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THE IDEALS OF A NUMERICAL SEMIGROUP WITH EMBEDDING DIMENSION TWO

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Abstract. Let S and Δ be numerical semigroups. We will say that S is an ideal of Δ if there exits $X \subseteq \Delta$ such that $S = (X + \Delta) \cup \{0\}$. In this work, we will study the ideals of a numerical semigroup of the form $\langle a, b \rangle$ with a and b positive integers such that $\gcd\{a, b\} = 1$. The main results that we have obtained are the following:

1. Given a numerical semigroup S and $\{a, b\} \subseteq \mathbb{N}$ such that $\gcd\{a, b\} = 1$, we present an algorithm that allows us to determine if S is an ideal of $\langle a, b \rangle$.
2. If S is a numerical semigroup, we show an algorithmic procedure to compute the set $\{\{a, b\} \subseteq \mathbb{N} \mid \gcd\{a, b\} = 1 \text{ and } S \text{ is an ideal of } \langle a, b \rangle\}$.
3. We obtain formulas to compute the multiplicity, Frobenius number and genus of the numerical semigroups of the form $(X + \langle a, b \rangle) \cup \{0\}$ in terms of X, a and b.

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1. Introduction

Let $\mathbb{Z}$ be the set of integer numbers and $\mathbb{N} = \{x \in \mathbb{Z} \mid x \geq 0\}$. A *numerical semigroup* is a subset S of $\mathbb{N}$ which is closed under addition, $0 \in S$ and $\mathbb{N} \backslash S$ is finite.

If A is a nonempty subset of $\mathbb{N}$, we denote by $\langle A \rangle$ the submonoid of $(\mathbb{N}, +)$ generated by A, that is, $\langle A \rangle = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid n \in \mathbb{N} \backslash \{0\}, \{a_1, \ldots, a_n\} \subseteq A \text{ and } \{\lambda_1, \ldots, \lambda_n\} \subseteq \mathbb{N}\}$. It is well known (for example, see [10, Lema 2.1]) that $\langle A \rangle$ is a numerical semigroup if and only if $\gcd(A) = 1$.

If S is a numerical semigroup and $S = \langle A \rangle$, then we say that A is a *system of generators* of S. Moreover, if $S \neq \langle B \rangle$ for all $B \subsetneq A$, then we will say that A is a *minimal system of generators* of S. In [10, Corollary 2.8] it is shown that every numerical semigroup has a unique minimal system of generators and besides this system is finite. We denote by $\mathrm{msg}(S)$ the minimal system of generators of S. The cardinality of $\mathrm{msg}(S)$ is called the *embedding dimension* of S and will be denoted by $\mathrm{e}(S)$.

If A and B are nonempty subsets of $\mathbb{Z}$, we write $A + B = \{a + b \mid a \in A, b \in B\}$. Let S be a numerical semigroup. An *ideal* of S is a nonempty set $I \subseteq S$ satisfying $I + S \subseteq I$.

If S is a numerical semigroup and I is an ideal of S, then $I \cup \{0\}$ is a numerical semigroup. This fact induces us to give the following definition. An $\mathrm{I}(S)$-*semigroup* is a numerical semigroup T such that $T\backslash\{0\}$ is an ideal of S. Observe that the study of the ideals of S is equivalent to the study of $\mathrm{I}(S)$-semigroups since I is an ideal of S if and only if $I \cup \{0\}$ is an $\mathrm{I}(S)$-semigroup.

In all this work, a and b will denote two integers greater than or equal to two such that $\gcd\{a, b\} = 1$. Note that $\langle a, b\rangle$ is a numerical semigroup with embedding dimension two and every numerical semigroups with embedding dimension two is of this form.

If S is a numerical semigroup, then $\mathrm{m}(S) = \min(S\backslash\{0\})$, $\mathrm{F}(S) = \max(\mathbb{Z}\backslash S)$ and $\mathrm{g}(S) = \sharp(\mathbb{N}\backslash S)$ (where $\sharp(X)$ denotes the cardinality of a set X), are three important invariants called the *multiplicity*, the *Frobenius number* and the *genus* of S, respectively.

The Frobenius problem for numerical semigroups (see [6]) focuses on finding formulas to the Frobenius number and the genus of a numerical semigroup from its minimum system of generators. This problem was solved in [11] for numerical semigroups with embedding dimension two.

Specifically, Sylvester proves that $\mathrm{F}(\langle a, b\rangle) = ab - a - b$ and $\mathrm{g}(\langle a, b\rangle) = \frac{(a-1)(b-1)}{2}$. Nowadays the Frobenius problem is open for numerical semigroups with embedding dimension equal or greater than three.

Let S be a numerical semigroup. A *S-incomparable set* is a nonempty subset X of S verifying that $x - y \notin S$ for all $(x, y) \in X \times X$ such that $x \neq y$. In [5, Theorem 5], it is shown that $\{(X + S) \cup \{0\} \mid X \text{ is a } S\text{-incomparable set}\} = \{T \mid T \text{ is an } \mathrm{I}(S)\text{-semigroup}\}$. Moreover, if X and Y are different S-incomparable sets, then $(X + S) \cup \{0\} \neq (Y + S) \cup \{0\}$. The cardinality of X we will called *ideal dimension* of $(X + S) \cup \{0\}$.

We will comment on the contents of this work. In Section 2, we introduce some basic concepts and results from the Theory of Ideals. We present an algorithm that allows us to determine if a numerical semigroup is or is not an $\mathrm{I}(\langle a, b\rangle)$-semigroup and we will give another algorithm such that given a numerical semigroup S, computes the set $\{\Delta \mid \Delta$ is a numerical semigroup, $\mathrm{e}(\Delta) = 2$ and S is an $\mathrm{I}(\Delta)$-semigroup$\}$.

In Section 3, we will study the $\mathrm{I}(\langle a, b\rangle)$-semigroups with ideal dimension one. The results of [8], allow us to give formulas to calculate the multiplicity, Frobenius number and genus of this kind of numerical semigroups. Also we will present an algorithm which computes the minimal generator system of these numerical semigroups.

In Section 4, we will study the $\mathrm{I}(\langle a, b\rangle)$-semigroups with arbitrary ideal dimension. In particular, we will give formulas to calculate the multiplicity, the Frobenius number and the genus of this class of numerical semigroups. Besides, also we will give an algorithm which calculates its minimal generator system.

In Section 5, we particularize and improve the results of the previous section for $\mathrm{I}(\langle a, b\rangle)$-semigroups with ideal dimension two. Moreover, we will give some algorithms which calculate all the $\mathrm{I}(\langle a, b\rangle)$-semigroups with a fixed genus or with a fixed Frobenius number.

2. Basic concepts and first results

We begin this section by presenting some concepts and basic results of Theory of Ideals of numerical semigroups. The following result appears in [5, Proposition 1].

Proposition 1. *If S is a numerical semigroup and X is a nonempty subset of S, then $X + S$ is an ideal of S.*

If S is a numerical semigroup, then we can define over $\mathbb{Z}$ the following order relation: $x \leq_S y$ if and only if $y - x \in S$.

