{\prod_{D\in\mathbf{K}_O}} \left( \overset{\odot}{\coprod_{Q\in D}} Q \right). \tag{48}$$

*From now on, we denote by $\mathrm{Gens}(O)$ the family of all generators for $O$.*

**Lemma 18.** *If $A \in \mathrm{Ob}(\Omega[\mathbf{DCat}])$ is nonempty, for every $O \in A$ there is at least one generator.*

*Proof.* The set $\big\{\{O\}\big\} \subseteq \wp_F(A)$ satisfies the equation

$$O = \prod_{D \in \big\{\{O\}\big\}}^{\odot} \left( \coprod_{Q \in D}^{\odot} Q \right) = \coprod_{Q \in \{O\}}^{\odot} Q = O$$

because both $\big\{\{O\}\big\}$ and $\{O\}$ are singletons. See (43) and (46).

**Definition 9.** *The class* $\mathrm{Ob}\,(\Omega[\mathbf{DCat}])$ *is the object one of a category* $\Omega[\mathbf{DCat}]$ *such that, for every pair of objects* $(A_1, A_2)$*, the homset* $\hom(A_1, A_2)$ *consists of every map* $f : A_1 \longrightarrow A_2$ *that satisfies the following conditions.*

(1) $\exists \mathbf{K}_O \in \mathrm{Gens}(O): \quad \mathbf{K}_{f(O)} = \big\{ f[D] \mid D \in \mathbf{K}_O \big\} \in \mathrm{Gens}(f(O))$,

(2) $$f\left( \prod_{D \in \mathbf{K}_O}^{\odot} \left( \coprod_{Q \in D}^{\odot} Q \right) \right) = \prod_{D \in \mathbf{K}_O}^{\odot} f\left( \coprod_{Q \in D}^{\odot} Q \right) = \prod_{D \in \mathbf{K}_O}^{\odot} \left( \coprod_{Q \in D}^{\odot} f(Q) \right) = \prod_{D \in \mathbf{K}_O}^{\odot} \left( \coprod_{Q \in f[D]}^{\odot} Q \right) = \prod_{D \in \mathbf{K}_{f(O)}}^{\odot} \left( \coprod_{Q \in D}^{\odot} Q \right).$$

For every object $O$, the $O$-identity satisfies the definition above. Likewise, morphism-composition and its associativity are a straightforward consequence of it. We say that the underlying generator in each morphism is one associated with it.

In ordinary categories, morphism domains and codomains are objects with similar structures. By contrast, we can define $\Omega[\mathbf{DCat}]$-morphisms between objects of different categories; hence, they can be heterogeneous. Although $\Omega[\mathbf{DCat}]$-morphisms are maps, this heterogeneous category includes $\mathfrak{C}^{\preceq}$ objects and morphisms. These are posets closely related with $\Omega$-categories [4]. This is why we use the prefix $\Omega$ in our notation.

**Lemma 19.** *The value of an* $\Omega[\mathbf{DCat}]$*-morphism* $f : A_1 \to A_2$ *at each member* $O$ *of its domain is uniquely determined by its values in any associated generator* $\mathbf{K}_O \in \mathrm{Gens}(O)$.

*Proof.* Let $f : A_1 \longrightarrow A_2$ and $g : A_1 \longrightarrow A_2$ be two morphisms and $\mathbf{K}_O$ a generator for $O \in A_1$ such that

$$\forall D \in \mathbf{K}_O, \forall Q \in D: \quad f(Q) = g(Q).$$

By assumption,

$$f(O) = f\left(\prod_{D\in\mathbf{K}_O}^{\odot}\left(\coprod_{Q\in D}^{\odot} Q\right)\right) = \prod_{D\in\mathbf{K}_O}^{\odot}\left(\coprod_{Q\in D}^{\odot} f(Q)\right) = \prod_{D\in\mathbf{K}_O}^{\odot}\left(\coprod_{Q\in D}^{\odot} g(Q)\right) = g\left(\prod_{D\in\mathbf{K}_O}^{\odot}\left(\coprod_{Q\in D}^{\odot} Q\right)\right) = g(O). \quad (49)$$

**Theorem 20.** *Let $A \in \mathrm{Ob}\,(\Omega[\mathbf{DCat}])$ be a nonempty attribute class and $D$ the subclass of $\mathrm{Ob}\,(\Omega[\mathbf{DCat}])$ of all objects defined by members of $A$. The map $f : A \longrightarrow D$, sending each attribute $p(x) \in A$ into the object that it defines, is an $\Omega[\mathbf{DCat}]$-morphism.*

*Proof.* Let $p(x)$ be a member of $A$ and $\mathbf{K}_{p(x)}$ a generator of it; hence,

$$p(x) = \prod_{D\in\mathbf{K}_{p(x)}}^{\ll}\left(\coprod_{q(x)\in D}^{\ll} q(x)\right).$$

By Lemma 7, the equation above leads to

$$f\left(p(x)\right) = f\left(\prod_{D\in\mathbf{K}_{p(x)}}^{\ll}\left(\coprod_{q(x)\in D}^{\ll} q(x)\right)\right) = \prod_{D\in\mathbf{K}_{p(x)}}^{\preceq} f\left(\coprod_{q(x)\in D}^{\ll} q(x)\right)$$

and by Theorem 11 this equation leads to

$$f\left(\prod_{D\in\mathbf{K}_{p(x)}}^{\ll}\left(\coprod_{q(x)\in D}^{\ll} q(x)\right)\right) = \prod_{D\in\mathbf{K}_{p(x)}}^{\preceq}\left(\coprod_{q(x)\in D}^{\preceq} f\left(q(x)\right)\right);$$

therefore,

$$f\left(\prod_{D\in\mathbf{K}_{p(x)}}^{\odot}\left(\coprod_{q(x)\in D}^{\odot} q(x)\right)\right) = \prod_{D\in\mathbf{K}_{p(x)}}^{\odot}\left(\coprod_{q(x)\in D}^{\odot} f\left(q(x)\right)\right).$$

### 3.1. Morphism extensions

Information extension is a substantial intelligent system capability. Extending maps from samples of their values is a powerful skill. We consider heterogeneous extensions arisen from $\Omega[\mathbf{DCat}]$-morphisms.

The map being easier to extend is the identity one and the like. Since different objects can have common attributes, maps between definitions can be easier to extend.

**Definition 10.** *Let $f : A_1 \longrightarrow A_2$ be an $\Omega[\mathbf{DCat}]$-morphism and $O$ a member of $A_1$ such that the associated generator $\mathbf{K}_O$ contains a singleton $\{Q\}$. Assume that there is an $\Omega[\mathbf{DCat}]$-morphism $g : B_1 \longrightarrow B_2$ such that $\mathbf{H}_Q \in Gens(Q)$ and*

$$f(Q) = g\left(\overset{\preceq}{\prod_{W\in\mathbf{H}_Q}}\left(\overset{\preceq}{\coprod_{V\in W}} V\right)\right) = \overset{\preceq}{\prod_{W\in\mathbf{H}_Q}}\left(\overset{\preceq}{\coprod_{V\in W}} g(V)\right). \tag{50}$$

*With these assumptions, we term $Q$-sub-extension the result of substituting $\{Q\}$ with $\mathbf{H}_Q$ in $\mathbf{K}_O$, $\{f(Q)\}$ with $f[\mathbf{H}_Q]$ in $f[\mathbf{K}_O]$, and extending $f$ with $g$ according to equation (50). Likewise, we term $g$-sub-extension the result of all $Q$-sub-extensions for every $Q \in \mathrm{dom}(g)$.*

**Theorem 21.** *Let $A$ and $B$ be two discernible classes with determining subsets $D_A$ and $D_B$, respectively. Let $f : D_A \to B$ be a map that satisfies the following conditions.*

**C1:** *There is an $\Omega[\mathbf{DCat}]$-morphism $g : \complement^{\curlyvee}_{\preceq A} A^{\curlyvee} \longrightarrow \complement^{\curlyvee}_{\preceq B} B^{\curlyvee}$ that, for every $O \in D_A$, sends $Q_O = \complement^{\curlyvee}_{\preceq O} A^{\curlyvee}$ into $Q_{f(O)} = \complement^{\curlyvee}_{\preceq f(O)} B^{\curlyvee}$.*

