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GROWTH OF MEROMORPHIC SOLUTIONS OF COMPLEX LINEAR DIFFERENTIAL-DIFFERENCE EQUATION WITH MEROMORPHIC COEFFICIENTS OF LOGARITHMIC ORDER

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Abstract. In this paper, we deal with the growth and the value distribution of meromorphic solutions of complex linear differential-difference equation

$$L(z,f)=\sum_{i=0}^{n}\sum_{j=0}^{m}A_{ij}(z)f^{(j)}(z+c_i)=F(z)$$

with meromorphic coefficients of logarithmic order. We improve some precedent results by weakening the relative conditions, and obtain some estimates on the lower bound of logarithmic order of these solutions.

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1. Introduction and Main Results

Throughout this paper, we assume that the readers are familiar with the fundamental results and the standard notations of Nevanlinna value distribution theory of meromorphic functions (see e.g., [13,14,23,24]). Let $f(z)$ be a meromorphic function in the complex plane $\mathbb{C}$ in the whole paper. Then, we use the notations $\rho(f)$ and $\mu(f)$ to denote the order and the lower order of $f(z)$, use $\tau(f)$ and $\underline{\tau}(f)$ to denote the type and the lower type of $f(z)$, and use $\tau_M(f)$ and $\underline{\tau}_M(f)$ to denote the M-type and the lower M-type of $f(z)$ respectively. We also use $\delta(a,f)$ to denote the deficiency of a with respect to $f(z)$, where $a\in\mathbb{C}\bigcup\{\infty\}$.

Recently, the study on the properties of meromorphic solutions of complex difference equations has arisen a great interest (see e.g., [6,15,16,18,19,22]), since Halburd-Korhonen [12] and Chiang-Feng [9] obtained the difference analogues of the Logarithmic Derivative Lemma separately. Especially, by using these variants of Nevanlinna theory, many scholars focused on meromorphic solutions of complex linear equations, and got many good results in the case of complex linear difference equations (see e.g., [1,3,7,17,20,25]) and in the case of complex linear differential-difference equations (see e.g., [2,4,21,26]).

Chiang and Feng in [9] investigated the linear difference equation

$$A_n(z)f(z+n) + \cdots + A_1(z)f(z+1) + A_0(z)f(z) = 0, \tag{1}$$

where $A_j(z)(j = 0, 1, \cdots, n)$ are entire functions. When (1) has an unique dominating coefficient, they obtained the relationship between the order of meromorphic solutions of (1) and the orders of the coefficients.

Theorem 1. *(see [9])* Let $A_j(z)(j = 0, 1, \cdots, n)$ be entire functions satisfying that there exists an integer $l(0 \le l \le n)$ such that

$$\max\{\rho(A_j) : 0 \le j \le n, j \ne l\} < \rho(A_l).$$

Then every meromorphic solution $f(z)(\not\equiv 0)$ of (1) satisfies $\rho(f) \ge \rho(A_l) + 1$.

As is well known that Theorem 1 is a good result, since the estimate on the lower bound of the order of meromorphic solutions of (1) is sharp. And Theorem 1 also holds for the more general equation

$$A_n(z)f(z+c_n) + \cdots + A_1(z)f(z+c_1) + A_0(z)f(z) = 0, \tag{2}$$

where $c_i(i = 1, 2, \cdots, n)$ are distinct complex numbers.

Next, Laine and Yang in [15] considered the case that more than one coefficient has the maximal order and among which only one has the maximal type, and got the following result.

Theorem 2. *(see [15])* Let $A_j(z)(j = 0, 1, \cdots, n)$ be entire functions of finite order such that among those having the maximal order $\rho = \max\{\rho(A_j) : 0 \le j \le n\}$, exactly one has its type strictly greater than the others. Then for every meromorphic solution $f(z)(\not\equiv 0)$ of (2), we have $\rho(f) \ge \rho + 1$.

For the case when there is more than one coefficient having the maximal lower order and among which only one has the maximal lower type, Zheng and Tu in [25] proved the following result.

Theorem 3. *(see [25])* Let $A_j(z)(j = 0, 1, \cdots, n)$ be entire functions satisfying that there exists an integer $l(0 \le l \le n)$ such that

$$\max\{\rho(A_j) : 0 \le j \le n, j \ne l\} \le \mu(A_l)(0 < \mu(A_l) < +\infty)$$

and

$$\max\{\tau(A_j) : \rho(A_j) = \mu(A_l), 0 \le j \le n, j \ne l\} < \underline{\tau}(A_l).$$

Then every meromorphic solution $f(z)(\not\equiv 0)$ of (1) satisfies $\mu(f) \geq \mu(A_l) + 1$.

Later, Luo and Zheng in [17] studied the linear difference equation (2), where $A_j(z)(j = 0, 1, \cdots, n)$ are entire or meromorphic functions. They weakened the conditions of Theorem 3 and obtained the following two theorems.