A *S-incomparable set* is a nonempty subset X of S verifying that $x - y \notin S$ for all $(x, y) \in X \times X$ such that $x \neq y$. The following result is Theorem 5 from [5].

Proposition 2. *Let S be a numerical semigroup. Then the set $\{X + S \mid X$ is a S-incomparable set$\}$ is the formed by all the ideals of S. Moreover, if X and Y are different S-incomparable sets, then $X + S \neq Y + S$.*

The following result appears in [5, Proposition 7].

Proposition 3. *Let S be a numerical semigroup and let X be a S-incomparable set, then the cardinality of X is less than or equal to* $\mathrm{g}(S) + 1$.

If I is an ideal of a numerical semigroup S, then by Proposition 2, there is a unique S-incomparable set, X, such that $I = X + S$. In this case, we will say that X is the *ideal minimal system of generator* of I in S and we will denote it by $\mathrm{imsg_S}(I)$. In [5, Proposition 6] appears the following result.

Proposition 4. *Let S be a numerical semigroup and let I be an ideal of S, then* $\mathrm{imsg_S}(I) = \mathrm{Minimals}_{\leq_S}(I)$.

The cardinality of $\mathrm{imsg_S}(I)$ is called *ideal dimension* of I in S and it will be denoted by $\dim_S(I)$.

The following result appears in [5, Proposition 9].

Proposition 5. *Let S be a numerical semigroup and let I be an ideal of S such that $I \neq S$. Then* $\mathrm{imsg_S}(I) = \mathrm{Minimals}_{\leq_S}(\mathrm{msg}(I \cup \{0\}))$.

As an immediate consequence of the previous proposition, we have the following result.

Corollary 1. *Let S be a numerical semigroup and let I be an ideal of S. Then* $\dim_S(I) \leq \mathrm{e}(I \cup \{0\})$.

If T is an $\mathrm{I}(S)$-semigroup then, taking advance of notation, we will write $\mathrm{imsg_S}(T)$ instead of $\mathrm{imsg_S}(T \backslash \{0\})$ and $\dim_S(T)$ instead of $\dim_S(T \backslash \{0\})$.

Next, our aim is to give an algorithm which allows us to determine if a numerical semigroup is or is not an $\mathrm{I}(\langle a, b\rangle)$-semigroup. The key to this algorithm is the following proposition.

Proposition 6. *Let S and T be numerical semigroups. Then T is an $\mathrm{I}(S)$-semigroup if and only if* $\mathrm{msg}(T) + \mathrm{msg}(S) \subseteq T$ *and* $\mathrm{msg}(T) \subseteq S$.

Proof. The necessary condition is trivial. So let us focus on the sufficiency. If $\mathrm{msg}(T) \subseteq S$, then $T \backslash \{0\} \subseteq S$. As $\mathrm{msg}(T) + \mathrm{msg}(S) \subseteq T$, then we can easily deduce that $(T \backslash \{0\}) + S \subseteq T \backslash \{0\}$. Therefore, $T \backslash \{0\}$ is an ideal of S and so T is an $\mathrm{I}(S)$-semigroup.

We have now all the ingredients needed to present the announced algorithm.

Algorithm 1

INPUT: A numerical semigroup S.
OUTPUT: TRUE, if S is an $\mathrm{I}(\langle a, b\rangle)$-semigroup, and FALSE otherwise.

1) If $\mathrm{msg}(S) \nsubseteq \langle a, b\rangle$, then return FALSE.

2) If $\mathrm{msg}(S) + \{a, b\} \nsubseteq S$, then return FALSE.

3) Return TRUE.

We are going to show how the previous algorithm works with an example.

Example 1. *We want to know if* $S = \langle 4, 9, 14, 19\rangle$ *is or is not an* $\mathrm{I}(\langle 4,5\rangle)$*-semigroup. By using Algorithm 1, we have*

1) $\mathrm{msg}(S) = \{4, 9, 14, 19\} \subseteq \langle 4, 5\rangle$.

2) $\{4, 9, 14, 19\} + \{4, 5\} = \{8, 9, 13, 14, 18, 19, 23, 24\} \subseteq S$.

3) TRUE.

Example 2. *We want to know if* $S = \langle 4, 9, 14\rangle$ *is or is not an* $\mathrm{I}(\langle 4,5\rangle)$*-semigroup. By using Algorithm 1, we have*

1) $\mathrm{msg}(S) = \{4, 9, 14\} \subseteq \langle 4, 5\rangle$.

2) $\{4, 9, 14\} + \{4, 5\} = \{8, 9, 13, 14, 18, 19\} \nsubseteq S$ *because* $19 \notin S$*. Therefore, the algorithm returns FALSE.*

We end this section by showing an algorithm such that, given a numerical semigroup S, it computes the set $\{\Delta \mid \Delta$ is a numerical semigroup, $\mathrm{e}(\Delta) = 2$ and S is an $\mathrm{I}(\Delta)$-semigroup$\}$. For this purpose, we need to introduce some concepts and results.

Let S be a numerical semigroup. Following the notation introduced in [9], we say that an integer x is a *pseudo-Frobenius number* if $x \notin S$ and $x + s \in S$ for all $s \in S\backslash\{0\}$. We will denote by $\mathrm{PF}(S)$ the set of pseudo-Frobenius numbers of S. The cardinality of $\mathrm{PF}(S)$ is an important invariant of S (see [2]) called *type* of S, denoted by $\mathrm{t}(S)$. The following result can be consulted in [10, Corollary 2.23].

Proposition 7. *If* S *is a numerical semigroup and* $S \neq \mathbb{N}$*, then* $\mathrm{t}(S) \leq \mathrm{m}(S) - 1$.

Observe that if S is a numerical semigroup, $S \neq \mathbb{N}$ and $\{x, y\} \subseteq \mathrm{PF}(S)$, then $x + y \in S$ or $x + y \in \mathrm{PF}(S)$. Therefore, we can enounce the following result.

Proposition 8. *If* S *is a numerical semigroup and* $S \neq \mathbb{N}$*, then* $S \cup \mathrm{PF}(S)$ *is also a numerical semigroup.*

The following result is easy to prove and it appears in [5, Introduction].

Proposition 9. *Let* S *and* T *be numerical semigroups. Then* T *is an* $\mathrm{I}(S)$*-semigroup if and only if* $T \subseteq S \subseteq T \cup \mathrm{PF}(T)$.

Let S be a numerical semigroup and $n \in S\backslash\{0\}$. We define the ApÃ©ry set of n in S (named so in honour of [1]) as $\mathrm{Ap}(S, n) = \{s \in S \mid s - n \notin S\}$. The following result appears in [10, Lemma 2.4].

Proposition 10. *If* S *is a numerical semigroup and* $n \in S\backslash\{0\}$*, then the set* $\mathrm{Ap}(S, n)$ *has cardinality* n*. Moreover,* $\mathrm{Ap}(S, n) = \{0 = w(0), w(1), \ldots, w(n-1)\}$*, where* $w(i)$ *is the least element of* S *congruent with* i *modulo* n*, for all* $i \in \{0, \ldots, n-1\}$.