**C2:** *$D_B \subseteq f[D_A]$.*

*With these assumptions,there is an extension $f^* : A \longrightarrow B$ of $f$ to $A$ being an $\Omega[\mathbf{DCat}]$-morphism that can be sub-extended with $g$.*

*Proof.* By hypothesis, the coproducts $\overset{\odot}{\coprod\limits_{P\in D_A}} \mathrm{def}_P(x)$ and $\overset{\odot}{\coprod\limits_{P\in D_B}} \mathrm{def}_P(x)$ define $A$ and $B$, respectively. These equations, together with Theorem 11, lead to

$$A^{\curlyvee} = \overset{\odot}{\coprod_{P\in D_A}} P \quad \textit{and} \quad B^{\curlyvee} = \overset{\odot}{\coprod_{P\in D_B}} P.$$

By definition 5,

$$\forall O \in A: \quad O = \left(\overset{\odot}{\coprod_{P\in D_A}} P\right) \overset{\odot}{\prod} \complement^{\curlyvee}_{\preceq O} A^{\curlyvee}.$$

Thus, taking into account **C2**, if for every $O \in A \setminus D_A$,

$$f^*(O) = \left( \coprod_{P \in D_A}^{\odot} f(P) \right) \coprod^{\odot} g\left( \complement^{\curlyvee}_{\preceq O} A^{\curlyvee} \right); \tag{51}$$

then

$$\forall O \in A : f^*\left( \complement^{\curlyvee}_{\preceq O} A^{\curlyvee} \right) = \complement^{\curlyvee}_{\preceq f^*(O)} B^{\curlyvee}; \tag{52}$$

hence,

$$f^*(O) = f^*\left( \left( \coprod_{P \in D_A}^{\odot} P \right) \coprod^{\odot} \left( \complement^{\curlyvee}_{\preceq O} A^{\curlyvee} \right) \right) = \left( \coprod_{P \in f[D_A]}^{\odot} P \right) \coprod^{\odot} \left( \complement^{\curlyvee}_{\preceq f^*(O)} B^{\curlyvee} \right). \tag{53}$$

If for each $O \in A$, $Q_O = \complement^{\curlyvee}_{\preceq O} A^{\curlyvee}$ and $\mathbf{K}_O = \left\{ D_A, \{Q_O\} \right\}$, the equation above leads to

$$f^*(O) = \coprod_{D \in \mathbf{K}_O}^{\odot} \left( \coprod_{P \in D}^{\odot} f^*(P) \right) = \coprod_{D \in \mathbf{K}_O}^{\odot} \left( \coprod_{P \in f^*[D]}^{\odot} P \right); \tag{54}$$

hence, $f^*$ is an $\Omega[\mathbf{DCat}]$-morphism. Since, for each $O \in A$, $\mathbf{K}_O$ contains the singleton $\{Q_O\}$, there is the $g$-sub-extension of $f^*$. To this end, the suitable generator is $\mathbf{K}'_O = \left\{ D_A, \mathbf{H}_O \right\}$, for each $O \in A$. This is the result of replacing the singleton $\{Q_O\}$ with $\mathbf{H}_O$ in each generator $\mathbf{K}_O$.

## 4. Non-algorithmic procedures

In this section, we are concerned with non-algorithmic procedures. To this end, we need an algorithm concept definition [3]. The following ones are the most generally accepted.

**D1** An algorithm is an effective procedure to solve a problem.

**D2** Algorithms are computational processes defined by Turing machines [5].

**D3** An algorithm is a set of rules transforming a symbol-sequence denoting a problem into another that denotes its solution.

**D4** An algorithm is a finite procedure, written in a fixed symbolic vocabulary, governed by precise instructions, moving in discrete steps, whose execution requires no insight, cleverness, intuition, intelligence, or perspicuity, and that sooner or later comes to an end [2].

**D5** As a consequence of **D4**, an algorithm is a finite procedure created by an intelligent system that knows its aim and understands each of its steps.

Definition **D1** is ambiguous and does not indicate who can handle the procedure. By contrast, definition **D2** states that it is a Turing machine process. In **D3**, actions work through symbol sequences of a language. In definition **D4**, the procedure is performed blindly, ignoring its goal and the meaning of the involved actions.

Asking for the solution of a problem, to somebody that knows how to solve it, fits into definition **D1**. It is a statement too ambiguous. The second definition is restricted to Turing machines. An infinite set of rules fits into definition **D3**. By condition **D5**, we can know no algorithm until it is created. However, the algorithm construction can be the result of observing properties of the Real World. For example, after observing that appending two objects to some collection of five, we obtain seven, we can build the algorithm consisting of the substitution of the symbols $5 + 2$ with 7. According to **D4**, we assume each algorithm to satisfy the following conditions.

**C1** Algorithms can be denoted by finite symbol sequences in a fixed language generated by a symbolic vocabulary.

**C2** Algorithms work blindly, handling symbol sequences and ignoring their meanings and aims.

**C3** Every algorithm, from an initial symbol sequence (input), obtains a final one (output), denoting an entity that satisfies some required properties.

The conditions above give rise to the following algorithm attributes.

$Cnd_1(x) =$ *"x is a procedure that can be denoted by finite symbol sequences in a fixed language generated by a symbolic vocabulary."*

$Cnd_2(x) =$ *"x is a procedure that works blindly, handling symbol sequences and ignoring their meanings and aims. It must be created by a system that understands its structure."*

$Cnd_3(x) =$ *"x is a procedure that, from an initial symbol sequence (input), obtains a final one (output), denoting an entity that satisfies some required properties."*

As a consequence, if $\mathbf{M}$ contains the attributes $Cnd_1(x)$, $Cnd_2(x)$, and $Cnd_3(x)$, for every algorithm $\mathcal{A}$, the relation below holds.

$$\Big\{Cnd_1(x), Cnd_2(x), Cnd_3(x)\Big\} \subseteq \mathcal{F}_{\mathbf{M}}(\mathcal{A}). \tag{55}$$

We do not consider a procedure as an algorithm when it is language independent and its image under $\mathcal{F}_{\mathbf{M}}$ does not satisfy condition (55). Unlike algorithms, non-algorithmic procedures need not be successful. They work frequently as the scientific method consisting of test, error, correction. Thus, we can define the non-algorithmic-procedure concept as follows.

**Definition 11.** *A procedure $\mathfrak{P}$ is non-algorithmic whenever it depends on the attributes of the entities that it involves, and it is language independent.*

**Notation 7.** We denote the input-output classes of any procedure $\mathfrak{P}$ by a function-like notation. Thus, $\mathrm{dom}(\mathfrak{P})$ is the class of all objects that we can apply $\mathfrak{P}$ and $\mathrm{img}(\mathfrak{P})$ the corresponding objects that $\mathfrak{P}$ builds.

**Definition 12.** *We term* eulerithm *each non-algorithmic procedure $\mathfrak{P}$ satisfying the conditions below.*

1. *From each object $O$ in its domain, $\mathfrak{P}$ can find the corresponding output $\mathfrak{P}(O) \in \mathrm{img}(\mathfrak{P})$ in a finite step-sequence.*

2. *If $O_1, O_2 \dots O_k$ is the object sequence that $\mathfrak{P}$ constructs starting from $O_1$ up to $O_k \in \mathrm{img}(\mathfrak{P})$, for every index $n < k$, $\mathfrak{P}$ determines $O_{n+1}$ from the attributes of both objects $O_n$ and $O_{n+1}$.*

**Example 4.** *Let $\vec{v}$ be a vector such that $|\vec{v}| > 0$. For each $n \in \mathbb{N}$, let $\vec{v}^{\,n}$ be the power defined as follows.*

$$\vec{v}^{\,n} = \begin{cases} |\vec{v}|^n & \text{if } n \text{ is even,} \\ |\vec{v}|^{\,(n-1)} \cdot \vec{v} & \text{if } n \text{ is odd.} \end{cases}$$

*This power definition has some common properties with the powers of $i$. The result of every even power is of different nature from odd ones. From these common properties we can generalize equation (2) as follows.*

$$e^{i\cdot\vec{v}} = cos(|\vec{v}|) + sin(|\vec{v}|)\frac{i\vec{v}}{|\vec{v}|}.$$

**Remark 3.** *It is a consequence of Theorem 15, those procedures based on $\preceq$-products and $\preceq$-coproducts can be stated as eulerithms. For instance, consider a procedure $P$*

*solving a problem class $K$. To consider $P$ as one procedure instead of a collection of them, it can only accept problems lying in a homogeneous class* $\mathrm{Input}(K)$*. If $P_1$ and $P_2$ are members of* $\mathrm{Input}(K)$*, they must have a similar structure characterized by some common attribute set $Q$; hence, for every suitable attribute class $\mathbf{M}$, $Q \subseteq \mathcal{F}_{\mathbf{M}}\left(P_1 \coprod^{\odot} P_2\right)$.*

**Remark 4.** *Since algorithms work blindly, ignoring the meaning of the symbol sequences that they handle, at least from a remote instance, they must be constructed by a non-algorithmic procedure. Thus, every algorithm is the result of a finite sequence of procedures starting from a non-algorithmic method.*

The situation is similar using patterns. These are procedures containing variable occurrences to be substituted by objects or some symbols denoting them. Nevertheless, patterns only can accept occurrences of those objects satisfying some required properties. Thus, procedures based on patterns depend on property knowledge too. At least, the procedure creator must know them.

