STABILITY OF SLANT AND SEMI-SLANT SUBMANIFOLDS IN SASAKI MANIFOLDS

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Abstract. In this paper we study the first and the second variation of slant and semi-slant submanifolds in Sasaki manifolds. We define the notions of Legendre, Hamiltonian and harmonic variations in the case of these submanifolds and we study some problems of minimality and stability.

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1. Introduction

The differential geometry of slant submanifolds has intensely been studied since B. Y. Chen defined and studied slant immersions in complex manifolds. J. L. Cabrerizo, A. Carriazo, L. M. Fernandez and M. Fernandez [2] studied and characterized slant submanifolds in the case of K-contact and Sasaki manifolds. The notion of semi-slant submanifold was introduced by N. Papaghiuc [14] in the case of the almost Hermitian manifolds and the class of slant submanifolds appears as a particular case of semi-slant submanifolds. Finally, we study some aspects concerning variational problems for slant and semi-slant submanifolds in Sasaki manifolds. Similar problems were studied by H. B. Lawson [9], H. B. Lawson and J. Simons [10], B. Y. Chen, P. F. Leung and T. Nagano [6], Y. G. Ohnita [11] , B. Palmer [12], [13] for Lagrange submanifolds and B. Y. Chen, J. M. Morvan [7] and others for isotropic submanifolds in Kahler manifolds.

2. Preliminaries

Let $\widetilde{M}$ be an almost contact manifold, C^∞–differentiabile with dimension $2m+1$. Let (F,ξ,η,g) its almost contact structure, where F is a tensor field

of type $(1,1)$, η is a 1-form and g is a Riemannian metric on $\widetilde{M}$, all these tensors satisfying the following conditions:

$$F^2 = -I + \eta \otimes \xi; \qquad \eta(\xi) = 1; \tag{1}$$

$$g(FX,FY) = g(X,Y) - \eta(X)\eta(Y) \tag{2}$$

for all $X, Y \in \chi(\widetilde{M})$, where $\chi(\widetilde{M})$ is the set of all vector fields. We consider Ω the fundamental (or the Sasaki) 2–form of $\widetilde{M}$, given by $\Omega(X,Y) = g(X,FY)$. If we denote by

$$N_F(X,Y) = F^2[X,Y] + [FX,FY] - F[FX,Y] - F[X,FY] \tag{3}$$

the Nijenhius tensor of F, then we consider the tensor field

$$N^{(1)} = N_F \oplus 2d\eta \otimes \xi. \tag{4}$$

We know that the almost contact manifold $\widetilde{M}$ is Sasaki if and only if

$$d\eta = \Omega; \qquad N^{(1)} = 0 \tag{5}$$

or equivalently

$$(\widetilde{\nabla}_X F)Y = g(X,Y)\xi - \eta(Y)X \tag{6}$$

for all $X, Y \in \chi(M)$, where $\widetilde{\nabla}$ is the Levi-Civita connection associated to the metric g. From (1), (4) and (6), it results the well-known equality:

$$\widetilde{\nabla}_X \xi = -FX. \tag{7}$$

Let M be a submanifold of the Sasaki manifold $\widetilde{M}$, ∇ the Levi-Civita connection induced by $\widetilde{\nabla}$ on M, $\nabla^{\perp}$ the connection in the normal bundle $T^{\perp}(M)$, h the second fundamental form of M and $A_{\vec{n}}$ the Weingarten operator. We recall the Gauss-Weingarten formulas on M:

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{8}$$

$$\widetilde{\nabla}_X \vec{n} = -A_{\vec{n}} X + \nabla^{\perp}_X \vec{n} \tag{9}$$

for all $X, Y \in \chi(M)$ and $\vec{n} \in \chi^{\perp}(M)$.

The submanifold M of $\widetilde{M}$, tangent to ξ, is a *slant* submanifold if

$$\theta = \angle(FX_x, T_xM) = constant \tag{10}$$

for all $x \in M$, $X_x \in T_xM$, X_x non collinear with ξ. Taking into account the definition of the angle between a vector and a subspace in the Euclidean space, this is equivalent with

$$\cos\theta = \frac{g(FX,Y)}{\|FX\|\,\|Y\|} = constant \tag{11}$$

for all $Y \in \chi(M)$, $X \in D$, X, Y nowhere zero, where D is the orthogonal distribution of ξ in $\chi(M)$. In this case, θ is the *slant angle* of the submanifold M and the distribution D is *the slant distribution* of M.

The submanifold M of the Sasaki manifold $\widetilde{M}$ is a *semi-slant submanifold* if there are D_1, D_2 two distributions on M so that:
i) $\chi(M) = D_1 \oplus D_2 \oplus <\xi>$
ii) D_1 is invariant, i.e. $FD_1 = D_1$
iii) D_2 is slant with the slant angle θ.