Theorem 4. *(see [17])* Let $A_j(z)(j = 0, 1, \cdots, n)$ be entire functions, and let $k, l \in \{0, 1, \cdots, n\}$. If the following three assumptions hold simultaneously:
(1) $\max\{\mu(A_k), \rho(A_j), j \neq k, l\} \leq \mu(A_l)(0 < \mu(A_l) < +\infty)$;
(2) $\underline{\tau}_M(A_k) < \underline{\tau}_M(A_l)$, when $\mu(A_k) = \mu(A_l)$;
(3) $\max\{\tau_M(A_j) : \rho(A_j) = \mu(A_l), j \neq k, l\} < \underline{\tau}_M(A_l)$, when $\max\{\rho(A_j), j \neq k, l\} = \mu(A_l)$.
Then every meromorphic solution $f(z)(\not\equiv 0)$ of (2) satisfies $\rho(f) \geq \mu(A_l) + 1$.

Theorem 5. *(see [17])* Let $A_j(z)(j = 0, 1, \cdots, n)$ be meromorphic functions, and let $k, l \in \{0, 1, \cdots, n\}$. If the following four assumptions hold simultaneously:
(1) $\max\{\mu(A_k), \rho(A_j), j \neq k, l\} \leq \mu(A_l)(0 < \mu(A_l) < +\infty)$;
(2) $\underline{\tau}(A_k) < \delta\underline{\tau}(A_l)$, when $\mu(A_k) = \mu(A_l)$;
(3) $\max\{\tau(A_j) : \rho(A_j) = \mu(A_l), j \neq k, l\} < \delta\underline{\tau}(A_l)$, when $\max\{\rho(A_j), j \neq k, l\} = \mu(A_l)$;
(4) $\delta(\infty, A_l) = \delta > 0$.
Then every meromorphic solution $f(z)(\not\equiv 0)$ of (2) satisfies $\rho(f) \geq \mu(A_l) + 1$.

Recently, Belaïdi and Bellaama in [3] extended Theorems 4 and 5 to the linear difference equation

$$A_n(z)f(z + c_n) + \cdots + A_1(z)f(z + c_1) + A_0(z)f(z) = A_{n+1}(z), \tag{3}$$

where $A_j(z)(j = 0, 1, \cdots, n + 1)$ are entire or meromorphic functions and $c_i(i = 1, 2, \cdots, n)$ are distinct complex numbers. By comparing their results and Theorems 4 and 5, we find that the results for the case of the non-homogeneous equation (3) are weaker than the ones for the case of the homogeneous equation (2).

Generally, for the case of the linear differential-difference equations, Wu and Zheng in [21] investigated the growth of meromorphic solutions of the homogeneous equation

$$L(z, f) = \sum_{i=0}^{n} \sum_{j=0}^{m} A_{ij}(z) f^{(j)}(z + c_i) = 0 \tag{4}$$

and the non-homogeneous equation

$$L(z, f) = \sum_{i=0}^{n} \sum_{j=0}^{m} A_{ij}(z) f^{(j)}(z + c_i) = F(z), \tag{5}$$

where $A_{ij}(z)(i = 0, 1, \cdots, n; j = 0, 1, \cdots, m)$ are meromorphic functions such that $A_{00}(z)A_{n0}(z)F(z) \not\equiv 0, c_i(i = 0, 1, \cdots, n)$ are distinct complex numbers.

Then, a natural question arises: when the coefficients $A_{ij}(z)(i = 0, 1, \cdots, n; j = 0, 1, \cdots, m)$ are of slow growth in the sense that are of order zero, how to express the growth and the value distribution of meromorphic solutions of (4) and (5)? Belaïdi in [2] considered the question for the case of the homogeneous equation (4) by using the concept of finite logarithmic order due to Chern [8].

The main purpose of this paper is to consider the case of the non-homogeneous equation (5) and our results weaken the relative conditions in [2]. Unfortunately, the results for the case of the homogeneous equation (4) under similar conditions are rough and not good. So, we give up the case of the corresponding homogeneous equation in this paper.

Firstly, We recall the following definitions.

Definition 1. *(see [8])* The logarithmic order of a meromorphic function $f(z)$ is defined as

$$\rho_{\log}(f) = \overline{\lim_{r\to+\infty}} \frac{\log T(r, f)}{\log\log r}.$$

Remark 1. *(see [1])* Clearly, the logarithmic order of any non-constant rational function is one, and any transcendental meromorphic function has logarithmic order no less than one. Conversely, a meromorphic function with logarithmic order one is not always a rational function. The logarithmic order of any constant function is zero. That is to say, there are no meromorphic functions having logarithmic order between zero and one. Furthermore, any meromorphic function with finite logarithmic order is of order zero.

