ABOUT APPROXIMATION OF B-CONTINUOUS FUNCTIONS OF THREE VARIABLES BY GBS OPERATORS OF BERNSTEIN TYPE ON A TETRAHEDRON

Mircea D. Farcaş

Abstract. In this article the sequence of GBS operators of Bernstein type on a tetrahedron for B-continuous functions of three variables is constructed and some approximation properties of this sequence are established.

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1. Preliminaries

In this section, we recall some results which we will use in this article. In the following, let X, Y and Z be compact real intervals and $D = X \times Y \times Z$. A function $f : D \to \mathbb{R}$ is called a B-continuous function at $(x_0, y_0, z_0) \in D$ iff for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|\Delta f[(x, y, z), (x_0, y_0, z_0)]| < \epsilon$$

for any $(x, y, z) \in D$, with $|x - x_0| < \delta$, $|y - y_0| < \delta$ and $|z - z_0| < \delta$. Here

$$\Delta f[(x, y, z), (x_0, y_0, z_0)] = f(x, y, z) - f(x, y, z_0) - f(x, y_0, z) - f(x_0, y, z) + \\ + f(x, y_0, z_0) + f(x_0, y, z_0) + f(x_0, y_0, z) - f(x_0, y_0, z_0)$$

denote a so-called mixed difference of f. The function f is B-continuous on D iff f is B-continuous at each point $(x_0, y_0, z_0) \in D$. These notions were introduced by K. Bögel in [4], [5] and [6]. The function $f : D \to \mathbb{R}$ is B-bounded on D iff there exists $K > 0$ such that

$$|\Delta f[(x, y, z), (x_0, y_0, z_0)]| \leq K$$

for any $(x, y, z), (x_0, y_0, z_0) \in D$.

We shall use the function sets

$$B(D)=\{f/f:D\to\mathbb{R},\text{f bounded on }D\},\text{with the norm }\|\bullet\|_\infty$$
$$B_b(D)=\{f/f:D\to\mathbb{R},f\text{ B}-\text{bounded on }D\},$$
$$\text{with the norm}\|f\|_B=\sup_{(x,y,z),(u,v,w)\in D}|\Delta f[(x,y,z),(u,v,w)]|$$
$$C_b(D)=\{f/f:D\to\mathbb{R},f\text{ B}-\text{continuous on }D\}.$$

Let $f\in B_b(D)$. The function

$$\omega_{mixed}(f;\cdot,\cdot,\cdot):[0,\infty)\times[0,\infty)\times[0,\infty)\to\mathbb{R}$$

defined by

$$\omega_{mixed}(f;\delta_1,\delta_2,\delta_3)=\sup\{|\Delta f[(x,y,z),(x_0,y_0,z_0)]|:|x-x_0|\le\delta_1,$$
$$|y-y_0|\le\delta_2,|z-z_0|\le\delta_3\}$$

for any $\delta_1,\delta_2,\delta_3\in[0,\infty)$ is called the mixed modulus of smoothness and was introduced by I. Badea in [1] for functions of two variables. Important properties of ω_{mixed} were established by C. Badea, I. Badea, C. Cottin and H.H. Gonska in the papers [2] and [3].

Lemma 1 *Let $f\in B_b(D)$. Then*

$$\omega_{mixed}(f;\delta_1,\delta_2,\delta_3)\le\omega_{mixed}(f;\delta_1',\delta_2',\delta_3') \tag{1.1}$$

for any $\delta_1,\delta_2,\delta_3,\delta_1',\delta_2',\delta_3'\in[0,\infty)$ such that $\delta_1\le\delta_1'$, $\delta_2\le\delta_2'$ and $\delta_3\le\delta_3'$;

$$\Delta f[(x,y,z),(u,v,w)]\le\omega_{mixed}(f;|x-u|,|y-v|,|z-w|); \tag{1.2}$$

$$\Delta f[(x,y,z),(u,v,w)]\le\left(1+\frac{|x-u|}{\delta_1}\right)\left(1+\frac{|y-v|}{\delta_2}\right)\cdot$$

$$\cdot\left(1+\frac{|z-w|}{\delta_3}\right)\omega_{mixed}(f;\delta_1,\delta_2,\delta_3) \tag{1.3}$$

for $\delta_1,\delta_2,\delta_3>0$;

$$\omega_{mixed}(f;\lambda_1\delta_1,\lambda_2\delta_2,\lambda_3\delta_3)\le(1+\lambda_1)(1+\lambda_2)(1+\lambda_3)\cdot$$
$$\cdot\omega_{mixed}(f;\delta_1,\delta_2,\delta_3),\lambda_1,\lambda_2,\lambda_3>0. \tag{1.4}$$