For M a slant or semi-slant submanifold in a Sasaki manifold $\widetilde{M}$, we consider the decompositions

$$FX = TX + NX; \quad F\vec{n} = t\vec{n} + n\vec{n} \tag{12}$$

for all $X \in \chi(M)$, $\vec{n} \in \chi^{\perp}(M)$, where TX is the tangent component and NX the normal component of FX and $t\vec{n}$ is the tangent component and $n\vec{n}$ is the normal component of $F\vec{n}$ in $\chi(\widetilde{M})$.

Moreover, if M is a semi-slant submanifold of a Sasaki manifold $\widetilde{M}$, then we consider

$$X = P_1X + P_2X + \eta(X)\xi \tag{13}$$

for all $X \in \chi(M)$, where P_1 is the projection on D_1 and P_2 is the projection on D_2.

We recall some known results for slant and semi-slant submanifolds [2], [3]:

Proposition 2.1. *Let M be a submanifold of the almost contact manifold $\widetilde{M}$ tangent to the Reeb vector field $\xi \in \chi(M)$. Then M is slant if and only if there is $\lambda \in [0, 1]$ so that:*

$$T^2 = -\lambda(I - \eta \otimes \xi). \tag{14}$$

Moreover, in this case, the slant angle θ of M satisfies the condition $\lambda = \cos^2\theta$.

Proposition 2.2. *Let M be a slant submanifold in an almost contact manifold $\widetilde{M}$ with the slant angle θ. Then:*

$$g(TX, TY) = \cos^2\theta[g(X, Y) - \eta(X)\eta(Y)] \tag{15}$$

and

$$g(NX, NY) = \sin^2\theta[g(X, Y) - \eta(X)\eta(Y)] \tag{16}$$

for all $X, Y \in \chi(M)$.

Proposition 2.3. *Let M be a semi-slant submanifold of the almost contact manifold $\widetilde{M}$ with the slant angle θ. Then:*

$$g(TX, TP_2Y) = \cos^2\theta g(X, P_2Y) \tag{17}$$

and

$$g(NX, NP_2Y) = \sin^2\theta g(X, P_2Y) \tag{18}$$

for all $X, Y \in \chi(M)$.

3. First order deformations for slant and semi-slant submanifolds

Let M be a n–dimensional slant submanifold of the Sasaki manifold $\widetilde{M}$, D the slant distribution and θ the slant angle. We consider

$$B_1^M = \{e_1, e_2, ..., e_{n-1}, \xi\} \tag{19}$$

a local orthonormal basis in $\chi(M)$, so that $\{e_1, e_2, ..., e_{n-1}\}$ is a basis in D. From (16) we obtain

$$g(Ne_i, Ne_j) = \sin^2\theta[g(e_i, e_j) - \eta(e_i)\eta(e_j)] = 0, \quad i \neq j, \quad i, j = \overline{1, n-1},$$

that is $\{Ne_i\}_{i=1,n-1}$ are liniar independent. Moreover, $g(Ne_i, Ne_i) = \sin^2\theta$. Let $e_{n+1} = \frac{Ne_1}{\sin\theta}, ..., e_{2n-1} = \frac{Ne_{n-1}}{\sin\theta}$ and $\Gamma NFTM =< e_{n+1}, ..., e_{2n-1} >$ be the subspace spanned by $e_{n+1}, ..., e_{2n-1}$. We observe that $\Gamma NFTM \subset \chi^{\perp}(M)$ and $\{e_{n+1}, ..., e_{2n-1}\}$ is a basis in $\Gamma NFTM$. Moreover, $g(e_{n+j}, e_i) = 0$, for $j = \overline{1, n-1}, i = \overline{1, n}$. Let $\Gamma(\tau(M))$ be the orthogonal complement of $\Gamma NFTM$ in $\chi^{\perp}(M)$ and $B_1^{\tau(M)} = \{e_{2n}, ..., e_{2m+1}\}$ a local orthonormal base in $\Gamma NFTM$. We obtain that
$B_1 = \{e_1, ...e_{n-1}, \xi, e_{n+1}, ..., e_{2n-1}, e_{2n}, ..., e_{2m+1}\}$ is a basis in $\chi(\widetilde{M})$.