In those sets enriched with topologies, map extensions require the underlying functions to be continuous. Since $\Omega[\mathbf{DCat}]$-morphisms are maps, we need some kind of continuity to state extensions properly.

**Definition 13.** *We say that a procedure $\mathfrak{P}$ is continuous in a pair $(O_1, O_2) \in \mathrm{dom}(\mathfrak{P}) \times \mathrm{dom}(\mathfrak{P})$ when it satisfies the following conditions.*

**A:** *There is the morphism $O_1 \xrightarrow{\preceq} O_2$.*

**B:** *For every $O \in \mathrm{dom}(\mathfrak{P})$ the relation*

$$\left(O \coprod^{\preceq} O_1\right) \xrightarrow{\preceq} \left(O_1 \coprod^{\preceq} O_2\right),$$

*leads to*

1. $\left(\mathfrak{P}(O) \coprod^{\preceq} \mathfrak{P}(O_1)\right) \xrightarrow{\preceq} \left(\mathfrak{P}(O_1) \coprod^{\preceq} \mathfrak{P}(O_2)\right)$.
2. *There are the following complements (definition 5).*

$$\complement^{\curlyvee}_{\preceq O}\left(O_1 \coprod^{\preceq} O_2\right) \quad \text{and} \quad \complement^{\curlyvee}_{\preceq \mathfrak{P}(O)}\left(\mathfrak{P}(O_1) \coprod^{\preceq} \mathfrak{P}(O_2)\right).$$

**Theorem 22.** *Let $\mathfrak{P}$ be a continuous procedure between $O_1$ and $O_2$. Let* $\mathbf{K}$ *and* $\mathbf{H}$ *be the classes*

$$\mathbf{K} = \Big\{ O \in \mathrm{Ob}\,(\Omega[\mathbf{DCat}]) \mid O \xrightarrow{\preceq} O_1 \coprod^{\curlyvee} O_2 \Big\} \tag{56}$$

$$\mathbf{H} = \Big\{ O \in \mathrm{Ob}\,(\Omega[\mathbf{DCat}]) \mid O \xrightarrow{\preceq} \mathfrak{P}(O_1) \coprod^{\curlyvee} \mathfrak{P}(O_2) \Big\}. \tag{57}$$

*If the map $g$ that sends $\big(\mathtt{C}^{\curlyvee}_{\preceq O}\mathbf{K}^{\curlyvee}\big)$ into $\mathtt{C}^{\curlyvee}_{\preceq \mathfrak{P}(O)}\mathbf{H}^{\curlyvee}$ is an $\Omega[\mathbf{DCat}]$-morphism, the following statements hold.*

1. *The determining sets for* $\mathbf{K}$ *and* $\mathbf{H}$ *are $\big\{O_1, O_2\big\}$ and $\big\{\mathfrak{P}(O_1), \mathfrak{P}(O_2)\big\}$, respectively.*

2. *There is an extension $f^*$ of the map $f$ that sends $O_1$ and $O_2$ into $\mathfrak{P}(O_1)$ and $\mathfrak{P}(O_2)$, respectively, and it is an $\Omega[\mathbf{DCat}]$-morphism.*

3. *The map $f^*$ can be sub-extended with $g$.*

*Proof.* 1. Statement (1) is a straightforward consequence of equations (56) and (57).

2. By definition, $f$ sends the determining set $\{O_1, O_2\}$ of $\mathbf{K}$ into the subset $\big\{\mathfrak{P}(O_1), \mathfrak{P}(O_2)\big\}$ of $\mathbf{H}$. Thus, taking into account definition 13, both maps $f$ and $g$ satisfy the conditions of Theorem 21; hence, there is the extension $f^*$ of $f$.

3. It is a consequence of the statement above.

**Remark 5.** *Continuous procedures satisfying the conditions of the theorem above are eulerithms because their underlying maps are $\Omega[\mathbf{DCat}]$-morphisms consisting of products and coproducts in the category $\mathfrak{C}^{\preceq}$; hence, they are based on attributes. The example below illustrates this topic.*

**Example 5.** *Consider the equation-classes $P = \big\{K(x) = c \mid c \in \mathbb{R}\big\}$ and $S = \big\{x = K^{-1}(c) \mid c \in \mathbb{R}\big\}$, where $K : X \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is a bijective function. Assume that* $\mathbf{M}$ *contains every attribute in this example. Consider the attributes below.*

$$
\begin{aligned}
p_1(x) &= \text{“}x \textit{ is an equation with one unknown } z,\text{”}\\
p_2(x) &= \text{“\textit{the left-hand of} } x \textit{ consists of a bijection with unknown argument,}\text{”}\\
p_3(x) &= \text{“\textit{the right-hand of} } x \textit{ consists of an occurrence of a real number}\\
&\quad \textit{explicitly stated,}\text{”}\\
q_1(x) &= \text{“\textit{the left-hand of} } x \textit{ consists of only one occurrence of } z\text{ ,”}\\
q_2(x) &= \text{“\textit{the right-hand of} } x \textit{ consists of a real number as the argument of}\\
&\quad \textit{the inverse of a bijection } g : X \subseteq \mathbb{R} \to \mathbb{R},\text{”}\\
h_1(x) &= \text{“\textit{the known real number in the expression} } x \textit{ is } 9,\text{”}\\
h_2(x) &= \text{“\textit{the known real number in the expression} } x \textit{ is } 3,\text{”}\\
h_3(x) &= \text{“\textit{the known real number in the expression} } x \textit{ is } 2,\text{”}
\end{aligned}
$$

$$
\begin{aligned}
r_1(x) &= \text{“\textit{the bijection that} } x \textit{ involves is } g(z) = \frac{1}{z-1} \textit{ and its inverse is}\\
&\quad g^{-1}(z) = \frac{1}{z} + 1,\text{”}\\
r_2(x) &= \text{“\textit{the bijection that} } x \textit{ involves is } g(z) = \frac{4}{z} \textit{ and its inverse is}\\
&\quad g^{-1}(z) = \frac{4}{z},\text{”}\\
r_3(x) &= \text{“\textit{the bijection that} } x \textit{ involves is } g(z) = 5z \textit{ and its inverse is}\\
&\quad g^{-1}(z) = \frac{z}{5}.\text{”}
\end{aligned}
$$

*Let $\mathbf{Q}^{\curlyvee}$ denote the generic object of the class $\mathbf{Q}$ that $(h_1(x) \wedge r_1(x))$ defines. The conjunction $P(x) = p_1(x) \wedge p_2(x) \wedge p_3(x) \wedge h_1(x) \wedge r_1(x)$ defines the equation $\frac{1}{z-1} = 9$. The conjunction $S(x) = q_1(x) \wedge q_2(x) \wedge h_1(x) \wedge r_1(x)$ defines its solution $z = \frac{1}{9} + 1$. An eulerithm can consist of a procedure sending $P(x)$ into $S(x)$. By Theorem 22 we can extend a sample as $\Omega[\mathbf{DCat}]$-morphisms do. For instance, consider the sample below of the map $f : P \cup \mathbf{Q} \longrightarrow S \cup \mathbf{Q}$ sending each equation $E \in P$ into its solution $f(E) \in S$ such that its restriction to $\mathbf{Q}$ is the identity.*

$$
\begin{cases}
f\left(\frac{4}{z} = 3\right) = \left(z = \frac{4}{3}\right),\\
f\left(5z = 2\right) = \left(z = \frac{2}{5}\right),\\
\forall Q \in \mathbf{Q} :\ f(Q) = Q
\end{cases}
$$