Lemma 2 *Let $f \in C_b(D)$. Then*

$$\lim_{\delta_1,\delta_2,\delta_3\to 0} \omega_{mixed}(f;\delta_1,\delta_2,\delta_3) = 0. \tag{1.5}$$

Let L be a positive operator of three variables, applying the space $\mathbb{R}^{[a,b]\times[a',b']\times[a'',b'']}$ into itself. The operator

$$UL : \mathbb{R}^{[a,b]\times[a',b']\times[a'',b'']} \to \mathbb{R}^{[a,b]\times[a',b']\times[a'',b'']}$$

defined by

$$(ULf)(x,y,z) = L[f(\bullet,y,z) + f(x,*,z) + f(x,y,\circ) -$$

$$-f(\bullet,*,z) - f(\bullet,y,\circ) - f(x,*,\bullet) + f(\bullet,*,\circ);x,y,z] \tag{1.6}$$

is called GBS (generalized boolean sum) operator associated to L, where $\bullet, *, \circ$ stand for the first, the second and the third variable. The term of GBS operator was introduced by C. Badea and C. Cottin in the paper [2]. For estimating the rate of the convergence we need the following result.

Lemma 3 *For any $f \in C_b(D)$ and any $\delta_1,\delta_2,\delta_3 > 0$ holds the inequality*

$$\begin{aligned}
&|(ULf)(x,y,z) - f(x,y,z)| \le \\
&\le \left(1 + \frac{\sqrt{(L(\bullet - x)^2)(x,y,z)}}{\delta_1} + \frac{\sqrt{(L(* - y)^2)(x,y,z)}}{\delta_2} + \right.\\
&+ \frac{\sqrt{(L(\circ - z)^2)(x,y,z)}}{\delta_3} + \frac{\sqrt{(L(\bullet - x)^2(* - y)^2)(x,y,z)}}{\delta_1\delta_2} + \\
&+ \frac{\sqrt{(L(* - y)^2(\circ - z)^2)(x,y,z)}}{\delta_2\delta_3} + \frac{\sqrt{(L(\circ - z)^2(\bullet - x)^2)(x,y,z)}}{\delta_3\delta_1} + \\
&\left. + \frac{\sqrt{(L(\bullet - x)^2(* - y)^2)(\circ - z)^2(x,y,z)}}{\delta_1\delta_2\delta_3}\right) \cdot \omega_{mixed}(f;\delta_1,\delta_2,\delta_3)
\end{aligned} \tag{1.7}$$

where $L : B(D) \to B(D)$ is a positive linear operator which reproduces the constants.

Proof. After (1.3), for $(x,y,z),(u,v,w)\in D$ we have

$$\Delta f[(x,y,z),(u,v,w)]\le\left(1+\frac{|x-u|}{\delta_1}\right)\cdot\left(1+\frac{|y-v|}{\delta_2}\right)\cdot$$
$$\cdot\left(1+\frac{|z-w|}{\delta_3}\right)\omega_{mixed}(f;\delta_1,\delta_2,\delta_3),$$

so that we can write

$$|f(x,y,z)-(ULf)(x,y,z)|\le|(L\Delta f[(x,y,z)(\bullet,*,\circ)])(x,y,z)|\le$$
$$\le\left(1+\frac{(L|\bullet-x|)(x,y,z)}{\delta_1}+\frac{(L|*-y|)(x,y,z)}{\delta_2}+\frac{(L|\circ-z|)(x,y,z)}{\delta_3}+\right.$$
$$+\frac{(L|\bullet-x||*-y|)(x,y,z)}{\delta_1\delta_2}+\frac{(L|*-y||\circ-z|)(x,y,z)}{\delta_2\delta_3}+$$
$$\left.+\frac{(L|\circ-z||\bullet-x|)(x,y,z)}{\delta_3\delta_1}+\frac{(L|\bullet-x||*-y||\circ-z|)(x,y,z)}{\delta_1\delta_2\delta_3}\right)\cdot$$
$$\cdot\,\omega_{mixed}(f,\delta_1,\delta_2,\delta_3).$$

Applying the Cauchy-Schwarz inequality for positive linear operators the estimation from theorem results.