Definition 2. *(see [8])* The logarithmic lower order of a meromorphic function $f(z)$ is defined as

$$\mu_{\log}(f) = \underline{\lim_{r\to+\infty}} \frac{\log T(r, f)}{\log\log r}.$$

Definition 3. *(see [5])* The logarithmic type of a meromorphic function $f(z)$ with $1 \le \rho_{\log}(f) < +\infty$ is defined as

$$\tau_{\log}(f) = \overline{\lim_{r\to+\infty}} \frac{T(r, f)}{(\log r)^{\rho_{\log}(f)}}.$$

Remark 2. *(see [1])* Obviously, any non-constant polynomial $P(z)$ has logarithmic type $\deg(P)$, any non-constant rational function has finite logarithmic type, and any transcendental meromorphic function with logarithmic order one must have infinite logarithmic type.

Definition 4. *(see [5])* The logarithmic lower type of a meromorphic function $f(z)$ with $1 \leq \mu_{\log}(f) < +\infty$ is defined as

$$\underline{\tau}_{\log}(f) = \varliminf_{r\to+\infty} \frac{T(r,f)}{(\log r)^{\mu_{\log}(f)}}.$$

Definition 5. *(see [5,8])* The logarithmic exponent of convergence of a-points of a meromorphic function $f(z)$ is equal to the logarithmic order of $n(r, \frac{1}{f-a})$, that is

$$\lambda_{\log}(f-a) = \lambda_{\log}(f,a) = \varlimsup_{r\to+\infty} \frac{\log n(r, \frac{1}{f-a})}{\log\log r},$$

where $a \in \mathbb{C}$. And the logarithmic exponent of convergence of poles of a meromorphic function $f(z)$ is defined as

$$\lambda_{\log}(\frac{1}{f}) = \lambda_{\log}(f,\infty) = \varlimsup_{r\to+\infty} \frac{\log n(r,f)}{\log\log r}.$$

Remark 3. *(see [1])* As is well known, the order of $n(r, \frac{1}{f-a})$ is equal to the order of $N(r, \frac{1}{f-a})$, where $a \in \mathbb{C}\bigcup\{\infty\}$ (see e.g., [23,24]). However, it does not hold for the case of logarithmic order. In fact, the logarithmic order of $N(r, \frac{1}{f-a})$ is equal to $\lambda_{\log}(f-a)+1$, where $a \in \mathbb{C}\bigcup\{\infty\}$([8]).

Next, we state our main results as follows.

Theorem 6. Let $A_{ij}(z)(i = 0,1,\cdots,n; j = 0,1,\cdots,m), F(z)(\not\equiv 0)$ be meromorphic functions and assume that there exists an integer $l \in \{0,1,\cdots,n\}$ such that

$$\max\{\rho_{\log}(A_{ij}), (i,j) \neq (l,0)\} = \sigma \leq \mu_{\log}(A_{l0}) < +\infty$$

and

$$\lambda_{\log}(\frac{1}{A_{l0}}) + 1 < \mu_{\log}(A_{l0}).$$

Then the following results hold.

(i) If one of the following three assumptions hold:

(a) $\mu_{\log}(F) < \mu_{\log}(A_{l0})$ and $\sum\limits_{\substack{\rho_{\log}(A_{ij})=\mu_{\log}(A_{l0})\\ (i,j)\neq(l,0)}} \tau_{\log}(A_{ij}) < \underline{\tau}_{\log}(A_{l0}) < +\infty$,

(b) $\mu_{\log}(F) = \mu_{\log}(A_{l0})$ and $\sum\limits_{\substack{\rho_{\log}(A_{ij})=\mu_{\log}(A_{l0})\\ (i,j)\neq(l,0)}} \tau_{\log}(A_{ij}) + \underline{\tau}_{\log}(F) < \underline{\tau}_{\log}(A_{l0}) < +\infty$,

(c) $\mu_{\log}(F) = \mu_{\log}(A_{l0})$ and $\sum\limits_{\substack{\rho_{\log}(A_{ij})=\mu_{\log}(A_{l0})\\ (i,j)\neq(l,0)}} \tau_{\log}(A_{ij}) + \underline{\tau}_{\log}(A_{l0}) < \underline{\tau}_{\log}(F) < +\infty$,

then every meromorphic solution $f(z)$ of (5) satisfies $\rho_{\log}(f) \geq \mu_{\log}(A_{l0})$.

(ii) If $\mu_{\log}(F) > \mu_{\log}(A_{l0})$, then every meromorphic solution $f(z)$ of (5) satisfies $\rho_{\log}(f) \geq \mu_{\log}(F)$.

We also consider the value distribution of meromorphic solutions of (5), and obtain the following result.

Theorem 7. Let $A_{ij}(z)(i = 0, 1, \cdots, n, j = 0, 1, \cdots, m), F(z)(\not\equiv 0)$ be meromorphic functions, and $f(z)$ be a meromorphic solution of (1.5) satisfying

$$\max\{\rho_{\log}(F), \rho_{\log}(A_{ij}), i = 0, 1, \cdots, n; j = 0, 1, \cdots, m\} < \rho_{\log}(f) < +\infty,$$

then $\rho_{\log}(f) = \lambda_{\log}(f)+1$. Further, if $\varphi(z)$ is a meromorphic function with $\rho_{\log}(\varphi) < \rho_{\log}(f)$ and $\varphi(z)$ is not a solution of (5) , then $\rho_{\log}(f) = \lambda_{\log}(f - \varphi) + 1$.