The following result can be consulted in [10, Proposition 2.20].

Proposition 11. *If S is a numerical semigroup and $n \in S\backslash\{0\}$, then* $\mathrm{PF}(S) = \{w - n \mid w \in \mathrm{Maximals}_{\leq_S}(\mathrm{Ap}(S,n))\}$.

Theorem 2. *A numerical semigroup S is an $\mathrm{I}(\langle a,b\rangle)$-semigroup if and only if one of the following conditions is verified:*

1) $S = \langle a,b\rangle$.

2) $\mathrm{msg}(S) \subseteq \langle a,b\rangle$ *and* $\{a,b\} \subseteq \mathrm{PF}(S)$.

3) $\mathrm{msg}(S) \subseteq \langle a,b\rangle, a \in \mathrm{msg}(S)$ *and* $b \in \mathrm{PF}(S)$.

Proof. Necessity. If S is an $\mathrm{I}(\langle a,b\rangle)$-semigroup, then by Proposition 9, we know that $S \subseteq \langle a,b\rangle \subseteq S \cup \mathrm{PF}(S)$. Therefore, $\mathrm{msg}(S) \subseteq \langle a,b\rangle$. We distinguish three cases.

1) If $\{a,b\} \subseteq S$, then $S = \langle a,b\rangle$.

2) If $a \in S$ and $b \notin S$, then $a \in \mathrm{msg}(S)$ and $b \in \mathrm{PF}(S)$.

3) If $\{a,b\} \cap S = \emptyset$, then $\{a,b\} \subseteq \mathrm{PF}(S)$.

Sufficiency). If 1), 2) or 3) is verified, we have that $S \subseteq \langle a,b\rangle \subseteq S \cup \mathrm{PF}(S)$ and by applying Proposition 9, we can assert that S is an $\mathrm{I}(\langle a,b\rangle)$-semigroup.

We are already able to provide the announced algorithm.

Algorithm 2

INPUT: A numerical semigroup S such that $S \neq \mathbb{N}$.
OUTPUT: The set $\{\Delta \mid \Delta$ is a numerical semigroup, $\mathrm{e}(\Delta) = 2$ and S is an $\mathrm{I}(\Delta)$-semigroup$\}$.

1) Compute $\mathrm{msg}(S)$ and $\mathrm{PF}(S)$.

2) $A = \{X \subseteq \mathrm{PF}(S)\backslash\{1\} \mid \#X = 2 \text{ and } \mathrm{msg}(S) \subseteq \langle X\rangle\}$.

3) $B = \{\{a,b\} \mid a \in \mathrm{msg}(S), b \in \mathrm{PF}(S)\backslash\{1\} \text{ and } \mathrm{msg}(S) \subseteq \langle a,b\rangle\}$.

4) $C = \begin{cases} \mathrm{msg}(S) & \text{if} \quad \mathrm{e}(S) = 2 \\ \emptyset & \text{otherwise.} \end{cases}$

5) Return $\{\langle X\rangle \mid X \in A \cup B \cup C\}$.

Example 3. *We are going to apply the Algorithm 2 to the numerical semigroup* $S = \langle 4, 5, 6, 7 \rangle$.

1) $\mathrm{msg}(S) = \{4, 5, 6, 7\}$ *and* $\mathrm{PF}(S) = \{1, 2, 3\}$.

2) $A = \{\{2, 3\}\}$.

3) $B = \{\{2, 5\}\}$.

4) $C = \emptyset$.

5) $\{\Delta \mid \Delta$ *is a numerical semigroup,* $\mathrm{e}(\Delta) = 2$ *and* S *is an* $\mathrm{I}(\Delta)$*-semigroup*$\} = \{\langle 2, 3\rangle, \langle 2, 5\rangle\}$.

Example 4. *We are going to apply the Algorithm 2 to the numerical semigroup* $S = \langle 5, 7, 9 \rangle$.

1) $\mathrm{msg}(S) = \{5, 7, 9\}$ *and* $\mathrm{PF}(S) = \{11, 13\}$.

2) $A = \emptyset$.

3) $B = \emptyset$.

4) $C = \emptyset$.

5) $\{\Delta \mid \Delta$ *is a numerical semigroup,* $\mathrm{e}(\Delta) = 2$ *and* S *is an* $\mathrm{I}(\Delta)$*-semigroup*$\} = \emptyset$.

The return of Algorithm 2, induces to give the following definition.

We say that a numerical semigroup S is *bipartite* if there is a numerical semigroup Δ such that $\mathrm{e}(\Delta) = 2$ and S is an $\mathrm{I}(\Delta)$-semigroup. In Example 4, we can see that there are numerical semigroups which are not bipartite.

Following the notation introduced in [7], a *R-Frobenius variety* is a nonempty family $\mathcal{F}$ of numerical semigroups that fulfills the following conditions.

1. $\mathcal{F}$ has a maximum element, denoted by $\Delta(\mathcal{F})$, with respect to the inclusion order.
2. If $\{S, T\} \subseteq \mathcal{F}$ then $S \cap T \in \mathcal{F}$.
3. If $S \in \mathcal{F}$ and $S \neq \Delta(\mathcal{F})$, then $S \cup \{\max(\Delta(\mathcal{F}) \backslash S)\} \in \mathcal{F}$.

In [7, Example 2.3] it is proven that if S and Δ are numerical semigroups such that $S \subseteq \Delta$, then the set $[S, \Delta] = \{T \mid T$ is a numerical semigroup and $S \subseteq T \subseteq \Delta\}$ is a R-Frobenius variety.

The following result is an immediate consequence from Proposition 9.

Proposition 12. *A numerical semigroup* S *is bipartite if and only if the R-Frobenius variety* $[S, S \cup \mathrm{PF}(S)]$ *contains at least one numerical semigroup with embedding dimension two.*

3. Principal I($\langle a, b\rangle$)-semigroups

An ideal I of a numerical semigroup S is *principal* if there is $x \in S$ such that $I = \{x\} + S$. This fact induce us to give the following definition. We will say that an I($\langle a, b\rangle$)-semigroup S is *principal* if $\dim_{\langle a,b\rangle}(S) = 1$.

The following result is deduced from Proposition 9 of [8].

Proposition 13. *Let S be a numerical semigroup, $x \in S\backslash\{0, 1\}$ and $T = (\{x\} + S) \cup \{0\}$. Then $\mathrm{m}(T) = x$, $\mathrm{F}(T) = \mathrm{F}(S) + x$ and $\mathrm{g}(T) = \mathrm{g}(S) + x - 1$.*

The following result can be deduced from [10, Proposition 2.10].

Proposition 14. *Let S be a numerical semigroup. Then $\mathrm{e}(S) \leq \mathrm{m}(S)$.*

A MED-*semigroup* (*numerical semigroup with maximal embedding dimension*), is a numerical semigroup such that $\mathrm{e}(S) = \mathrm{m}(s)$.

The following result can be deduced from Section 1 of [8].

Proposition 15. *Let S be a numerical semigroup, $n \in S\backslash\{0, 1\}$ and $T = (\{n\} + S) \cup \{0\}$. Then T is a* MED-*semigroup. Moreover,* $\mathrm{msg}(T) = \mathrm{Ap}(S, n) + \{n\}$.