If M is a semi-slant submanifold of the Sasaki manifold $\widetilde{M}$, with D_1 the invariant distribution, D_2 the slant distribution with slant angle θ, then we consider $\{e_1, ..., e_p\}$ a basis in D_1, $\{e_{p+1}, ..., e_{n-1}\}$ a basis in D_2 so that $B_2^M = \{e_1, ..., e_p, e_{p+1}, ..., e_{n-1}, e_n = \xi\}$ is a local orthonormal basis in $\chi(M)$. Also, we consider

$$e_{n+1} = \frac{Ne_{p+1}}{\sin\theta}, ..., e_{2n-p-1} = \frac{Ne_{n-1}}{\sin\theta}.$$

Taking into account Proposition 2.3, we deduce that $\{e_{n+1}, ..., e_{2n-p-1}\}$ are orthonormal vectors. Let $\Gamma NFD_2 =< e_{n+1}, ..., e_{2n-p-1} >$ be the subspace spanned by $\{e_{n+1}, ..., e_{2n-p-1}\}$ and $\Gamma(\tau(M))$ the orthogonal complement of ΓNFD_2 in $\chi^{\perp}(M)$, so that $\{e_{2n-p}, ..., e_{2m+1}\}$ is a basis in $\Gamma(\tau(M))$. Thus, in the case of semi-slant submanifolds in Sasaki manifolds, we have:

$$\chi(\widetilde{M}) = \chi(M) \oplus \Gamma NFD_2 \oplus \Gamma(\tau(M)).$$

If M is a slant or a semi-slant submanifold of the Sasaki manifold $\widetilde{M}$, then we consider the dual 1-form to the vector $\vec{n} \in \chi^{\perp}(M)$, defined by

$$\alpha_{\vec{n}} : \chi(M) \to F(M), \quad \alpha_{\vec{n}}(X) = g(F\vec{n}, X) \tag{20}$$

for all $X \in \chi(M)$ and we denote by

$$\mathbf{L} = \left\{\vec{n} \in \chi^{\perp}(M) : d\alpha_{\vec{n}} = 0\right\} \tag{21}$$

the set of *Legendre variations*, by

$$\mathbf{E} = \left\{\vec{n} \in \chi^{\perp}(M) : (\exists) f \in F(M) : \alpha_{\vec{n}} = df\right\} \tag{22}$$

the set of *Hamiltonian variations* and by

$$\mathbf{H} = \left\{\vec{n} \in \chi^{\perp}(M) : d\alpha_{\vec{n}} = \delta\alpha_{\vec{n}} = 0\right\} \tag{23}$$

the set of *harmonic variations.*

Example: We consider a slant submanifold in $\mathbb{R}^5$ [2] and we find harmonic variations.
Let $\widetilde{M} = \mathbb{R}^5$ with local coordinates (x^1, x^2, y^1, y^2, z) and the Sasaki structure given by

$$\eta = \frac{1}{2}(dz - y^1 dx^1 - y^2 dx^2); \quad \xi = 2\frac{\partial}{\partial z};$$

$$g = \eta \otimes \eta + \frac{1}{4}(dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dy^1 \otimes dy^1 + dy^2 \otimes dy^2);$$

and $F : \chi(R^5) \to \chi(R^5)$ a tensor field of type (1,1) so that

$$F(\frac{\partial}{\partial x^1}) = -\frac{\partial}{\partial y^1}; \quad F(\frac{\partial}{\partial x^2}) = -\frac{\partial}{\partial y^2}; \quad F(\frac{\partial}{\partial z}) = 0;$$

$$F(\frac{\partial}{\partial y^1}) = \frac{\partial}{\partial x^1} + y^1\frac{\partial}{\partial z}; \quad F(\frac{\partial}{\partial y^2}) = \frac{\partial}{\partial x^2} + y^2\frac{\partial}{\partial z}$$

where $\left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \frac{\partial}{\partial z}\right\}$ is a base in $\chi(R^5)$. The matrix of g is

$$\begin{pmatrix} \frac{y_1^2+1}{4} & \frac{y^1y^2}{4} & 0 & 0 & -\frac{y^1}{4} \\ \frac{y^1y^2}{4} & \frac{y_2^2+1}{4} & 0 & 0 & -\frac{y^2}{4} \\ 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 \\ -\frac{y^1}{4} & -\frac{y^2}{4} & 0 & 0 & \frac{1}{4} \end{pmatrix}.$$

For $\theta \in [0, \frac{\pi}{2}]$ we consider the submanifold

$$M : x(u, v, t) = (2u\cos\theta, 2u\sin\theta, 2v, 0, 2t)$$

with

$$x^1(u, v, t) = 2u\cos\theta; \quad x^2(u, v, t) = 2u\sin\theta;$$
$$y^1(u, v, t) = 2v; \quad y^2(u, v, t) = 0; \quad z(u, v, t) = 2t.$$

We have

$$\frac{\partial}{\partial u} = 2\cos\theta\frac{\partial}{\partial x^1} + 2\sin\theta\frac{\partial}{\partial x^2}; \quad \frac{\partial}{\partial v} = 2\frac{\partial}{\partial y^1}; \quad \frac{\partial}{\partial t} = 2\frac{\partial}{\partial z} = \xi.$$