2. Preparations for proofs

Firstly, we need some definitions of measures (see e.g., [12]). The linear measure of a set $E \subset (0, +\infty)$ is defined as $mE = \int_0^{+\infty} \mathcal{X}_E(t)dt$ and the logarithmic measure of a set $F \subset (1, +\infty)$ is defined as $m_l F = \int_1^{+\infty} \frac{\mathcal{X}_F(t)}{t}dt$, where $\mathcal{X}_H(t)$ is the characteristic function of a set H.

Secondly, we need the following lemmas.

Lemma 8. *(see [2])* Let $f(z)$ be a meromorphic function with $1 \leq \mu_{\log}(f) < +\infty$. Then there exists a set $E \subset (1, +\infty)$ of infinite logarithmic measure such that for any given $\varepsilon > 0$ and $r \in E$, we have

$$T(r, f) < (\log r)^{\mu_{\log}(f)+\varepsilon}.$$

Lemma 9. *(see [10])* Let $f(z)$ be a meromorphic function with $1 \leq \mu_{\log}(f) < +\infty$ and $0 \leq \underline{\tau}_{\log}(f) < +\infty$. Then there exists a set $E \subset (1, +\infty)$ of infinite logarithmic measure such that for any given $\varepsilon > 0$ and $r \in E$, we have

$$T(r, f) < (\underline{\tau}_{\log}(f) + \varepsilon)(\log r)^{\mu_{\log}(f)}.$$

Lemma 10. *(see [1])* Let η_1, η_2 be two arbitrary complex numbers such that $\eta_1 \neq \eta_2$ and let $f(z)$ be a meromorphic function with finite logarithmic order ρ. Then for any given $\varepsilon > 0$, we have

$$m(r, \frac{f(z + \eta_1)}{f(z + \eta_2)}) = O(\frac{(\log r)^{\rho+\varepsilon}}{r}).$$

Lemma 11. *(see [1])* Let $f(z)$ and $g(z)$ be non-constant meromorphic functions with logarithmic orders $\rho_{\log}(f)$ and $\rho_{\log}(g)$ respectively. Then we have

$$\rho_{\log}(f + g) \leq \max\{\rho_{\log}(f), \rho_{\log}(g)\}$$

and

$$\rho_{\log}(fg) \le \max\{\rho_{\log}(f), \rho_{\log}(g)\}.$$

Furthermore, if $\rho_{\log}(f) > \rho_{\log}(g)$, then we have

$$\rho_{\log}(f+g) = \rho_{\log}(fg) = \rho_{\log}(f).$$

Lemma 12. Let $f(z)$ be a meromorphic function with logarithmic order $\rho_{\log}(f)$, and $g(z)$ be a meromorphic function with logarithmic lower order $\mu_{\log}(g)$. Then we have

$$\mu_{\log}(f+g) \le \max\{\rho_{\log}(f), \mu_{\log}(g)\}$$

and

$$\mu_{\log}(fg) \le \max\{\rho_{\log}(f), \mu_{\log}(g)\}.$$

Proof. Similarly to [23, P.33]. Without loss of generality, we let

$$\rho_{\log}(f) < +\infty, \quad \mu_{\log}(f) < +\infty.$$

By Definition 2, there exists an increasing sequence $\{r_n\}(r_n \to +\infty)$ satisfying

$$\mu_{\log}(g) = \lim_{r_n \to +\infty} \frac{\log T(r_n, g)}{\log\log r_n}.$$

Then for any given $\varepsilon > 0$, there exists a positive integer n_0 such that for all $n > n_0$,

$$T(r_n, g) < (\log r_n)^{\mu_{\log}(g)+\varepsilon}.$$

By Definition 1, for the above ε, there exists $R > 0$ such that for all $r > R$,

$$T(r, f) < (\log r)^{\rho_{\log}(f)+\varepsilon}.$$

Since $r_n \to +\infty$, there exists a positive integer n_1 such that for all $n > n_1, r_n > R$ holds and

$$T(r_n, f) < (\log r_n)^{\rho_{\log}(f)+\varepsilon}.$$

So, when $n > \max\{n_0, n_1\}$, we have

$$T(r_n, fg) \le T(r_n, f) + T(r_n, g) \le (\log r_n)^{\rho_{\log}(f)+\varepsilon} + (\log r_n)^{\mu_{\log}(g)+\varepsilon} \le 2(\log r_n)^{\lambda+\varepsilon},$$

where $\lambda = \max\{\rho_{\log}(f), \mu_{\log}(g)\}$. Then, for the above $\varepsilon > 0$,