Example 5. *Let $S = \langle 5, 7\rangle$ and $T = (\{10\} + \langle 5, 7\rangle) \cup \{0\}$. It is clear that* $\mathrm{Ap}\left(\langle 5, 7\rangle, 10\right) = \{0, 5, 7, 12, 14, 19, 21, 26, 28, 33\}$. *Therefore, by applying Proposition 15, we have that* $\mathrm{msg}(T) = \{10, 15, 17, 22, 24, 29, 31, 36, 38, 43\}$.

The following result has an immediate proof.

Proposition 16. *Let S be a numerical semigroup and $n \in S\backslash\{0\}$. Then for all $z \in \mathbb{Z}$ there is a unique $(k, w) \in \mathbb{Z} \times \mathrm{Ap}(S, n)$ such that $z = kn + w$. Moreover, $z \in S$ if and only if $k \in \mathbb{N}$.*

As a consequence of the previous proposition we have the following result.

Corollary 3. *Let S be a numerical semigroup, $n \in S\backslash\{0\}$, $\mathrm{Ap}(S, n) = \{0 = w(0), w(1), w(2), \cdots, w(n-1)\}$ and $x \in \mathbb{Z}$. Then $x \in S$ if and only if $x \geq w(x \bmod n)$.*

The following result is well known and it appears in Example 2.22 from [10].

Proposition 17. *With the above notation,*

$$\mathrm{Ap}\left(\langle a, b\rangle, a\right) = \{kb \mid k \in \{0, 1, \cdots, a-1\}\}.$$

As a consequence of Propositions 16 and 17, we have the following result.

Corollary 4. *With the above notation, the correspondence* $f : \mathbb{N} \times \{0, 1, \cdots, a - 1\} \longrightarrow \langle a, b\rangle$*, defined by* $f(\lambda, \mu) = \lambda a + \mu b$ *is a bijective map.*

Let $(\lambda, \mu) \in \mathbb{N} \times \{0, 1, \cdots, a - 1\}$, our next aim in this section will be to prove the Theorem 6 which allows us to calculate $\mathrm{Ap}\,(\langle a, b\rangle, \lambda a + \mu b)$ quickly. To prove such theorem it is necessary to apply the following result.

Lemma 5. *Let* (λ, μ) *and* (λ', μ') *be two elements which belong to* $\mathbb{N} \times \{0, 1, \cdots, a - 1\}$*. Then* $(\lambda' a + \mu' b) - (\lambda a + \mu b) \in \langle a, b\rangle$ *if and only if* $(\lambda, \mu) \leq (\lambda', \mu')$ *or* $\lambda' - \lambda \geq b$*.*

Proof. We distinguish two cases.

1) If $\mu \leq \mu'$, then $(\lambda' a + \mu' b) - (\lambda a + \mu b) \in \langle a, b\rangle$ if and only if $(\lambda' - \lambda)a + (\mu' - \mu)b \in \langle a, b\rangle$. By applying Propositions 16 and 17, we have that $(\lambda' - \lambda)a + (\mu' - \mu)b \in \langle a, b\rangle$ if and only if $\lambda \leq \lambda'$. Therefore, $(\lambda' a + \mu' b) - (\lambda a + \mu b) \in \langle a, b\rangle$ if and only if $(\lambda, \mu) \leq (\lambda', \mu')$.

2) If $\mu' < \mu$, then by applying Corollary 3 and Proposition 17, we have that $(\lambda' a + \mu' b) - (\lambda a + \mu b) \in \langle a, b\rangle$ if and only if $(\lambda' a + \mu' b) - (\lambda a + \mu b) \geq (a + \mu' - \mu)b$. Therefore, $(\lambda' a + \mu' b) - (\lambda a + \mu b) \in \langle a, b\rangle$ if and only if $\lambda' - \lambda \geq b$.

Theorem 6. *If* $(\lambda, \mu) \in \mathbb{N} \times \{0, 1, \cdots, a - 1\}$*, then* $\mathrm{Ap}\,(\langle a, b\rangle, \lambda a + \mu b) = \{\alpha a + \beta b \mid (\alpha, \beta) \in \{0, 1, \cdots, \lambda + b - 1\} \times \{0, 1, \cdots, \mu - 1\} \cup \{0, 1, \cdots, \lambda - 1\} \times \{\mu, \mu + 1, \cdots, a - 1\}\}$*.*

Proof. It is enough to observe that by Corollary 4 and Lemma 5, we have that $\mathrm{Ap}\,(\langle a, b\rangle, \lambda a + \mu b) = \{\alpha a + \beta b \mid (\alpha, \beta) \in \mathbb{N} \times \{0, 1, \cdots, a - 1\}, (\lambda, \mu) \not\leq (\alpha, \beta)$ and $\alpha - \lambda < b\}$.

As a consequence of Propositions 13 and 15 and Theorem 6, we have the following result.

Corollary 7. *If* $(\lambda, \mu) \in \mathbb{N} \times \{0, 1, \cdots, a - 1\}$*,* $(\lambda, \mu) \neq (0, 0)$ *and* $T = (\{\lambda a + \mu b\} + \langle a, b\rangle) \cup \{0\}$*, then*

1) $\mathrm{F}(T) = ab + (\lambda - 1)a + (\mu - 1)b$.

2) $\mathrm{g}(T) = \dfrac{(a - 1)(b - 1)}{2} + \lambda a + \mu b - 1$.

3) $\mathrm{m}(T) = \lambda a + \mu b$.

4) $\mathrm{e}(T) = \lambda a + \mu b$.

5) $\mathrm{msg}(T) = \{(1+\alpha)a + (\mu+\beta)b \mid (\alpha,\beta) \in \{0,1,\cdots,\lambda+b-1\} \times \{0,1,\cdots,\mu-1\} \cup \{0,1,\cdots,\lambda-1\} \times \{\mu,\mu+1,\cdots,a-1\}\}$.

Example 6. *Let* $T = (\{17\} + \langle 5,7\rangle) \cup \{0\}$. *As* $17 = 2\cdot 5 + 1\cdot 7$, *then by applying Corollary 7 for* $a = 5$, $b = 7$, $\lambda = 2$ *and* $\mu = 1$, *we obtain that* $\mathrm{F}(T) = 40$, $\mathrm{g}(T) = 28$, $\mathrm{m}(T) = 17$, $\mathrm{e}(T) = 17$ *and* $\mathrm{msg}(T) = \{(2+\alpha)5 + (1+\beta)7 \mid (\alpha,\beta) \in \{0,1,2,3,4,5,6,7,8\}\times\{0\}\cup\{0,1\}\times\{1,2,3,4\}\} = \{17,22,27,32,37,42,47,52,57,24,31,38,45,29,36,43,50\}$.

4. I($\langle a,b\rangle$)-semigroups in general

The aim of this section is to study the I($\langle a,b\rangle$)-semigroups which have the form $(\{n_1,\cdots,n_p\} + \langle a,b\rangle) \cup \{0\}$ being $\{n_1,\cdots,n_p\}$ a $\langle a,b\rangle$-incomparable set and $p \in \mathbb{N}\backslash\{0,1\}$. We are specially interested in calculating the genus, the Frobenius number and the embedding dimension of this kind of numerical semigroups.