2. Main results

Let the sets $\Delta_3=\{(x,y,z)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}:x,y,z\ge 0,x+y+z\le 1\}$ and $F(\Delta_3)=\{f/f:\Delta_3\to\mathbb{R}\}$. For m a non-negative integer, let the operator $B_m:F(\Delta_3)\to F(\Delta_3)$ defined for any function $f\in F(\Delta_3)$ by

$$(B_mf)(x,y,z)=\sum_{\substack{i,j,k=0\\ i+j+k\le m}}p_{m,i,j,k}(x,y,z)f\left(\frac{i}{m},\frac{j}{m},\frac{k}{m}\right)\tag{2.1}$$

for any $(x,y,z)\in\Delta_3$, where

$$p_{m,i,j,k}(x,y,z)=\frac{m!}{i!j!k!(m-i-j-k)!}x^iy^jz^k\cdot\tag{2.2}$$
$$\cdot(1-x-y-z)^{m-i-j-k}.$$

The operators are named Bernstein polynomials of three variables and they are linear and positive on $F(\Delta_3)$, see [8].
The following result can be found in the paper [7].

Lemma 4 *If m, p_1, p_2, p_3, k are positive given integers then*

$$(B_m e_{p_1p_2p_3})(x,y,z) = \frac{1}{m^{p_1+p_2+p_3}} \sum_{n_1=1}^{p_1} \sum_{n_2=1}^{p_2} \sum_{n_3=1}^{p_3} m^{[n_1+n_2+n_3]}. \tag{2.3}$$

$$\cdot S(p_1,n_1)S(p_2,n_2)S(p_3,n_3)x^{n_1}y^{n_2}z^{n_3},$$

where $e_{p_1p_2p_3}(x,y,z) = x^{p_1}y^{p_2}z^{p_3}$, $(x,y,z) \in \Delta_3$, $m^{[k]} = m(m-1)\dots(m-k+1)$ and $S(m,k)$ denoted the Stirling numbers of second kind.

Lemma 5 *If $(x,y,z) \in \Delta_3$ then*

$$xy \le \frac{1}{4},\ 3x^2y^2 - xy(x+y) + xy \le \frac{3}{16}; \tag{2.4}$$

$$xyz \le \frac{xy+yz+zx}{9} \le \frac{x+y+z}{27} \le \frac{1}{27}; \tag{2.5}$$

$$xy + yz + zx - \frac{9}{4}xyz \le \frac{1}{4}; \tag{2.6}$$

$$17xyz - 5xyz(x+y+z) \le \frac{4}{9} \tag{2.7}$$

$$-2xyz(x+y+z) + \frac{xy+yz+zx}{9} + 2xyz \le \frac{1}{27}; \tag{2.8}$$

$$5x^2y^2z^2 - xyz(xy+yz+zx) + \frac{xyz(x+y+z)}{3} - \frac{xyz}{3} + \frac{4}{729} \ge 0; \tag{2.9}$$

$$\frac{-5m+6}{3}xyz(x+y+z) + \frac{17m-18}{3}xyz - \frac{4m-4}{27} \le 0, m \ge 1. \tag{2.10}$$

Proof. We have $4xy \le (x+y)^2 \le 1$;

$$3x^2y^2 - xy(x+y) + xy = 2\left(x^2y^2 - \frac{1}{16}\right) + x(1-x)y(1-y) - \frac{1}{16} + \frac{3}{16} \le \frac{3}{16}$$

because $4x(1-x) \le (x+1-x)^2 = 1$;

$$xy + yz + zx \le \frac{(x+y+z)^2}{3} \le \frac{x+y+z}{3} \le \frac{1}{3};$$

$$xyz \le \frac{(x+y+z)^3}{27} \le \frac{x+y+z}{27} \le \frac{1}{27};$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq (x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq 9 \text{ so that } xyz \leq \frac{xy+yz+zx}{9};$$

$$xy+yz+zx-\frac{9}{4}xyz = xy\left(1-\frac{9}{4}\right)+(x+y)z;$$

let $z=\min\{x,y,z\}$; then $1-3z \geq 0$ so that $4-9z \geq 0$ and

$$xy\left(1-\frac{9}{4}z\right)+(x+y)z \leq \frac{(x+y)^2(4-9z)}{16}+(1-z)z \leq$$