$$\underline{\lim}_{r \to +\infty} \frac{\log T(r, fg)}{\log\log r} \le \underline{\lim}_{r_n \to +\infty} \frac{\log T(r_n, fg)}{\log\log r_n} \le \lambda + \varepsilon,$$

that is,

$$\mu_{\log}(fg) \leq \max\{\rho_{\log}(f), \mu_{\log}(g)\}.$$

Similarly, by $T(r, f+g) \leq T(r, f) + T(r, g) + \log 2$, we can obtain

$$\mu_{\log}(f+g) \leq \max\{\rho_{\log}(f), \mu_{\log}(g)\}.$$

Remark 4. It is shown in [11, P.66] and [9, P.106] that for an arbitrary complex number $c \neq 0$, the following inequalities

$$(1+o(1))T(r-|c|, f(z)) \leq T(r, f(z+c)) \leq (1+o(1))T(r+|c|, f(z))$$

hold as $r \to +\infty$ for a general meromorphic function $f(z)$. Therefore, it's easy to obtain that

$$\rho_{\log}(f(z+c)) = \rho_{\log}(f), \quad \mu_{\log}(f(z+c)) = \mu_{\log}(f).$$

Remark 5. Following Yang and Yi [23, P.37-39], the inequalities

$$T(r, f^{(n)}) \leq (n+1)T(r, f) + S(r, f)$$

and

$$T(r, f) < O(T(2^n r, f^{(n)}) + \log r)$$

hold as $r \to +\infty$ for an arbitrary meromorphic function $f(z)$. Therefore, by Lemmas 11 and 12, for a transcendent meromorphic function $f(z)$, it is easy to see

$$\rho_{\log}(f^{(n)}) = \rho_{\log}(f), \quad \mu_{\log}(f^{(n)}) = \mu_{\log}(f).$$

3. Proofs of Theorems 6 and 7

3.1 Proof of Theorem 6

If $f(z)$ has infinite logarithmic order, then the results hold. Now, we suppose $\rho_{\log}(f) < +\infty$.

(i) We divide (5) by $f(z+c_l)$ to obtain

$$-A_{l0}(z) = \sum_{\substack{i=0 \\ i\neq l}}^{n} \sum_{j=0}^{m} A_{ij}(z) \frac{f^{(j)}(z+c_i)}{f(z+c_i)} \frac{f(z+c_i)}{f(z+c_l)} + \sum_{j=1}^{m} A_{lj}(z) \frac{f^{(j)}(z+c_l)}{f(z+c_l)} - \frac{F(z)}{f(z+c_l)}. \tag{6}$$

By (6) and Remark 4, for sufficiently large r, we have

$$\begin{aligned}
&T(r,A_{l0}) = m(r,A_{l0}) + N(r,A_{l0}) \\
&\leq \sum_{\substack{i=0\\ i\neq l}}^{n}\sum_{j=0}^{m} T(r,A_{ij}) + \sum_{j=1}^{m} T(r,A_{lj}) + \sum_{i=0}^{n}\sum_{j=1}^{m} m(r,\frac{f^{(j)}(z+c_i)}{f(z+c_i)}) + T(r,F) \\
&+ O(\sum_{\substack{i=0\\ i\neq l}}^{n} m(r,\frac{f(z+c_i)}{f(z+c_l)})) + (1+o(1))T(r+|c_l|,f) + N(r,A_{l0}) + O(1).
\end{aligned} \tag{7}$$

By Lemma 10, for any given $\varepsilon > 0$, we have

$$m(r,\frac{f(z+c_i)}{f(z+c_l)}) = O(\frac{(\log r)^{\rho_{\log}(f)+\varepsilon}}{r}), \quad i=0,1,\cdots,n, i\neq l. \tag{8}$$

By the Logarithmic Derivative Lemma and Remarks 1 and 4, for the above ε and sufficiently large r, we have

$$m(r,\frac{f^{(j)}(z+c_i)}{f(z+c_i)}) = O(\log r), \quad i=0,1,\cdots,n, j=1,\cdots,m. \tag{9}$$

By Definition 4, for the above ε and sufficiently large r, we have

$$T(r,A_{l0}) \geq (\underline{\tau}_{\log}(A_{l0}) - \varepsilon)(\log r)^{\mu_{\log}(A_{l0})}. \tag{10}$$

By Remark 3, for the above ε and sufficiently large r, we have

$$N(r,A_{l0}) \leq (\log r)^{\lambda_{\log}(\frac{1}{A_{l0}})+1+\varepsilon}. \tag{11}$$

By Definition 1, for the above ε and sufficiently large r, we have

$$T(r+|c_l|,f) \leq (\log(r+|c_l|))^{\rho_{\log}(f)+\frac{\varepsilon}{2}} \leq (\log r)^{\rho_{\log}(f)+\varepsilon}. \tag{12}$$