*Since the restriction $f|_{\mathbf{Q}}$ of $f$ to $\mathbf{Q}$ is the identity, it is an $\Omega[\mathbf{DCat}]$-morphism. By Theorem 15, the following relations hold.*

$$(p_1(x) \wedge p_2(x) \wedge p_3(x)) \in \mathcal{F}_{\mathbf{M}}\left(\left(\frac{4}{z}=3\right)\coprod^{\ll}(5z=2)\right), \tag{58}$$

$$(q_1(x) \wedge q_2(x)) \in \mathcal{F}_{\mathbf{M}}\left(\left(z=\frac{4}{3}\right)\coprod^{\ll}\left(z=\frac{2}{5}\right)\right), \tag{59}$$

$$\forall Q \in \mathbf{Q}: \quad (h_1(x) \wedge r_1(x)) \in \mathcal{F}_{\mathbf{M}}(Q). \tag{60}$$

*The involved classes* $\mathbf{K}$ *and* $\mathbf{H}$ *are*

$$\mathbf{K} = \left\{O \in \mathrm{Ob}\left(\Omega[\mathbf{DCat}]\right) \mid O \xrightarrow{\preceq} \left(\frac{4}{z}=3\right)\coprod^{\ll}(5z=2)\right\},$$
$$\mathbf{H} = \left\{O \in \mathrm{Ob}\left(\Omega[\mathbf{DCat}]\right) \mid O \xrightarrow{\preceq} \left(z=\frac{4}{3}\right)\coprod^{\ll}\left(z=\frac{2}{5}\right)\right\}.$$

*The equations above lead to the extension* $f^*$ *of* $f$ *such that*

$$\begin{aligned} f^*\left(\frac{1}{z-1}=9\right) = f\left(\left(\left(\frac{4}{z}=3\right)\coprod^{\preceq}(5z=2)\right)\prod^{\preceq}\mathbf{Q}^{\curlyvee}\right) = \\ \left(f\left(\frac{4}{z}=3\right)\coprod^{\preceq} f\,(5z=2)\right)\prod^{\preceq} f\left(\mathbf{Q}^{\curlyvee}\right) = \\ \left(\left(z=\frac{4}{3}\right)\coprod^{\preceq}\left(z=\frac{5}{2}\right)\right)\prod^{\preceq}\mathbf{Q}^{\curlyvee} = \left(z=\frac{1}{9}+1\right) \end{aligned} \tag{61}$$

*because, as a consequence of (60),*

$$p_1(x) \wedge p_2(x) \wedge p_3(x) \wedge h_1(x) \wedge r_1(x) \in \mathcal{F}_{\mathbf{M}}\left(\left(\left(z=\frac{4}{3}\right)\coprod^{\preceq}\left(z=\frac{5}{2}\right)\right)\prod^{\preceq}\mathbf{Q}^{\curlyvee}\right),$$

*and, if* $P_1 = \left(\frac{1}{z-1}=9\right)$ *and* $P_2 = \left(z=\frac{1}{9}+1\right)$, *the relations below hold.*

$$\mathbf{Q}^{\curlyvee} = \complement^{\curlyvee}_{\preceq P_1}\mathbf{K}^{\curlyvee} = \complement^{\curlyvee}_{\preceq P_2}\mathbf{H}^{\curlyvee}.$$

*In equation (61), we extend* $f$ *as an* $\Omega[\mathbf{DCat}]$*-morphism (Theorem 21). The procedure is an eulerithm because, like* $\Omega[\mathbf{DCat}]$*-morphism extensions, it handles attributes instead of symbol sequences. As a consequence of Theorem 22, we can obtain the same result with attributes defining each equation instead of the equations themselves.*

Procedures based on attributes build pairs *equation* $\mapsto$ *solution*, algorithm *input* $\mapsto$ *output*, *theorem* $\mapsto$ *proof*, eulerithm *input* $\mapsto$ *output*, etc. Procedure classification depends on common attributes of the members of domains and codomains. We cannot consider as the same procedure a rule set to solve an equation class and an integration method. As a consequence, we require both classes $\mathrm{dom}(\mathfrak{P})$ and $\mathrm{img}(\mathfrak{P})$, of every for every intrinsic procedure $\mathfrak{P}$, to be discernible (definition 6).

Every procedure $\mathfrak{P}$ that preserves $\preceq$-products and $\preceq$-coproducts is equivalent to an $\Omega[\mathbf{DCat}]$-morphism because, for every $\mathbf{K} \subseteq \wp_F(\mathrm{dom}(\mathfrak{P}))$, the relation below holds when the involved products and coproducts exist.

$$\mathfrak{P}\left(\overset{\preceq}{\prod_{D\in\mathbf{K}}}\left(\overset{\preceq}{\coprod_{O\in D}} O\right)\right) = \overset{\preceq}{\prod_{D\in\mathbf{K}}}\left(\overset{\preceq}{\coprod_{O\in D}}\mathfrak{P}(O)\right).$$

## 5. Algorithm creation

A proper method for eulerithm research is the creation of algorithms. Information extension is the main aim of researching and learning. When we learn a language, knowing a finite set $D$ of sentence-meaning pairs, we can understand and build sentences lying in a superset $D^*$ of $D$. This is possible because $D^*$ is not a random extension of $D$. By contrast, each sentence-meaning pair in $D^* \setminus D$ preserves some logic structure of members of $D$. This is why we are concerned with eulerithms and algorithms to build information extensions.

Although $\Omega[\mathbf{DCat}]$-object structures need not be the same, their logical bases consist of attributes; hence, they are predicate sets. Thus, universal methods must be based on attributes.

**Remark 6.** *Let $A$ be a finite symbol-set (alphabet), $A^*$ the free-monoid that it generates, and $A^{**}$ the one generated by $A^*$. As $\mathfrak{C}^{\preceq}$-objects, the products and coproducts of the members of $A^*$ and $A^{**}$ depend on the symbol-sequence attributes instead of monoid structure. For example, consider the sequences $S_1 = ABCxyz$ and $S_2 = xyz55$. Both have the subsequence $xyz$; hence, the attribute*

$$p(x) = \text{“}x \text{ is a symbol-sequence containing the subsequence } xyz.\text{”}$$

*Thus, by Theorem 15, for every attribute set $\mathbf{M}$ containing $p(x)$,*

$$p(x) \in \mathcal{F}_{\mathbf{M}}\left(S_1 \overset{\bigstar}{\coprod} S_2\right).$$

*Under some restricting conditions, the relation above leads to $S_1 \overset{\bigstar}{\coprod} S_2 = xyz$.*

*As usual, we term symbol each member of $A$, we denote word every member of $A^*$ and phrase each sequence in $A^{**}$.*

**Definition 14.** *For each alphabet $A$, we term $A$-vocabulary each pair $(A^*, \preccurlyeq)$ such that $A^*$ is the free-monoid generated by $A$ equipped with a total order relation $\preccurlyeq$. An $A$-vocabulary is syntactic or partial when not every member of $A^*$ belongs to it.*

**Definition 15.** *We term $(A^*, \preccurlyeq)$-phrase every $\preccurlyeq$-ordered finite subset of $A^*$ having no duplicate occurrence of any word.*

Since each $(A^*, \preccurlyeq)$-phrase $W$ is an ordered finite-set, we can consider it as a finite sequence; however, $W$ can contain, at most, one occurrence of any word.

**Notation 8.** For each $A$-vocabulary $(A^*, \preccurlyeq)$, we denote by $\Sigma(A^*, \preccurlyeq)$ the category of all $(A^*, \preccurlyeq)$-phrases and $\mathfrak{C}^{\preceq}$-morphisms among them.