$$\leq \frac{(1-z)^2(4-9z)}{16}+(1-z)z = \frac{-9z^3+6z^2-z+4}{16} \leq \frac{1}{4};$$

the last inequality is equivalent to $z(1-3z)^2 \geq 0$. For (2.7) we can write

$$17xyz-5xyz(x+y+z) \leq \frac{4}{9} \Leftrightarrow 5xyz\left(\frac{4}{3}-x-y-z\right)+\frac{31}{3}xyz \leq \frac{4}{9},$$

which results from $5xyz(4/3-x-y-z) \leq 1/81$ and $xyz \leq 1/27$. For (2.8) we have

$$2xyz(1-x-y-z) \leq \frac{1}{27}-\frac{xy+yz+zx}{9} \Leftrightarrow$$

$$\Leftrightarrow 18xyz(1-x-y-z) \leq \frac{1}{3}-(xy+yz+zx);$$

but

$$18xyz(1-x-y-z) \leq 2(xy+yz+zx)(1-x-y-z) \leq \frac{1}{3}-(xy+yz+zx) \Leftrightarrow$$

$$\Leftrightarrow (xy+yz+zx)(3-2(x+y+z)) \leq \frac{1}{3};$$

but

$$(xy+yz+zx)(3-2(x+y+z)) \leq \frac{(x+y+z)^2}{3}(3-2(x+y+z))$$

and

$$\frac{(x+y+z)^2}{3}(3-2(x+y+z)) \leq \frac{1}{3} \Leftrightarrow 2(x+y+z)^3-3(x+y+z)^2+1 \geq 0 \Leftrightarrow$$

$\Leftrightarrow (1-x-y-z)^2(2(x+y+z)+1) \geq 0$ and the last inequality holds true. For (2.9) we can write

$$5x^2y^2z^2 - xyz(xy+yz+zx) + \frac{xyz(x+y+z)}{3} - \frac{xyz}{3} + \frac{4}{729} =$$

$$= 4\left(\frac{xy+yz+zx}{9} - xyz\right)\left(\frac{1}{27} - xyz\right) + \frac{4}{9}\left(\frac{9}{4}xyz - (xy+yz+zx) + \frac{1}{4}\right) +$$

$$+\frac{1}{9}xyz\left(\frac{x+y+z}{3} - (xy+yz+zx)\right) +$$

$$+\frac{4}{27}\left(2xyz(x+y+z) - \frac{xy+yz+zx}{9} - 2xyz + \frac{1}{27}\right) \geq 0.$$

The inequality (2.10) can be proven by mathematical induction. For $m=1$ we have $xyz(x+y+z) \leq xyz$, holds true and the step two is in fact (2.7).

For m a non-negative integer, let the GBS operator of Bernstein type (see [1]) $UB_m : C_b(\Delta_3) \to B(\Delta_3)$ defined for any function $f \in C_b(\Delta_3)$ and any $(x,y,z) \in \Delta_3$ by

$$(UB_m f)(x,y,z) = (B_m(f(\bullet,y,z) + f(x,*,z) + f(x,y,\circ) - \tag{2.11}$$

$$- f(\bullet,*,z) - f(\bullet,y,\circ) - f(x,*,\circ) + f(\bullet,*,\circ))(x,y,z) =$$

$$= \sum_{\substack{i,j,k=0\\ i+j+k\leq m}} p_{m,i,j,k}\left(f\left(\frac{i}{m},y,z\right) + f\left(x,\frac{j}{m},z\right) + f\left(x,y,\frac{k}{m}\right) -\right.$$

$$\left. - f\left(\frac{i}{m},\frac{j}{m},z\right) - f\left(\frac{i}{m},y,\frac{k}{m}\right) - f\left(x,\frac{j}{m},\frac{k}{m}\right) + f\left(\frac{i}{m},\frac{j}{m},\frac{k}{m}\right)\right).$$