In [12], Wilf conjectures that if S is a numerical semigroup, then $\mathrm{g}(S) \leq \dfrac{\mathrm{e}(S)-1}{\mathrm{e}(S)}$ $(\mathrm{F}(S)+1)$. Nowadays, this question remains open and its resolution is one of the most important issues in the Theory of Numerical Semigroups. The conjecture has been tested for a large number of numerical semigroups families. A good work to know about the current status of the problem is [3]. One of the families for which Wilf's conjecture has been proved to be true is the family of MED-semigroups. Hence, the principal I($\langle a,b\rangle$)-semigroups, verify the Wilf's conjeture.

One of the current work aims has been to try to prove that the Wilf's conjecture is true for I($\langle a,b\rangle$)-semigroups with embedding dimension two, three, $\cdots$. We have not fulfilled this; however, we will present the results obtained around the three invariants involved in the Wilf's conjecture.

Proposition 18. *The following statements are equivalent.*

1) *X is a $\langle a,b\rangle$-incomparable set with cardinality p.*

2) $X = \{\lambda_1 a + \mu_1 b, \lambda_2 a + \mu_2 b, \cdots, \lambda_p a + \mu_p b\}$ *where* $\{\lambda_1,\mu_1,\cdots,\lambda_p,\mu_p\} \subseteq \mathbb{N}$, $\mu_1 < \mu_2 < \cdots < \mu_p < a$ *and* $\lambda_p < \cdots < \lambda_2 < \lambda_1 < \lambda_p + b$.

Proof. 1) implies 2). It is a consequence from Corollary 4 and Lemma 5.

2) implies 1). By applying Lemma 5, we deduce that X is a $\langle a,b\rangle$-incomparable set with cardinality p.

Throughout this section, $(\lambda_1,\mu_1),(\lambda_2,\mu_2),\cdots,(\lambda_p,\mu_p)$ will denote elements from $\mathbb{N}^2$ such that $\mu_1 < \mu_2 < \cdots < \mu_p < a$ and $\lambda_p < \cdots < \lambda_2 < \lambda_1 < \lambda_p + b$. Note that

$T = (\{\lambda_1 a + \mu_1 b, \lambda_2 a + \mu_2 b, \cdots, \lambda_p a + \mu_p b\} + \langle a, b\rangle) \cup \{0\}$ is an $\mathrm{I}(\langle a, b\rangle)$-semigroup and $\dim_{\langle a,b\rangle}(T) = p$. Moreover, as a consequence of Proposition 18, we also have that if T is an $\mathrm{I}(\langle a, b\rangle)$-semigroup and $\dim_{\langle a,b\rangle}(T) = p$, then T has this form.

The following result has an immediate proof.

Lemma 8. *With the above notation, we have that*

$$T = (\langle a, b\rangle \setminus (\mathrm{Ap}(\langle a, b\rangle, \lambda_1 a + \mu_1 b) \cap \cdots \cap \mathrm{Ap}(\langle a, b\rangle, \lambda_p a + \mu_p b))) \cup \{0\}.$$

By applying Theorem 6, we obtain the following result.

Lemma 9. *With the above notation, we have that*
$\mathrm{Ap}(\langle a, b\rangle, \lambda_1 a+\mu_1 b)\cap\cdots\cap\mathrm{Ap}(\langle a, b\rangle, \lambda_p a+\mu_p b) = \{\alpha a+\beta b \mid (\alpha, \beta) \in \{0, 1, \cdots, \lambda_p+ b-1\} \times \{0, 1, \cdots, \mu_1 - 1\} \cup \{0, 1, \cdots, \lambda_1 - 1\} \times \{\mu_1, \cdots, \mu_2 - 1\} \cup \{0, 1, \cdots, \lambda_2 - 1\} \times \{\mu_2, \cdots, \mu_3 - 1\}\} \cup \cdots \cup \{0, 1, \cdots, \lambda_p - 1\} \times \{\mu_p, \cdots, a - 1\}$.

As a consequence of Corollary 4 and Lemma 9, we obtain the following result.

Lemma 10. *The cardinality of* $\mathrm{Ap}(\langle a, b\rangle, \lambda_1 a + \mu_1 b) \cap \cdots \cap \mathrm{Ap}(\langle a, b\rangle, \lambda_p a + \mu_p b)$ *is* $(\lambda_p + b)\mu_1 + \lambda_1(\mu_2 - \mu_1) + \cdots + \lambda_{p-1}(\mu_p - \mu_{p-1}) + \lambda_p(a - \mu_p)$.

By using now Lemmas 8, 9 and 10, we obtain the following result.

Theorem 11. *With the previous notation, the following condition holds:*

1) $\mathrm{m}(T) = \min\{\lambda_1 a + \mu_1 b, \cdots, \lambda_p a + \mu_p b\}$.

2) $\mathrm{g}(T) = \dfrac{(a-1)(b-1)}{2} + (\lambda_p + b)\mu_1 + \lambda_1(\mu_2 - \mu_1) + \cdots + \lambda_{p-1}(\mu_p - \mu_{p-1}) + \lambda_p(a - \mu_p) - 1$.

3) $\mathrm{F}(T) = \max\{(\lambda_p + b - 1)a + (\mu_1 - 1)b, (\lambda_1 - 1)a + (\mu_2 - 1)b, \cdots, (\lambda_{p-1} - 1)a + (\mu_p - 1)b, (\lambda_p - 1)a + (a - 1)b\}$.

Example 7. *Let* $a = 5$, $b = 7$, $0 = \mu_1 < \mu_2 = 1 < \mu_3 = 2$ *and* $1 = \lambda_3 < \lambda_2 = 2 < \lambda_1 = 3$. *Then* $T = (\{15, 17, 19\} + \langle 5, 7\rangle) \cup \{0\}$. *By applying Theorem 11, we have* $\mathrm{m}(T) = 15$, $\mathrm{g}(T) = \frac{4\cdot 6}{2} + (1 + 7)0 + 3(1 - 0) + 2(2 - 1) + 1(5 - 2) - 1 = 19$ *and* $\mathrm{F}(T) = \max\{7 \cdot 5 - 7, (3 - 1)5 + 0 \cdot 7, (2 - 1)5 + (2 - 1)7, (1 - 1)5 + 4 \cdot 7\} = \max\{28, 10, 12, 28\} = 28$.

Our next aim in this section will be to present an algorithm to compute $\mathrm{msg}(T)$.

Lemma 12. *Let* Δ *be a numerical semigroup,* $\{x_1, \cdots, x_p\} \subseteq \Delta$ *and* $S = (\{x_1, \cdots, x_p\} +\Delta) \cup \{0\}$. *Then* $(\mathrm{Ap}(\Delta, x_1) + \{x_1\}) \cup \cdots \cup (\mathrm{Ap}(\Delta, x_p) + \{x_p\})$ *is a system of generators of numerical semigroup* S.