The attributes defining members of $\mathrm{Ob}\,(\Sigma(A^*, \preccurlyeq))$ are expressions denoting each of their words. For every word $w \in A^*$, by $p(x, w)$ we denote the attribute

$$p(x, w) = \text{“}x \textit{ is a phrase containing a unique occurrence of the word } w\text{”}$$

Likewise,

$$\begin{aligned} p_{\oslash}(x) &= \text{“}x \textit{ is a void phrase,”} \\ p_{\preccurlyeq}(x) &= \text{“}x \textit{ is word sequence ordered by } \preccurlyeq\text{,”} \\ p^*(x) &= \text{“}x \textit{ is a member of } \mathrm{Ob}\,(\Sigma(A^*, \preccurlyeq))\text{.”} \end{aligned}$$

By the attributes above, the conjunction

$$\mathrm{def}_W(x) = p^*(x) \wedge p_{\preccurlyeq}(x) \wedge p(x, w_1) \wedge p(x, w_2) \wedge \cdots \wedge p(x, w_k). \tag{62}$$

defines the $(A^*, \preccurlyeq)$-phrase $W = w_1 w_2 \cdots w_k$ when $i \le j$ leads to $w_i \preccurlyeq w_j$.

By the infix symbol $\vartriangle$ we denote the binary composition law

$$\vartriangle\colon \Sigma(A^*, \preccurlyeq) \times \Sigma(A^*, \preccurlyeq) \to \Sigma(A^*, \preccurlyeq)$$

defined as follows. For every sequence $W = w_1 w_2 \ldots w_n$ in $\Sigma(A^*, \preccurlyeq)$ and each one-word phrase $v$,

$$W \vartriangle v = \begin{cases} W \textit{ if } v = \oslash, \\ v \textit{ if } p_{\oslash}(W), \\ W \textit{ if } v \in \{w_1, w_2 \ldots w_n\}, \\ w_1 w_2 \ldots \ldots w_n\, v, \textit{ if } w_n \preccurlyeq v, \\ v\, w_1 w_2 \ldots \ldots w_n, \textit{ if } v \preccurlyeq w_1, \\ w_1 w_2 \ldots w_j\, v\, w_{j+1} \ldots w_n, \textit{ if } w_j \preccurlyeq v \preccurlyeq w_{j+1}, \end{cases} \tag{63}$$

Analogously, we define the composition of two sequences $W = w_1 w_2 \dots w_n$ and $V = v_1 v_2 \dots v_m$ of $\Sigma(A^*, \preccurlyeq)$ iterating the procedure above; hence,

$$\begin{cases} W_1 = W \vartriangle v_1, \\ W_2 = W_1 \vartriangle v_2, \\ W_3 = W_2 \vartriangle v_3, \\ \dots\dots\dots\dots \\ W \vartriangle V = W_{m-1} \vartriangle v_m. \end{cases}$$

Since the $\Sigma(A^*, \preccurlyeq)$-objects are ordered finite sets, the composition

$$W_1 \vartriangle W_2 \vartriangle \cdots W_j$$

of a set of them $\mathbf{K} = \big\{W_1, W_2 \dots W_j\big\}$ is order-independent (63). Thus, for every bijection $\sigma : \big\{1, 2, \dots j\big\} \longrightarrow \big\{1, 2, \dots j\big\}$, the equation below holds.

$$W_1 \vartriangle W_2 \vartriangle \cdots W_j = W_{\sigma(1)} \vartriangle W_{\sigma(2)} \cdots W_{\sigma(j)}.$$

From now on, the expression $\underset{V \in \mathbf{K}}{\vartriangle} V$ denotes the $\vartriangle$-composition of a family $\mathbf{K}$ of $\Sigma(A^*, \preccurlyeq)$-objects. Thus, if $\mathbf{K} = \big\{W_i \mid i \in \mathbf{I} = \{1, 2 \dots j\}\big\}$,

$$\underset{i \in \mathbf{I}}{\bigtriangleup} W_i = W_1 \vartriangle W_2 \vartriangle \cdots W_j.$$

When $\#(\mathbf{I}) = 1$

$$\underset{i \in \mathbf{I}}{\bigtriangleup} W_i = W_1,$$

and $\underset{i \in \mathbf{I}}{\bigtriangleup} W_i = \emptyset$ when $\mathbf{K}$ is empty.

As in equation (62), we can define every $\Sigma(A^*, \preccurlyeq)$-object $W$ by the conjunction of $p^*(x) \wedge p_{\preccurlyeq}(x)$ and a subset of

$$\mathfrak{P}(A^*, \preccurlyeq) = \big\{p(x, w) \mid w \in A^*\big\} \tag{64}$$

**Lemma 23.** *For every couple of* $\Sigma(A^*, \preccurlyeq)$*-objects,* $W$ *and* $V$*, there is the morphism* $W \xrightarrow{\preceq} V$ *if and only if* $V$ *is a subsequence of* $W$*.*

*Proof.* Suppose that there is the arrow $W \xrightarrow{\preceq} V$. For some attribute $p(x) = \mathrm{def}_U(x)$ the following equivalence holds.

$$\mathrm{def}_W(x) \Longleftrightarrow \mathrm{def}_V(x) \wedge \mathrm{def}_U(x). \tag{65}$$

If $W$ is a void phrase, so must be $V$ and $U$. In this case, the lemma is obvious. Assume that $W = w_1 w_2 \dots w_j$ with $j > 0$. Thus, there is an attribute $\mathrm{def}_U(x)$ such that

$$\mathrm{def}_W(x) = p^*(x) \wedge p_{\preccurlyeq}(x) \wedge p(x, w_1) \wedge \dots p(x, w_j) = \mathrm{def}_V(x) \wedge \mathrm{def}_U(x); \tag{66}$$

hence, if $\mathbf{I} = \{1, 2 \dots j\}$ and $j \geq 2$, there is a partition of $\mathbf{I}$

$$\mathbf{I}_1 \cup \mathbf{I}_2 = \mathbf{I} \ \mathit{and}\ \mathbf{I}_1 \cap \mathbf{I}_2 = \emptyset,$$

such that the following equations hold.

$$\mathrm{def}_V(x) = p^*(x) \wedge p_{\preccurlyeq}(x) \wedge \left( \bigwedge_{i \in \mathbf{I}_1} p\left(x, w_i\right) \right), \tag{67}$$

$$\mathrm{def}_U(x) = p^*(x) \wedge p_{\preccurlyeq}(x) \wedge \left( \bigwedge_{i \in \mathbf{I}_2} p\left(x, w_i\right) \right). \tag{68}$$

By the equations above, together with (65), $V$ is a subsequence of $W$.

Assuming that $V$ is a word subsequence of $W$, let $\mathbf{I}_2$ be the subset of $\mathbf{I}$ of all indices of the members of $W$ that $V$ does not contain, and $\mathbf{I}_1 = \mathbf{I} \setminus \mathbf{I}_2$. If $\mathbf{I}_2 = \emptyset$, then $W = V$ and the lemma is obvious. Assume that $\#(\mathbf{I}_2) > 0$ and let $U$ be the word-sequence defined in (68). Thus, $V$ is the $\Sigma(A^*, \preccurlyeq)$-object defined in (67) and

$$\mathrm{def}_W(x) \Longleftrightarrow \mathrm{def}_V(x) \wedge \mathrm{def}_U(x);$$

hence, $\mathrm{def}_W(x) \xrightarrow{\lll} \mathrm{def}_V(x)$, and $W \xrightarrow{\preceq} V$.

**Lemma 24.** *Let* $\mathbf{I}$ *be a nonempty subset of* $\mathbb{N}$*. Let* $\mathbf{I}_1$ *and* $\mathbf{I}_2$ *be a partition of it such that*

$$\mathrm{def}_W(x) = p^*(x) \wedge p_{\preccurlyeq}(x) \wedge \left( \bigwedge_{i \in \mathbf{I}_1} p\left(x, w_i\right) \right)$$

*and*

$$\mathrm{def}_V(x) = p^*(x) \wedge p_{\preccurlyeq}(x) \wedge \left( \bigwedge_{i \in \mathbf{I}_2} p\left(x, w_i\right) \right)$$

*define two* $\Sigma(A^*, \preccurlyeq)$*-objects* $W$ *and* $V$*, respectively. With these assumptions, the conjunction*

$$\mathrm{def}_W(x) \wedge \mathrm{def}_V(x) = p^*(x) \wedge p_{\preccurlyeq}(x) \wedge \left( \bigwedge_{i \in (\mathbf{I}_1 \cup \mathbf{I}_2)} p\left(x, w_i\right) \right)$$

*defines the* $\Sigma(A^*, \preccurlyeq)$*-object* $W \vartriangle V$*.*

*Proof.* By Definition 15, both objects $W$ and $V$ are finite word-sets ordered by $\preccurlyeq$, and by construction, $W \vartriangle V$ is the $\preccurlyeq$-ordered sequence of the members of $\big\{w_i \mid i \in (\mathbf{I}_1 \cup \mathbf{I}_2)\big\}$.