Denote

$$\sigma_1 = \max\{\rho_{\log}(A_{ij}) : \rho_{\log}(A_{ij}) < \mu_{\log}(A_{l0}), (i,j)\neq(l,0)\}$$

and

$$\tau_1 = \sum_{\substack{\rho_{\log}(A_{ij})=\mu_{\log}(A_{l0})\\ (i,j)\neq(l,0)}} \tau_{\log}(A_{ij}).$$

Then by Definitions 1 and 3, for the above ε and sufficiently large r, we have

$$T(r,A_{ij}) \leq \begin{cases} (\log r)^{\sigma_1+\varepsilon}, & \rho_{\log}(A_{ij}) < \mu_{\log}(A_{l0}) \\ (\tau_{\log}(A_{ij})+\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}, & \rho_{\log}(A_{ij}) = \mu_{\log}(A_{l0}) \end{cases}, \tag{13}$$

where $(i,j) \neq (l,0)$.

(a) If $\mu_{\log}(F) < \mu_{\log}(A_{l0})$ and $\tau_1 < \underline{\tau}_{\log}(A_{l0}) < +\infty$.

By Lemma 8, there exists a set $E_1 \subset (1,+\infty)$ with infinite logarithmic measure such that for the above ε and $r \in E_1$, we have

$$T(r,F) \leq (\log r)^{\mu_{\log}(F)+\varepsilon}. \tag{14}$$

Then we can choose $\varepsilon > 0$ sufficiently small to satisfy

$$\max\{\sigma_1, \lambda_{\log}(\frac{1}{A_{l0}})+1, \mu_{\log}(F)\} + 2\varepsilon < \mu_{\log}(A_{l0}), \quad \tau_1 + (k+1)\varepsilon < \underline{\tau}_{\log}(A_{l0}), \tag{15}$$

where $k = (n+1)(m+1)$. By substituting (8)-(14) into (7), for sufficiently large $r \in E_1$, we obtain

$$\begin{aligned}
&(\underline{\tau}_{\log}(A_{l0}) - \varepsilon)(\log r)^{\mu_{\log}(A_{l0})} \\
&\leq O((\log r)^{\sigma_1+\varepsilon}) + O(\frac{(\log r)^{\rho_{\log}(f)+\varepsilon}}{r}) + (\log r)^{\mu_{\log}(F)+\varepsilon} + O((\log r)^{\rho_{\log}(f)+\varepsilon}) \\
&\quad + (\log r)^{\lambda_{\log}(\frac{1}{A_{l0}})+1+\varepsilon} + \sum_{\substack{\rho_{\log}(A_{ij})=\mu_{\log}(A_{l0}) \\ (i,j)\neq(l,0)}} (\tau_{\log}(A_{ij})+\varepsilon)(\log r)^{\mu_{\log}(A_{l0})} + O(\log r) \\
&\leq (\log r)^{\sigma_1+2\varepsilon} + (\log r)^{\rho_{\log}(f)+2\varepsilon} + (\log r)^{\mu_{\log}(F)+\varepsilon} + (\tau_1 + (k-1)\varepsilon)(\log r)^{\mu_{\log}(A_{l0})} \\
&\quad + (\log r)^{\lambda_{\log}(\frac{1}{A_{l0}})+1+2\varepsilon}.
\end{aligned} \tag{16}$$

By (15) and (16), we get

$$\mu_{\log}(A_{l0}) \leq \rho_{\log}(f) + 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\rho_{\log}(f) \geq \mu_{\log}(A_{l0}).$$

(b) If $\mu_{\log}(F) = \mu_{\log}(A_{l0})$ and $\tau_1 + \underline{\tau}_{\log}(F) < \underline{\tau}_{\log}(A_{l0}) < +\infty$.

By Lemma 9, there exists a set $E_2 \subset (1,+\infty)$ with infinite logarithmic measure such that for the above ε and $r \in E_2$, we have

$$T(r,F) \leq (\underline{\tau}_{\log}(F)+\varepsilon)(\log r)^{\mu_{\log}(F)} = (\underline{\tau}_{\log}(F)+\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}. \tag{17}$$

Then we can choose $\varepsilon > 0$ sufficiently small to satisfy

$$\max\{\sigma_1, \lambda_{\log}(\frac{1}{A_{l0}})+1\} + 2\varepsilon < \mu_{\log}(A_{l0}), \quad \tau_1 + \underline{\tau}_{\log}(F) + (k+2)\varepsilon < \underline{\tau}_{\log}(A_{l0}). \tag{18}$$