Proof. For all $i \in \{1, \cdots, p\}$, we consider $S_i = (\{x_i\} + \Delta) \cup \{0\}$. Then it is clear that $S = \bigcup_{i=1}^{p} S_i$. The proof concludes noting that, by Proposition 15, we know that $\mathrm{Ap}(\Delta, x_i) + \{x_i\} = \mathrm{msg}(S_i)$ for all $i \in \{1, \cdots, p\}$.

Lemma 13. *Let Δ be a numerical semigroup, $\{x_1, \cdots, x_p\}$ a Δ-incomparable set, $S = (\{x_1, \cdots, x_p\} + \Delta) \cup \{0\}$ and $w \in \mathrm{Ap}(\Delta, x_1)$. Then $w + x_1 \in \mathrm{msg}(S)$ if and only if $w \in \mathrm{Ap}(\Delta, x_1) \cap \cdots \cap \mathrm{Ap}(\Delta, x_p)$ and $w + x_1 \in \bigcap_{\{i,j\} \subseteq \{1, \cdots, p\}} \mathrm{Ap}(\Delta, x_i + x_j)$.*

Proof. Necessity. If $w \notin \mathrm{Ap}(\Delta, x_i)$ for some $i \in \{1, \cdots, p\}$, then $w = n + x_i$ with $n \in \Delta$. Therefore, $w + x_1 = x_1 + (x_i + n)$ where $\{x_1, x_i + n\} \subseteq S\backslash\{0\}$ and so $w + x_1 \notin \mathrm{msg}(S)$.

If $w + x_1 \notin \mathrm{Ap}(\Delta, x_i + x_j)$ for some $\{i, j\} \subseteq \{1, \cdots, p\}$, then $w + x_1 = x_i + x_j + n$ with $n \in \Delta$. Therefore, $w + x_1 = x_i + x_j + n$ with $\{x_1, x_i + n\} \subseteq S\backslash\{0\}$ and so $w + x_1 \notin \mathrm{msg}(S)$.

Sufficieny. If $w + x_1 \notin \mathrm{msg}(S)$, then there is $\{s_1, s_2\} \subseteq S\backslash\{0\}$ such that $w + x_1 = s_1 + s_2$. As $\{s_1, s_2\} \subseteq S\backslash\{0\}$, then there is $\{i, j\} \subseteq \{1, \cdots, p\}$ and $\{n_1, n_2\} \subseteq \Delta$ such that $s_1 = x_i + n_1$ and $s_2 = x_j + n_2$. Thus $w + x_1 = x_i + x_j + n_1 + n_2$ and so $w + x_1 \notin \mathrm{Ap}(\Delta, x_i + x_j)$.

As an immediate consequence of Lemmas 12 and 13, we have the following result.

Proposition 19. *Let Δ be a numerical semigroup, $\{x_1, \cdots, x_p\}$ a Δ-incomparable set and $S = (\{x_1, \cdots, x_p\} + \Delta) \cup \{0\}$. Then $\mathrm{msg}(S) \subseteq [\mathrm{Ap}(\Delta, x_1) \cap \cdots \cap \mathrm{Ap}(\Delta, x_p)] + \{x_1, \cdots, x_p\}$. Moreover, if $w \in \mathrm{Ap}(\Delta, x_1) \cap \cdots \cap \mathrm{Ap}(\Delta, x_p)$, then $w + x_1 \in \mathrm{msg}(S)$ if and only if $w + x_1 \in \bigcap_{\{i,j\} \subseteq \{1, \cdots, p\}} \mathrm{Ap}(\Delta, x_i + x_j)$.*

It is easy to prove the following result and it appears in [5, Lemma 23].

Lemma 14. *Let Δ be a numerical semigroup, $n \in \Delta\backslash\{0\}$ and $k \in \mathbb{N}\backslash\{0\}$. Then*

$$\mathrm{Ap}(\Delta, kn) = \mathrm{Ap}(\Delta, n) + \{in \mid i \in \{0, 1, \cdots, k-1\}\}.$$

We are already able to provide the previously announced algorithm.

Algorithm 3

INPUT: A Δ-incomparable set, $\{x_1 < \cdots < x_p\}$.
OUTPUT: $\mathrm{msg}(T)$ where $T = (\{x_1, \cdots, x_p\} + \langle a, b\rangle) \cup \{0\}$.

1) Compute $\mathrm{Ap}(\langle a,b\rangle, x_1) \cup \{0\}$, by using Theorem 6.

2) Compute $B = \mathrm{Ap}(\langle a,b\rangle, x_1) \cap \cdots \cap \mathrm{Ap}(\langle a,b\rangle, x_p) = \{w \in \mathrm{Ap}(\langle a,b\rangle, x_1) \mid \{w - x_2, \cdots, x - x_p\} \cap \mathrm{Ap}(\langle a,b\rangle, x_1) = \emptyset\}$.

3) Compute $\mathrm{Ap}(\langle a,b\rangle, 2x_1) = \mathrm{Ap}(\langle a,b\rangle, x_1) + \{0, x_1\}$, by using Lemma 14

4) Compute $C = \{x_1, \cdots, x_p\} + \{x_1, \cdots, x_p\}$.

5) Compute $D = \bigcap_{\{i,j\} \subseteq \{1,\cdots,p\}} \mathrm{Ap}(\langle a,b\rangle, x_i + x_j) = \{w \in \mathrm{Ap}(\langle a,b\rangle, 2x_1) \mid \{w - c \mid c \in C\} \cap \mathrm{Ap}(\langle a,b\rangle, 2x_1) = \emptyset\}$.

6) Return $(B + \{x_1, \cdots, x_p\}) \cap D$.

Example 8. *We are going to compute the minimal system of generators of numerical semigroup $T = (\{10, 12, 14\} + \langle 5, 7\rangle) \cup \{0\}$. For this aim we using the Algorithm 3, taking $a = 5$, $b = 7$, $x_1 = 10$, $x_2 = 12$ and $x_3 = 14$.*

1) $\mathrm{Ap}(\langle 5,7\rangle, 10) = \{0, 5, 7, 12, 14, 19, 21, 26, 28, 33\}$.

2) $B = \{w \in \mathrm{Ap}(\langle 5,7\rangle, 10) \mid \{w - 12, w - 14\} \cap \mathrm{Ap}(\langle 5,7\rangle, 10) = \emptyset\} = \{0, 5, 7\}$.

3) $\mathrm{Ap}(\langle 5,7\rangle, 20) = \{0, 5, 7, 10, 12, 14, 15, 17, 19, 21, 22, 24, 26, 28, 29, 31, 33, 36, 38, 43\}$.

4) $C = \{20, 22, 24, 26, 28\}$.

5) $D = \{w \in \mathrm{Ap}(\langle 5,7\rangle, 20) \mid \{w - 22, w - 24, w - 26, w - 28\} \cap \mathrm{Ap}(\langle 5,7\rangle, 20) = \emptyset\} = \{0, 5, 7, 10, 12, 14, 15, 17, 19, 21\}$.

6) $\mathrm{msg}(T) = (B + \{10, 12, 14\}) \cap D = \{10, 12, 14, 15, 17, 19, 21\}$.