**Lemma 25.** *For every $\Sigma(A^*, \preccurlyeq)$-object pair $(W, V)$, there is the morphism $W \xrightarrow{\preceq} V$ if and only if, for some $U \in \Sigma(A^*, \preccurlyeq)$, the following statement holds.*

$$W = V \vartriangle U. \tag{69}$$

*Proof.* Assume that $W \xrightarrow{\preceq} V$. By Lemma 23, $V$ is a word subsequence of $W = w_1 w_2 \ldots w_n$. If $\mathbf{J} = \big\{k_1, k_2 \ldots k_m\big\}$ is the subset of $\mathbb{N}$ consisting of indices of those elements in $W$ that are not in $V$, then

$$\mathrm{def}_W(x) = \mathrm{def}_V(x) \wedge \left(\bigwedge_{k\in\mathbf{J}} p(x, w_k)\right);$$

hence, the phrase $U$ defined by the conjunction

$$p^*(x) \wedge p_{\preccurlyeq}(x) \wedge \left(\bigwedge_{k\in\mathbf{J}} p(x, w_k)\right) = \mathrm{def}_U(x)$$

satisfies the relation

$$\mathrm{def}_W(x) = \mathrm{def}_V(x) \wedge \mathrm{def}_U(x). \tag{70}$$

The equation above, together with Lemma 24, leads to (69).

Suppose that equation (69) holds. Since $\{w_k \mid k \in \mathbf{J}\}$ is the set of all words in $W$ that are not in $V$, then

$$\mathrm{def}_W(x) = \mathrm{def}_V(x) \wedge \mathrm{def}_U(x);$$

therefore, by definition 4, there is the arrow $W \xrightarrow{\preceq} V$.

**Theorem 26.** *For every set $\mathbf{K} = \big\{W_i \mid i \in \mathbf{I}\big\}$ of $\Sigma(A^*, \preccurlyeq)$-objects such that $\#(\mathbf{I}) \geq 2$, the following statements hold.*

1. $\displaystyle\prod_{i\in\mathbf{I}}^{\preceq} W_i = \bigtriangleup_{i\in\mathbf{I}} W_i.$

2. *If* $\displaystyle\mathbf{K} = \bigcap_{i\in\mathbf{I}} \big\{U \in \mathrm{Ob}\,(\Sigma(A^*, \preccurlyeq)) \mid W_i \xrightarrow{\preceq} U\big\}$, *then*

$$\coprod_{i\in\mathbf{I}}^{\preceq} W_i = \bigtriangleup_{U\in\mathbf{K}} U = \prod_{U\in\mathbf{K}}^{\preceq} U. \tag{71}$$

*Proof.*

1. It is a straightforward consequence of definition of the law $\vartriangle$ that

$$\bigtriangleup_{i\in\mathbf{I}} W_i = \left(\bigtriangleup_{\substack{i\in\mathbf{I}\\ i\neq j}} W_i\right) \vartriangle W_j;$$

therefore, by Lemma 25,

$$\forall j \in \mathbf{I}: \ \bigtriangleup_{i\in\mathbf{I}} W_i \xrightarrow{\preceq} W_j. \tag{72}$$

Thus, by the definition of product,

$$\bigtriangleup_{i\in\mathbf{I}} W_i \xrightarrow{\preceq} \overset{\preceq}{\prod_{i\in\mathbf{I}}} W_i. \tag{73}$$

and by Lemma 23, $\overset{\preceq}{\prod_{i\in\mathbf{I}}} W_i$ is a subsequence of $\bigtriangleup_{i\in\mathbf{I}} W_i$. Now, suppose that there is a word $w \in \bigtriangleup_{i\in\mathbf{I}} W_i$ that does not belong to $\overset{\preceq}{\prod_{i\in\mathbf{I}}} W_i$. For some $k \in \mathbf{I}$, $w \in W_k$. Since $\overset{\preceq}{\prod_{i\in\mathbf{I}}} W_i \xrightarrow{\preceq} W_k$, by Lemma 23, $W_k \ni w$ is a subsequence of $\overset{\preceq}{\prod_{i\in\mathbf{I}}} W_i$, which contradicts our assumption $w \notin \overset{\preceq}{\prod_{i\in\mathbf{I}}} W_i$, and Statement (1) holds.

2. The set $\mathbf{K}$ is nonempty because, at least, it contains the void sequence $\oslash$. By Lemma 23, every $U \in \mathbf{K}$ is a subsequence of each $W_i$, for every $i$ in $\mathbf{I}$. As a consequence of the definition of $\mathbf{K}$, together with Lemma 6 and Theorem 11, the conjunction $\bigwedge_{U\in\mathbf{K}} \mathrm{def}_U(x)$ defines $\coprod^{\preceq}_{i\in\mathbf{I}} W_i$; therefore,

$$\forall U \in \mathbf{K}: \quad \overset{\preceq}{\coprod_{i\in\mathbf{I}}} W_i \xrightarrow{\preceq} U. \tag{74}$$

Thus, by Lemma 23, every $U$ in $\mathbf{K}$ is a subsequence of $\coprod^{\preceq}_{i\in\mathbf{I}} W_i$; therefore, so is their composition $\bigtriangleup_{U\in\mathbf{K}} U$. Likewise, there is a word-sequence $T \in \mathrm{Ob}\,(\Sigma(A^*, \preccurlyeq))$

such that,

$$\coprod_{i\in\mathbf{I}}^{\preceq} W_i = U \vartriangle T; \tag{75}$$

hence, $T$ is a subsequence of $\coprod_{i\in\mathbf{K}}^{\preceq} W_i$. The relation above, together with Lemma 23, leads to

$$\coprod_{i\in\mathbf{I}}^{\preceq} W_i \xrightarrow{\preceq} T;$$

therefore, $T \in \mathbf{K}$ and Statement (2) holds.

**Remark 7.** *The similarity between (71) and equation (32) states a relation between coproduct and the generalized operator $\vartriangle$. To avoid any ambiguity in (71), we denote by $\triangledown$ the operator corresponding with the coproduct. Accordingly,*

$$\coprod_{i\in\mathbf{I}}^{\preceq} W_i = \bigtriangleup_{U\in\mathbf{K}} U = \bigtriangledown_{i\in\mathbf{i}} W_i. \tag{76}$$

*To simplify expressions, when $\mathbf{K} = \{W\}$ is a singleton,*

$$\bigtriangledown_{V\in\mathbf{K}} V = \bigtriangleup_{V\in\mathbf{K}} V = W.$$

*Analogously, if $\mathbf{K} = \emptyset$, then*

$$\bigtriangledown_{V\in\mathbf{K}} V = \bigtriangleup_{V\in\mathbf{K}} V = \oslash.$$

**Corollary 27.** *The category $\Sigma(A^*, \preccurlyeq)$ has products and coproducts.*

*Proof.* It is a straightforward consequence of Theorem 26 and equation (63).

**Corollary 28.** *If the map $f : A_1 \subseteq \mathrm{Ob}\,(\Omega[\mathbf{DCat}]) \longrightarrow A_2 \subseteq \mathrm{Ob}\,(\Sigma(A^*, \preccurlyeq))$ is an $\Omega[\mathbf{DCat}]$-morphism and $\mathbf{K}_O$ is a generator for $O \in A_1$, then*

$$f(O) = f\left(\prod_{D\in\mathbf{K}_O}^{\odot}\left(\coprod_{Q\in D}^{\odot} Q\right)\right) =$$

$$\prod_{D\in\mathbf{K}_O}^{\odot}\left(\coprod_{Q\in D}^{\odot} f(Q)\right) = \bigtriangleup_{D\in\mathbf{K}_O}\left(\bigtriangledown_{Q\in D} f(Q)\right) \tag{77}$$

*Proof.* By Corollary 27, $\Sigma(A^*, \preccurlyeq)$ has products and coproducts. Thus, equation (77) is a straightforward consequence of definition 9 and Theorem 26.