By substituting (8)-(13), (17) into (7), for sufficiently large $r \in E_2$, we obtain

$$\begin{aligned}
&(\underline{\tau}_{\log}(A_{l0}) - \varepsilon)(\log r)^{\mu_{\log}(A_{l0})} \\
&\leq O((\log r)^{\sigma_1+\varepsilon}) + O(\frac{(\log r)^{\rho_{\log}(f)+\varepsilon}}{r}) + (\log r)^{\lambda_{\log}(\frac{1}{A_{l0}})+1+\varepsilon} + O((\log r)^{\rho_{\log}(f)+\varepsilon}) \\
&\quad + \sum_{\substack{\rho_{\log}(A_{ij})=\mu_{\log}(A_{l0}) \\ (i,j)\neq(l,0)}} (\tau_{\log}(A_{ij}) + \varepsilon)(\log r)^{\mu_{\log}(A_{l0})} + (\underline{\tau}_{\log}(F) + \varepsilon)(\log r)^{\mu_{\log}(A_{l0})} \\
&\quad + O(\log r) \\
&\leq (\log r)^{\sigma_1+2\varepsilon} + (\tau_1 + \underline{\tau}_{\log}(F) + k\varepsilon)(\log r)^{\mu_{\log}(A_{l0})} + (\log r)^{\lambda_{\log}(\frac{1}{A_{l0}})+1+2\varepsilon} \\
&\quad + (\log r)^{\rho_{\log}(f)+2\varepsilon}.
\end{aligned} \tag{19}$$

By (18) and (19), we get

$$\mu_{\log}(A_{l0}) \leq \rho_{\log}(f) + 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\rho_{\log}(f) \geq \mu_{\log}(A_{l0}).$$

(c) If $\mu_{\log}(F) = \mu_{\log}(A_{l0})$ and $\tau_1 + \underline{\tau}_{\log}(A_{l0}) < \underline{\tau}_{\log}(F) < +\infty$.
By (5) and Remarks 4 and 5, for sufficiently large r, we have

$$\begin{aligned}
T(r,F) &\leq \sum_{(i,j)\neq(l,0)} T(r,A_{ij}) + T(r,A_{l0}) + \sum_{i=0}^{n}\sum_{j=0}^{m} T(r, f^{(j)}(z+c_i)) \\
&\leq \sum_{(i,j)\neq(l,0)} T(r,A_{ij}) + T(r,A_{l0}) + O(T(2r,f)) + O(\log r).
\end{aligned} \tag{20}$$

By Definition 4, for the above ε and sufficiently large r, we have

$$T(r,F) \geq (\underline{\tau}_{\log}(F) - \varepsilon)(\log r)^{\mu_{\log}(A_{l0})}. \tag{21}$$

By Definition 1, for the above ε and sufficiently large r, we have

$$T(2r,f) \leq (\log(2r))^{\rho_{\log}(f)+\frac{\varepsilon}{2}} \leq (\log r)^{\rho_{\log}(f)+\varepsilon}. \tag{22}$$

By Lemma 9, there exists a set $E_3 \subset (1,+\infty)$ with infinite logarithmic measure such that for the above ε and $r \in E_3$, we have

$$T(r,A_{l0}) \leq (\underline{\tau}_{\log}(A_{l0}) + \varepsilon)(\log r)^{\mu_{\log}(A_{l0})}. \tag{23}$$

Then we can choose $\varepsilon > 0$ sufficiently small to satisfy

$$\max\{\sigma_1, 1\} + 2\varepsilon < \mu_{\log}(A_{l0}), \quad \tau_1 + \underline{\tau}_{\log}(A_{l0}) + (k+2)\varepsilon < \underline{\tau}_{\log}(F). \tag{24}$$

By substituting (13), (21)-(23) into (20), for sufficiently large $r \in E_3$, we obtain

$$\begin{aligned} &(\underline{\tau}_{\log}(F) - \varepsilon)(\log r)^{\mu_{\log}(A_{l0})} \\ \leq &\sum_{\substack{\rho_{\log}(A_{ij})=\mu_{\log}(A_{l0}) \\ (i,j)\neq(l,0)}} (\tau_{\log}(A_{ij}) + \varepsilon)(\log r)^{\mu_{\log}(A_{l0})} + (\underline{\tau}_{\log}(A_{l0}) + \varepsilon)(\log r)^{\mu_{\log}(A_{l0})} \\ &+ O((\log r)^{\sigma_1+\varepsilon}) + O((\log r)^{\rho_{\log}(f)+\varepsilon}) + O(\log r) \\ \leq &(\tau_1 + \underline{\tau}_{\log}(A_{l0}) + k\varepsilon)(\log r)^{\mu_{\log}(A_{l0})} + (\log r)^{\sigma_1+2\varepsilon} + (\log r)^{\rho_{\log}(f)+2\varepsilon} + O(\log r). \end{aligned} \tag{25}$$

By (24) and (25), we get

$$\mu_{\log}(A_{l0}) \leq \rho_{\log}(f) + 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\rho_{\log}(f) \geq \mu_{\log}(A_{l0}).$$

(ii) If $\mu_{\log}(F) > \mu_{\log}(A_{l0})$.