5. $\mathrm{I}(\langle a,b\rangle)$-semigroups with ideal dimension two

In all this section (λ_1, μ_1) and (λ_2, μ_2) will be denote two elements of $\mathbb{N}^2$ such that $\mu_1 < \mu_2 < a$ and $\lambda_2 < \lambda_1 < \lambda_2 + b$. Note that $T = (\{\lambda_1 a + \mu_1 b, \lambda_2 a + \mu_2 b\} + \langle a, b\rangle) \cup \{0\}$ is an $\mathrm{I}(\langle a,b\rangle)$-semigroup and $\dim_{\langle a,b\rangle}(T) = 2$. Moreover, by Proposition

18, we know that every I($\langle a,b\rangle$)-semigroup with ideal dimension two has this form. As a consequence of Theorem 11, we have the following result.

Theorem 15. *With the previous notation, the following condition holds:*

1) $\mathrm{m}(T) = \min\{\lambda_1 a + \mu_1 b, \lambda_2 a + \mu_2 b\}$.

2) $\mathrm{g}(T) = \dfrac{(a-1)(b-1)}{2} + (\lambda_2 + b)\mu_1 + \lambda_1(\mu_2 - \mu_1) + \lambda_2(a - \mu_2) - 1$.

3) $\mathrm{F}(T) = \begin{cases} (\lambda_2 + b - 1)a + (\mu_1 - 1)b & \textit{if } ab \geq (\lambda_1 - \lambda_2)a + (\mu_2 - \mu_1)b \\ \textit{or} & \\ (\lambda_1 - 1)a + (\mu_2 - 1)b & \textit{otherwise.} \end{cases}$

Our main aim in this section will be to present some algorithms which allow us to compute all the I($\langle a,b\rangle$)-semigroups with ideal dimension two, with a fixed genus or a fixed Frobenius number.

The following result has an immediante proof.

Lemma 16. *Let S be a numerical semigroup, $X \subseteq S$ and $s \in S$. Then X is a S-incomparable set if and only if $X + \{s\}$ is a S-incomparable set.*

The following result has also an immediate proof.

Lemma 17. *Let S be a numerical semigroup, X a S-incomparable set and $s \in S$ such that $x - s \in S$ for all $x \in X$. Then $X + \{-s\}$ is also a S-incomparable set.*

We say that a subset X of a numerical semigroup S is S-*basic* if it is a S-incomparable set and for every $s \in S \backslash \{0\}$, there is $x \in X$ such that $x - s \notin S$.

As a consequence of Lemmas 16 and 17, we have the following result.

Proposition 20. *If S is a numerical semigroup, X is a S-basic set and $s \in S$, then $X + \{s\}$ is a S-incomparable set. Moreover, every S-incomparable set has this form.*

Lemma 18. *Let X be a $\langle a,b\rangle$-incomparable set. Then X is a $\langle a,b\rangle$-basic set if and only if there is $\{x,y\} \subseteq X$ such that $x - a \notin \langle a,b\rangle$ and $y - b \notin \langle a,b\rangle$.*

Proof. Necessity. If $x - a \in \langle a,b\rangle$ for all $x \in X$, then X is not a $\langle a,b\rangle$-basic set.

Sufficiency. If X is not a $\langle a,b\rangle$-basic set, then there is $s \in \langle a,b\rangle\backslash\{0\}$ such that $x - s \in \langle a,b\rangle$ for all $x \in X$. As $s \in \langle a,b\rangle\backslash\{0\}$, then $s - a \in \langle a,b\rangle$ or $s - b \in \langle a,b\rangle$. If $s - a \in \langle a,b\rangle$ then $x - a \in \langle a,b\rangle$ for all $x \in X$. Reasoning in the same way for $x - b$, we obtain the result.

As a consequence of Lemma 18, we have the following result.

Lemma 19. *Let X be a $\langle a,b\rangle$-incomparable set. Then X is a $\langle a,b\rangle$-basic set with cardinality two if and only if $X=\{\lambda a,\mu b\}$ for some $(\lambda,\mu)\in\{1,\cdots,b-1\}\times\{1,\cdots,a-1\}$.*

As a consequence of Proposition 20 and Lemma 19, we have the following result.

Lemma 20. *X is a $\langle a,b\rangle$-incomparable set with cardinality two if and only if $X=\{\lambda a+n,\mu b+n\}$ for some $(\lambda,\mu,n)\in\{1,\cdots,b-1\}\times\{1,\cdots,a-1\}\times\langle a,b\rangle$.*

As a consequence of Lemma 20, we have the following result.

Theorem 21. *Let $(\lambda,\mu,n)\in\{1,\cdots,b-1\}\times\{1,\cdots,a-1\}\times\langle a,b\rangle$. Then $T(\lambda,\mu,n)=(\{\lambda a+n,\mu b+n\}+\langle a,b\rangle)\cup\{0\}$ is an $\mathrm{I}(\langle a,b\rangle)$-semigroup and $\dim_{\langle a,b\rangle}(T(\lambda,\mu,n))=2$. Moreover, every $\mathrm{I}(\langle a,b\rangle)$-semigroup with ideal dimension two has this form.*

The following result has an esasy proof.

Lemma 22. *Let Δ be a numerical semigroup and $n\in\mathbb{N}$ such that $\{n\}+(\Delta\backslash\{0\})\subseteq\Delta$. Then $S=(\{n\}+(\Delta\backslash\{0\}))\cup\{0\}$ is a numerical semigroup. Moreover, $\mathrm{m}(S)=\mathrm{m}(\Delta)+n$, $\mathrm{F}(S)=\mathrm{F}(\Delta)+n$ and $\mathrm{g}(S)=\mathrm{g}(\Delta)+n$.*

We are going to illustrate the previous lemma with an example.

Example 9. *Let $\Delta=\langle 5,7,9\rangle=\{0,5,7,9,10,12,14,\longrightarrow\}$(the symbol $\longrightarrow$ means that every integer greater than 14 belongs to the set). Then $\mathrm{m}(\Delta)=5$, $\mathrm{F}(\Delta)=13$ and $\mathrm{g}(\Delta)=8$. If we take $n=5$ in the previous lemma, then $S=(\{5\}+(\Delta\backslash\{0\}))\cup\{0\}=\{0,10,12,14,15,17,19,\longrightarrow\}=\langle 10,12,14,15,17,19,21,23\rangle$. Note that $10=\mathrm{m}(S)=\mathrm{m}(\Delta)+5$, $18=\mathrm{F}(S)=\mathrm{F}(\Delta)+5$ and $13=\mathrm{g}(S)=\mathrm{g}(\Delta)+5$.*

Lemma 23. *Let Δ be a numerical semigroup, $X\subseteq\Delta\backslash\{0\}$, $S=(X+\Delta)\cup\{0\}$, $n\in\Delta$ and $S(n)=(\{n\}+(S\backslash\{0\}))\cup\{0\}$. Then $S(n)$ is a numerical semigroup, $\mathrm{m}(S(n))=\mathrm{m}(S)+n$, $\mathrm{F}(S(n))=\mathrm{F}(S)+n$ and $\mathrm{g}(S(n))=\mathrm{g}(S)+n$.*

Proof. Note that S is a numerical semigroup and $\{n\}+(S\backslash\{0\})\subseteq S$. The proof finishes by applying Lemma 22.