**Definition 16.** *A subclass* $\mathbf{K}$ *of* $\mathrm{Ob}\,(\Sigma(A^*, \preceq))$ *is* $\Sigma$*-definable, when there is a finite* $(A^*, \preccurlyeq)$*-phrase* $W$ *such that*

$$\mathbf{K} = \left\{ V \in \mathrm{Ob}\,(\Sigma(A^*, \preccurlyeq))\, \middle|\, V \xrightarrow{\preceq} W \right\}. \tag{78}$$

**Lemma 29.** *Every* $\Sigma$*-definable subclass of* $\mathrm{Ob}\,(\Sigma(A^*, \preccurlyeq))$ *is discernible.*

*Proof.* Let $W$ be the $\Sigma(A^*, \preccurlyeq)$-object defining $\mathbf{K}$: hence,

$$\mathbf{K} = \left\{ V \in \mathrm{Ob}\,(\Sigma(A^*, \preccurlyeq))\, \middle|\, V \xrightarrow{\preceq} W \right\}. \tag{79}$$

Since $W$ is finitely definable, by Lemma 25, the predicate

$$\mathrm{def}_{\mathbf{K}}(x) = \text{“}W \textit{ is a sub-phrase of } x\text{”}$$

defines $\mathbf{K}$. Likewise, for every $V \in \mathbf{K}$, there is $U \in \mathrm{Ob}\,(\Sigma(A^*, \preceq))$ such that $V = W \vartriangle U$; therefore, $U$ complements $W$ to $V$.

If $U_1$ and $U_2$ are two members of $\complement_{\mathrm{Ob}(\Sigma(A^*, \preceq))}\mathbf{K}$ that have no common word, then, by Theorem 26,

$$(W \vartriangle U_1) \coprod^{\preceq} (W \vartriangle U_2) = W;$$

consequently, $\mathrm{def}_W(x) = \mathrm{def}_{(W \vartriangle U_1)}(x) \coprod^{\ll} \mathrm{def}_{(W \vartriangle U_2)}(x)$ and the lemma holds.

**Theorem 30.** *Let* $\mathbf{C}$ *be a subcategory of* $\mathfrak{C}^{\ll}$ *such that* $\mathrm{Ob}(\mathbf{C})$ *is stable under conjunctions and disjunctions, and* $\mathbf{M}$ *a basis of* $\mathrm{Ob}(\mathbf{C})$ *(definition 3). Let* $\mathbf{K}$ *be a full subcategory of* $\Omega[\mathbf{DCat}]$ *each of its objects is definable by an attribute in* $\mathrm{Ob}(\mathbf{C})$*. With these assumptions there is a functor* $\mathcal{C}_{\mathbf{M}}$ *from* $\mathbf{K}$ *into* $\Sigma(A^*, \preccurlyeq)$*.*

*Proof.* Let $O^*$ be the set of all one-word phrases in $A^*$ and $\mathcal{D} : \mathbf{M} \longrightarrow O^*$ an injective map. Let $\mathcal{D}^*\colon \mathrm{Ob}(\mathbf{C}) \to \mathrm{Ob}\,(\Sigma(A^*, \preccurlyeq))$ be the extension of $\mathcal{D}$ defined as follows. By hypothesis, $\mathbf{M}$ is a basis of $\mathrm{Ob}(\mathbf{C})$; hence, for every $q(x) \in \mathrm{Ob}(\mathbf{C}) \setminus \mathbf{M}$, there is a subset $P$ of $\mathbf{M}$ such that $q(x) = \bigwedge\limits_{p(x) \in P} p(x)$. Thus, we define an extension $\mathcal{D}^*$ of $\mathcal{D}$ as follows.

$$\mathcal{D}^*\left(q(x)\right) = \mathcal{D}^*\left( \bigwedge_{p(x) \in P} p(x) \right) = \mathop{\triangle}_{p(x) \in P} \mathcal{D}\left(p(x)\right). \tag{80}$$

By assumption, for every $O \in \mathbf{K}$, there is an attribute $\mathrm{def}_O(x)$ in $\mathrm{Ob}(\mathbf{C})$ defining it. We can define the object-map of $\mathcal{C}_{\mathbf{M}}$ as follows.

$$\forall O \in \mathrm{Ob}(\mathbf{K}) : \quad \mathcal{C}_{\mathbf{M}}(O) = \mathcal{D}^*(\mathrm{def}_O(x)). \tag{81}$$

We show that $\mathcal{C}_{\mathbf{M}}$ preserves products and coproducts in both categories $\mathfrak{C}^{\ll}$ and $\mathfrak{C}^{\preceq}$. Let $\mathbf{Q}$ be a $\mathbf{K}$-object set of cardinality greater than 1. By the definition of $\mathcal{D}^*$,

$$\mathcal{D}^* \left( \bigwedge_{O \in \mathbf{Q}} \mathrm{def}_O(x) \right) = \mathop{\triangle}_{O \in \mathbf{Q}} \mathcal{D}^* \left( \mathrm{def}_O(x) \right). \tag{82}$$

By Lemma 7, Corollary 27, and Theorem 26, together with equation (81), the equation above leads to

$$\mathcal{C}_{\mathbf{M}} \left( \prod_{O \in \mathbf{Q}}^{\preceq} O \right) = \mathcal{D}^* \left( \mathrm{def}_{(\prod^{\ll}_{O \in \mathbf{Q}})}(x) \right) = \mathcal{D}^* \left( \prod_{O \in \mathbf{Q}}^{\ll} \mathrm{def}_O(x) \right) =$$
$$\mathcal{D}^* \left( \bigwedge_{O \in \mathbf{Q}} \mathrm{def}_O(x) \right) = \mathop{\triangle}_{O \in \mathbf{Q}} \mathcal{D}^* \left( \mathrm{def}_O(x) \right) =$$
$$\prod_{O \in \mathbf{Q}}^{\preceq} \mathcal{D}^* \left( \mathrm{def}_O(x) \right) = \prod_{O \in \mathbf{Q}}^{\preceq} \mathcal{C}_{\mathbf{M}}(O); \tag{83}$$

therefore, $\mathcal{C}_{\mathbf{M}}$ preserves products.

To show that $\mathcal{C}_{\mathbf{M}}$ preserves coproducts, we show that, for every couple of objects $(O, Q)$, if $O \xrightarrow{\preceq} Q$, then $\mathcal{D}^*(O) \xrightarrow{\preceq} \mathcal{D}^*(Q)$. By definition, the arrow $Q \xrightarrow{\preceq} Q$ leads to $\mathrm{def}_O(x) \xrightarrow{\ll} \mathrm{def}_Q(x)$. Again, by definition, there is an attribute $h(x)$ such that $\mathrm{def}_O(x) = \mathrm{def}_Q(x) \wedge h(x)$. Thus, $\mathcal{D}^*(\mathrm{def}_O(x)) = \mathcal{D}^*(\mathrm{def}_Q(x)) \vartriangle \mathcal{D}^*(h(x))$.