Then on the contrary, we suppose that $\rho_{\log}(f) < \mu_{\log}(F)$. By (5), Lemmas 11 and 12, Remarks 4 and 5, we obtain

$$\begin{aligned} \mu_{\log}\Big(\sum_{i=0}^{n}\sum_{j=0}^{m} A_{ij}(z) f^{(j)}(z+c_i)\Big) &\leq \max\{\mu_{\log}(A_{l0}), \rho_{\log}(f), \rho_{\log}(A_{ij}), (i,j)\neq(l,0)\} \\ &< \mu_{\log}(F), \end{aligned}$$

which is a contradiction. Hence, we have

$$\rho_{\log}(f) \geq \mu_{\log}(F).$$

Therefore, the proof of Theorem 6 is complete.

3.2 Proof of Theorem 7

By Remark 1, if $f(z)$ is a constant function, then $\rho_{\log}(f) = 0$, not satisfying the condition. If $f(z)$ is a non-constant rational function, then $\rho_{\log}(f) = 1, \lambda_{\log}(f) = 0$, that is, the result holds. Now, we suppose that $f(z)$ is a transcendent meromorphic function.

Let $c_0 = 0$ without loss of generality, then by (5) we have

$$\frac{1}{f(z)} = \frac{1}{F(z)}(\sum_{i=0}^{n}\sum_{j=1}^{m} A_{ij}(z)\frac{f^{(j)}(z+c_i)}{f(z+c_i)}\frac{f(z+c_i)}{f(z)} + \sum_{i=1}^{n} A_{i0}(z)\frac{f(z+c_i)}{f(z)} + A_{00}(z)). \tag{26}$$

By (26), we have

$$\begin{aligned} &T(r,f) = T(r,\frac{1}{f}) + O(1) = m(r,\frac{1}{f}) + N(r,\frac{1}{f}) + O(1) \\ &\leq \sum_{i=0}^{n}\sum_{j=1}^{m} m(r,\frac{f^{(j)}(z+c_i)}{f(z+c_i)}) + O(\sum_{i=1}^{n} m(r,\frac{f(z+c_i)}{f(z)})) + \sum_{i=0}^{n}\sum_{j=0}^{m} T(r,A_{ij}) \\ &+ T(r,F) + N(r,\frac{1}{f}) + O(1). \end{aligned} \tag{27}$$

Let $l = 0$, by substituting (8) and (9) into (27), for any given $\varepsilon > 0$ and sufficiently large r, we have

$$T(r,f) \leq O(\frac{(\log r)^{\rho_{\log}(f)+\varepsilon}}{r}) + O(\log r) + \sum_{i=0}^{n}\sum_{j=0}^{m} T(r,A_{ij}) + T(r,F) + N(r,\frac{1}{f}). \tag{28}$$

By (28) and the assumption that $\max\{\rho_{\log}(F), \rho_{\log}(A_{ij}), i = 0,1,\cdots,n; j = 0,1,\cdots,m\} < \rho_{\log}(f)$, for the above ε and sufficiently large r, we obtain

$$\rho_{\log}(f) \leq \lambda_{\log}(f) + 1.$$

By Remark 3, we have

$$\lambda_{\log}(f) + 1 \leq \rho_{\log}(f).$$

Hence,

$$\rho_{\log}(f) = \lambda_{\log}(f) + 1.$$

Set $g(z) = f(z) - \varphi(z)$. By Lemma 11 and the assumption that $\rho_{\log}(\varphi) < \rho_{\log}(f)$, we have

$$\rho_{\log}(g) = \rho_{\log}(f). \tag{29}$$

By substituting $f(z) = g(z) + \varphi(z)$ into (5), we obtain

$$\sum_{i=0}^{n}\sum_{j=0}^{m} A_{ij}(z)g^{(j)}(z+c_i) = G(z), \tag{30}$$

where $G(z) = F(z) - \sum_{i=0}^{n}\sum_{j=0}^{m} A_{ij}(z)\varphi^{(j)}(z+c_i)$. Since $\varphi(z)$ is not a solution of (5), then

$$G(z) \not\equiv 0.$$

By Lemma 11 and Remarks 4 and 5, we have

$$\begin{aligned}\rho_{\log}(G) &\le \max\{\rho_{\log}(F), \rho_{\log}(\varphi), \rho_{\log}(A_{ij}), i = 0,1,\cdots,n; j = 0,1,\cdots,m\}\\ &< \rho_{\log}(f).\end{aligned} \tag{31}$$

Then by (29) and (31), we have

$$\max\{\rho_{\log}(G), \rho_{\log}(A_{ij}), i = 0,1,\cdots,n; j = 0,1,\cdots,m\} < \rho_{\log}(f) = \rho_{\log}(g). \tag{32}$$

By (30) and (32), we obtain

$$\rho_{\log}(g) = \lambda_{\log}(g) + 1,$$

that is,

$$\rho_{\log}(f) = \lambda_{\log}(f - \varphi) + 1.$$

Therefore, the proof of Theorem 7 is complete.

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