Lemma 24. *Let Δ be a numerical semigroup, $X\subseteq\Delta\backslash\{0\}$, $S=(X+\Delta)\cup\{0\}$ and $n\in\Delta$. Then $S(n)=(\{n\}+X+\Delta)\cup\{0\}$ is a numerical semigroup, $\mathrm{m}(S(n))=\mathrm{m}(S)+n$, $\mathrm{F}(S(n))=\mathrm{F}(S)+n$ and $\mathrm{g}(S(n))=\mathrm{g}(S)+n$.*

Proof. It is enough to observe that $S(n) = (\{n\} + (S\backslash\{0\})) \cup \{0\}$ and to apply Lemma 23.

As a consequence of Theorem 15 and Lemma 24, we obtain the following result.

Proposition 21. *If* $(\lambda, \mu, n) \in \{1, \cdots, b-1\} \times \{1, \cdots, a-1\} \times \langle a, b\rangle$, *then*

1) $\mathrm{m}(T(\lambda, \mu, n)) = \min\{\lambda a + n, \mu b + n\}$.

2) $\mathrm{g}(T(\lambda, \mu, n)) = \dfrac{(a-1)(b-1)}{2} + \lambda\mu - 1 + n.$

3) $\mathrm{F}(T(\lambda, \mu, n)) = \max\{ab - a - b, (\lambda - 1)a + (\mu - 1)b\} + n.$

Example 10. *We are going to compute all the* $\mathrm{I}(\langle 5, 7\rangle)$*-semigroups with ideal dimension two and genus* 34*. By applying Theorem 21 and Proposition 21, this is equivalent to calculate all the triples* $(\lambda, \mu, n) \in \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4\} \times \langle 5, 7\rangle$*, verifying* $\lambda \cdot \mu + n = 23$*. If we take* $\lambda = 2$*,* $\mu = 3$ *and* $n = 17$*, then* $T(\lambda, \mu, n) = T(2, 3, 17) = (\{27, 38\} + \langle 5, 7\rangle) \cup \{0\} = \{0, 27, 32, 34, 37, 38, 39, 41, \longrightarrow\}$ *is an* $\mathrm{I}(\langle 5, 7\rangle)$*-semigroup with ideal dimension two and genus* 34.

Algorithm 4

INPUT: An integer g such that $g > \dfrac{(a-1)(b-1)}{2}$.
OUTPUT: $\left\{S \mid S \text{ is an } \mathrm{I}(\langle a, b\rangle)\text{-semigroup},\ \dim_{\langle a,b\rangle}(S) = 2 \text{ and } \mathrm{g}(S) = g\right\}$.

1) Compute $A = \{(\lambda, \mu, n) \in \{1, \cdots, b-1\} \times \{1, \cdots, a-1\} \times \langle a, b\rangle \mid \lambda \cdot \mu + n = g + 1 - \frac{(a-1)(b-1)}{2}\}$.

2) Return $\{T(\lambda, \mu, n) \mid (\lambda, \mu, n) \in A\}$.

Example 11. *We suppose that we want to built all the* $\mathrm{I}(\langle 5, 7\rangle)$*-semigroups with ideal dimension two and Frobenius number* 43*. By applying Theorem 21 and Proposition 21, this is equivalent to calculate all the triples* $(\lambda, \mu, n) \in \{1, \cdots, b-1\} \times \{1, \cdots, a-1\} \times \langle a, b\rangle$*, verifying one of the following conditions:*

1) $(\lambda - 1)5 + (\mu - 1)7 < 23$ *and* $n = 20$.

2) $(\lambda - 1)5 + (\mu - 1)7 > 23$ *and* $(\lambda - 1)5 + (\mu - 1)7 + n = 43$.

If we consider $(\lambda, \mu, n) = (2, 2, 20)$, *then* $T(\lambda, \mu, n) = T(2, 2, 20) = (\{30, 34\} + \langle 5, 7\rangle) \cup \{0\} = \{0, 30, 34, 35, 37, 39, 40, 41, 42, 44 \longrightarrow\}$ *is an* $\mathrm{I}(\langle 5, 7\rangle)$*-semigroup with ideal dimension two and Frobenius number* 43.

If we consider $(\lambda, \mu, n) = (4, 3, 14)$, *then* $T(\lambda, \mu, n) = T(4, 3, 14) = (\{34, 35\} + \langle 5, 7\rangle) \cup \{0\} = \{0, 34, 35, 39, 40, 41, 42, 44, \longrightarrow\}$ *is an* $\mathrm{I}(\langle 5, 7\rangle)$*-semigroup with ideal dimension two and Frobenius number* 43.

Algorithm 5

INPUT: An integer F such that $F \geq ab - a - b$.
OUTPUT: $\{S \mid S \text{ is an } \mathrm{I}(\langle a, b\rangle) - \text{semigroup}, \dim_{\langle a,b\rangle}(S) = 2 \text{ and } \mathrm{F}(S) = F\}$.

1) If $F - (ab - a - b) \notin \langle a, b\rangle$, then $A = \emptyset$.

2) If $F - (ab - a - b) \in \langle a, b\rangle$, then $A = \{(\lambda, \mu, F - (ab - a - b)) \mid (\lambda, \mu) \in \{1, \cdots, b-1\} \times \{1, \cdots, a-1\}$ and $(\lambda - 1)a + (\mu - 1)b < ab - a - b\}$.

3) $B = \{(\lambda, \mu, n) \in \{1, \cdots, b-1\} \times \{1, \cdots, a-1\} \times \langle a, b\rangle \mid (\lambda - 1)a + (\mu - 1)b > ab - a - b$ and $n = F - (\lambda - 1)a - (\mu - 1)b\}$.

4) Return $\{T(\lambda, \mu, n) \mid (\lambda, \mu, n) \in A \cup B\}$.

We will finish this work proposing the following problems. To find formulas that from a, b, F and g, compute the cardinality of the following sets:

1) $\{S \mid S \text{ is an } \mathrm{I}(\langle a, b\rangle)\text{-semigroup}, \dim_{\langle a,b\rangle}(S) = 2 \text{ and } \mathrm{g}(S) = g\}$.

2) $\{S \mid S \text{ is an } \mathrm{I}(\langle a, b\rangle)\text{-semigroup}, \dim_{\langle a,b\rangle}(S) = 2 \text{ and } \mathrm{F}(S) = F\}$.

3) $\{S \mid S \text{ is an } \mathrm{I}(\langle a, b\rangle)\text{-semigroup and } \mathrm{g}(S) = g\}$.

4) $\{S \mid S \text{ is an } \mathrm{I}(\langle a, b\rangle)\text{-semigroup and } \mathrm{F}(S) = F\}$.

In [4], the problem 3 is solved when $(a, b) = (2, 2k + 1)$, $k \in \mathbb{N}\backslash\{0\}$.

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