By Lemma 25, this relation leads to $\mathcal{D}^* \left( \mathrm{def}_O(x) \right) \xrightarrow{\ll} \mathcal{D}^* \left( \mathrm{def}_Q(x) \right)$; hence, $\mathcal{C}_{\mathbf{M}}(O) \xrightarrow{\preceq} \mathcal{C}_{\mathbf{M}}(Q)$. As a consequence, from Lemma 10, equation (20), and Theorem 20, the coproduct of an object set $\left\{ W_i \mid i \in \mathbf{I} \right\}$ coincides with the product of the members of

$$\mathbf{K} = \bigcap_{i \in \mathbf{I}} \left\{ U \in \mathrm{Ob}\left( \Sigma(A^*, \preccurlyeq) \right) \mid W_i \xrightarrow{\preceq} U \right\}.$$

Thus, since $\mathcal{C}_{\mathbf{M}}$ preserves products, so does it with coproducts. Consequently, for every $\mathbf{K}$-morphism

$$f \left( \prod_{H \in W}^{\preceq} \left( \coprod_{U \in H}^{\preceq} U \right) \right) = \prod_{H \in W}^{\preceq} \left( \coprod_{U \in H}^{\preceq} f(U) \right)$$

the following equation holds.

$$\mathcal{C}_{\mathbf{M}}\left(f\left(\prod_{H\in W}^{\preceq}\left(\coprod_{U\in H}^{\preceq} U\right)\right)\right) = \prod_{H\in W}^{\preceq}\left(\coprod_{U\in H}^{\preceq}\mathcal{C}_{\mathbf{M}}\left(f(U)\right)\right). \tag{84}$$

For every $f \in \mathrm{Mor}\,(\mathbf{K})$, let $f^\flat : \mathcal{C}_{\mathbf{M}}(\mathrm{dom}(f)) \to \mathrm{Ob}\,(\Sigma(A^*, \preccurlyeq))$ be the map such that, for every $U \in \mathrm{dom}(f)$, sends $\mathcal{C}_{\mathbf{M}}(U)$ into $\mathcal{C}_{\mathbf{M}}\left(f(U)\right)$. Thus, if $\mathcal{C}_{\mathbf{M}}(f) = f^\flat$, equation (84) leads to

$$\mathcal{C}_{\mathbf{M}}\left(f\right)\left(\prod_{H\in W}^{\preceq}\left(\coprod_{U\in H}^{\preceq}\mathcal{C}_{\mathbf{M}}(U)\right)\right) = \prod_{H\in W}^{\preceq}\left(\coprod_{U\in H}^{\preceq}\mathcal{C}_{\mathbf{M}}(f)\left(\mathcal{C}_{\mathbf{M}}(U)\right)\right).$$

Consequently, $\mathcal{C}_{\mathbf{M}}(f)$ is a $\Sigma(A^*. \preccurlyeq)$-morphism. It is a straightforward consequence of the equation above that $\mathcal{C}_{\mathbf{M}}$ preserves identities and morphism-composition; hence, $\mathcal{C}_{\mathbf{M}}$ is a functor.

The theorem above defines a functor that sends definable mathematical constructions into word-sequences. Since functors preserve morphisms, the association is not arbitrary but based on logic. Thus, it generates suitable languages to build algorithms.

By Theorem 21, $\Omega[\mathbf{DCat}]$-morphism and eulerithm extensions are the result of a combination of $\mathfrak{C}^{\preceq}$-products and $\mathfrak{C}^{\preceq}$-coproducts. By contrast, $\Sigma(A^*, \preccurlyeq)$-morphism extensions are the result of combinations of $\triangle$ and $\bigtriangledown$ working on word sequences; hence, they are algorithms defined in extendable languages. Thus, the functor $\mathcal{C}_{\mathbf{M}}$ constructs algorithms from eulerithms.

**Example 6.** *We apply the theorem above in example 5. Using the Latin alphabet, let* $\mathbf{M}$ *be the attribute set containing the subset*

$$\left\{p_1(x), p_2(x), p_3(x), q_1(x), q_2(x), h_1(x), h_2(x), h_3(x), r_1(x), r_2(x), r_3(x)\right\}$$

*and* $\mathcal{D}$ *the map defined as follows.*

$$\begin{aligned}
\mathcal{D}(p_1(x)) &= pon,\\
\mathcal{D}(p_2(x)) &= ptw,\\
\mathcal{D}(p_3(x)) &= pth,\\
\mathcal{D}(q_1(x)) &= qon,\\
\mathcal{D}(q_2(x)) &= qtw,
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}(h_1(x)) &= hon,\\
\mathcal{D}(h_2(x)) &= htw,\\
\mathcal{D}(h_3(x)) &= hth,\\
\mathcal{D}(r_1(x)) &= ron,\\
\mathcal{D}(r_2(x)) &= rtw,\\
\mathcal{D}(r_3(x)) &= rth.
\end{aligned}$$

*For short, the symbols $W$, $V$, $U$, $U_1$, and $U_2$ denote the word sequences*

$$\begin{aligned}
W &= pon\,ptw\,pth,\\
V &= qon\,qtw,\\
U_1 &= hon\,ron,\\
U_2 &= htw\,rtw,\\
U_3 &= hth\,rth.
\end{aligned}$$

*With these notations,*

$$\begin{cases}
\mathcal{C}_{\mathbf{M}}\left(\frac{1}{x-1} = 9\right) = W \bigtriangleup U_1\\
\mathcal{C}_{\mathbf{M}}\left(\frac{4}{x} = 3\right) = W \bigtriangleup U_2\\
\mathcal{C}_{\mathbf{M}}\left(5x = 2\right) = W \bigtriangleup U_3\\
\mathcal{C}_{\mathbf{M}}\left(x = \frac{1}{9} + 1\right) = V \bigtriangleup U_1\\
\mathcal{C}_{\mathbf{M}}\left(x = \frac{4}{3}\right) = V \bigtriangleup U_2\\
\mathcal{C}_{\mathbf{M}}\left(x = \frac{5}{2}\right) = V \bigtriangleup U_3
\end{cases}$$

*Let $f$ be the map such that*

$$\begin{cases}
f(U_1) = U_1,\\
f(U_2) = U_2,\\
f(U_3) = U_3,\\
f(W) = V.
\end{cases}$$

*With these relations, we can write example 5 as follows.*

$$\begin{aligned}
f(W \bigtriangleup U_1) = f\left((W \bigtriangleup U_2) \bigtriangledown (W \bigtriangleup U_3)) \bigtriangleup f(U_1)\right) &=\\
(f(W \bigtriangleup U_2) \bigtriangledown f(W \bigtriangleup U_3)) \bigtriangleup f(U_1) &=\\
((f(W) \bigtriangleup f(U_2)) \bigtriangledown (f(W) \bigtriangleup f(U_3))) \bigtriangleup f(U_1) &=\\
((V \bigtriangleup U_2) \bigtriangledown (V \bigtriangleup U_3)) \bigtriangleup U_1. &
\end{aligned} \tag{85}$$

*By Theorem 26,* $((V) \bigtriangleup U_2) \bigtriangledown (V \bigtriangleup U_3)) = V$*; therefore,*

$$f(W \bigtriangleup U_1) = V \bigtriangleup U_1 = hon\, pon\, pth\, ptw\, ron$$

*and* $\mathcal{C}_{\mathbf{M}}\left(x = \frac{1}{9} + 1\right) = V \bigtriangleup U_1$ *is the solution of* $\frac{1}{x-1} = 9$ *because the meaning of the phrase* $hon\, pon\, pth\, ptw\, ron$ *is the attribute*

$$P(x) = h_1(x) \wedge p_1(x) \wedge p_2(x) \wedge p_3(x) \wedge r_1(x)$$

*defining the expression* $\left(x = \frac{1}{9} + 1\right)$*. By Theorem 20, the map that sends each attribute into the object that it defines is an* $\Omega[\mathbf{DCat}]$*-morphism too.*

As in the example above, the operators $\vartriangle$ and $\triangledown$ work transforming symbol sequences blindly. Their actions fit into the algorithm concept. To this end, Theorem 30 builds a suitable language, the extensions of which we obtain by Theorem 21.

## 6. Conclusions

Noticeable scientific research methods consist of finding, from a behavior-sample, the laws ruling each procedure, to extend it to larger scenarios. Theorem 21 shows that those procedures that we can state as $\Omega[\mathbf{DCat}]$-morphisms we can extend them from behavior-samples involving discernible classes. They consist of eulerithms because they are based on attributes through combinations of products$^{\odot}$ and coproducts$^{\odot}$. Theorem 30 states a procedure to build suitable languages and algorithms to perform them. Since eulerithms depend on attributes, a deeper research would consist of methods to classify and find attributes.

Thus, we can discern three procedure–levels. By the first one, we find attributes to construct eulerithms. The second level consists of eulerithms building algorithms. The last level consists of algorithms solving problems.

*Summarizing:* Theorem 22 shows that when the involved classes are discernible, *problem–solution* maps are $\Omega[\mathbf{DCat}]$-morphisms; therefore, they are eulerithms. By Theorem 21, we can obtain them by extending samples of their behavior. Finally, by Theorem 30, we can assign languages and build algorithms evaluating $\Omega[\mathbf{DCat}]$-morphisms.

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