Lemma 6 *The operators $(B_m)_{m\geq 1}$ verify for any $(x,y,z) \in \Delta_3$ the following*

$$(B_m e_{000})(x,y,z) = 1 \tag{2.12}$$

$$m(B_m(\bullet - x)^2)(x,y,z) = x(1-x) \leq \frac{1}{4}, m \geq 1 \tag{2.13}$$

$$m^3(B_m(\bullet - x)^2(* - y)^2)(x,y,z) = \tag{2.14}$$

$$= 3(m-2)x^2y^2 - (m-2)xy(x+y) + (m-1)xy \leq \frac{3m-2}{16}, m \geq 2$$

$$m^5(B_m(\bullet - x)^2(* - y)^2(\circ - z)^2)(x,y,z) = \tag{2.15}$$
$$= (-15m^2 + 130m - 120)x^2y^2z^2 + (3m^2 - 26m + 24)xyz(xy + yz + zx)+$$
$$+(-m^2 + 7m - 6)xyz(x + y + z) + (m^2 - 3m + 2)xyz \leq$$
$$\leq \frac{12m^2 + 4m - 12}{729}, m \geq 8$$

Proof. We have

$$(B_m e_{000})(x,y,z) = \sum_{\substack{i,j,k \geq 0 \\ i+j+k \leq m}} \frac{m!}{i!j!k!(m-i-j-k)!} x^i y^j z^k \cdot$$
$$\cdot (1 - x - y - z)^{m-i-j-k} = (x + y + z + 1 - x - y - z)^m = 1.$$

From Lemma 2.1 we obtain the relations

$$m(B_m(\bullet - x)^2)(x,y,z) = x(1 - x),$$

$$m^3(B_m(\bullet - x)^2(* - y)^2)(x,y,z) = 3(m-2)x^2y^2 - (m-2)xy(x+y) + (m-1)xy$$

and

$$m^5(B_m(\bullet - x)^2(* - y)^2(\circ - z)^2)(x,y,z) = (-15m^2 + 130m - 120)x^2y^2z^2+$$
$$+(3m^2 - 26m + 24)xyz(xy + yz + zx) + (-m^2 + 7m - 6)xyz(x + y + z)+$$
$$+(m^2 - 3m + 2)xyz.$$

Further, we have $x(1 - x) \leq 1/4$ so that the relation (2.13) results;

$$3(m-2)x^2y^2 - (m-2)xy(x+y) + (m-1)xy - \frac{3m-2}{16} =$$
$$= (m-2)\left(3x^2y^2 - xy(x+y) + xy - \frac{3}{16}\right) + xy - \frac{1}{4} \leq 0$$

with (2.4), from where the relation (2.14) results; for (2.15) we apply (2.9) and (2.10):

$$(-15m^2 + 130m - 120)x^2y^2z^2 + (3m^2 - 26m + 24)xyz(xy + yz + zx)+$$
$$+(-m^2 + 7m - 6)xyz(x + y + z) + (m^2 - 3m + 2)xyz - \frac{12m^2 + 4m - 12}{729} =$$

$$= (3m^2-26m+24)\left(-5x^2y^2z^2+xyz(xy+yz+zx)-\frac{xyz(x+y+z)}{3}+\frac{xyz}{3}-\right.$$
$$\left.-\frac{4}{729}\right)+\frac{-5m+6}{3}xyz(x+y+z)+\frac{17m-18}{3}xyz-\frac{4m-4}{27}\le 0.$$

Theorem 1 *If $f \in C_b(\Delta_3)$ then*

$$|(UB_mf)(x,y,z)-f(x,y,z)| \le 4\omega_{mixed}\left(f;\frac{1}{\sqrt{m}},\frac{1}{\sqrt{m}},\frac{1}{\sqrt{m}}\right), \qquad (2.16)$$

for $m \ge 8$.

Proof. We apply Lemma 1.4 using the estimations from Lemma 2.3

$$(B_m(\bullet - x)^2(* - y)^2)(x,y,z) \le \frac{3}{16m^2}$$

and

$$(B_m(\bullet - x)^2(* - y)^2(\circ - z)^2) \le \frac{1}{25m^3};$$

by choosing $\delta_1 = \delta_2 = \delta_3 = 1/\sqrt{m}$ the relation (2.13) results.

Corollary 1 *If $f \in C_b(\Delta_3)$ then the sequence $(UB_m)_m \ge 1$ converges to f uniformly on Δ_3.*

Remark 1 *For analogous results about Bernstein bivariate operators, see [9].*

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Author:

Mircea Farcaş
National College "Mihai Eminescu"
Satu Mare
Romania
e-mail:*mirceafarcas2005@yahoo.com*