Using (11), it results that M is a minimal slant submanifold with the slant angle θ and slant distribution D, having the local orthonormal basis

$$\left\{\vec{v_1} = \frac{\partial}{\partial v}; \vec{v_2} = \frac{\partial}{\partial u} + 2v\cos\theta\frac{\partial}{\partial t}\right\}.$$

Moreover, $\chi(M) = D\oplus <\xi>$ and from the definition of Sasaki structure on $\mathbb{R}^5$, we obtain that

$$\left\{\vec{n_1} = 2\frac{\partial}{\partial y^2}; \vec{n_2} = 2\sin\theta\frac{\partial}{\partial x^1} - 2\cos\theta\frac{\partial}{\partial x^2} + 4v\sin\theta\frac{\partial}{\partial z}\right\}$$

is an orthonormal basis in $\chi^{\perp}(M)$. We also consider $\alpha_{\vec{n_1}} : \chi(M) \to F(M)$; $\alpha_{\vec{n_1}} = g(F\vec{n_1}, X)$ and $\alpha_{\vec{n_2}} : \chi(M) \to F(M)$; $\alpha_{\vec{n_2}} = g(F\vec{n_2}, X)$ for all $X = a_1\vec{v_1} + b_1\vec{v_2} + c_1\xi \in \chi(M)$. Using the definition of Sasaki structure on $\mathbb{R}^5$, we obtain $\alpha_{\vec{n_1}}(X) = b_1\sin\theta$ and $\alpha_{\vec{n_2}}(X) = -a_1\sin\theta$.
Taking into account the decomposition of Lie derivative for a vector field in a local chart of M, the definition of the exterior derivative of a 1–form and the definition of the co-differential operator, we obtain that $\vec{n_1}$ and $\vec{n_2}$ are harmonic variations.

Proposition 3.1. *Let M be a slant (or semi-slant) submanifold of the Sasaki manifold $\widetilde{M}$ with its slant angle θ. Then $\alpha_{\vec{n}}$ is closed if and only if*

$$g(A_{\vec{n}}X, TY) - g(A_{\vec{n}}Y, TX) = g(\nabla^{\perp}_{X}\vec{n}, NY) - g(\nabla^{\perp}_{Y}\vec{n}, NX) \tag{24}$$

for all $X, Y \in \chi(M)$.

Proof. We consider $X, Y \in \chi(M)$. From the definition of the exterior derivative and 1–form $\alpha_{\vec{n}}$, we obtain

$$\begin{aligned}(d\alpha_{\vec{n}})(X,Y) &= X(g(F\vec{n}, Y)) - Y(g(F\vec{n}, X)) - g(F\vec{n}, [X,Y]) \\ &= -X(g(\vec{n}, FY)) + Y(g(\vec{n}, FX)) + g(\vec{n}, F\widetilde{\nabla}_X Y - F\widetilde{\nabla}_Y X).\end{aligned}$$

Using (6) and the properties of Levi-Civita connection we obtain

$$\begin{aligned}(d\alpha_{\vec{n}})(X,Y) &= -g(\widetilde{\nabla}_X\vec{n}, FY) - g(\vec{n}, \widetilde{\nabla}_X(FY)) + g(\widetilde{\nabla}_Y\vec{n}, FX) \\ &+ g(\vec{n}, \widetilde{\nabla}_Y(FX)) + g(\vec{n}, F\widetilde{\nabla}_X Y) - g(\vec{n}, F\widetilde{\nabla}_Y X).\end{aligned}$$

From Gauss-Weingarten formulas, $(8), (9)$ and (6) we obtain

$$\begin{aligned}(d\alpha_{\vec{n}})(X,Y) &= g(A_{\vec{n}}X, FY) - g(A_{\vec{n}}Y, FX) - g(\nabla^{\perp}_X\vec{n}, FY) + g(\nabla^{\perp}_Y\vec{n}, FX) \\ &- g(X,Y)g(\vec{n},\xi) + \eta(Y)g(\vec{n}, X) - g(\vec{n}, F\widetilde{\nabla}_X Y) + g(X,Y)g(\vec{n},\xi) \\ &- \eta(X)g(\vec{n}, Y) + g(\vec{n}, F\widetilde{\nabla}_Y X) + g(\vec{n}, F\widetilde{\nabla}_X Y) - g(\vec{n}, F\widetilde{\nabla}_Y X) \\ &= g(A_{\vec{n}}X, FY) - g(A_{\vec{n}}Y, FX) - g(\nabla^{\perp}_X\vec{n}, FY) + g(\nabla^{\perp}_Y\vec{n}, FX)\end{aligned}$$

and then (24). □

Proposition 3.2. *Let M be a slant (or semi-slant) submanifold with slant angle θ of the Sasaki manifold $\widetilde{M}$. Then:*

i) $\Gamma(\tau(M)) \subset \boldsymbol{L}$;

ii) $\boldsymbol{H} \subset \boldsymbol{L}$;

iii) $\boldsymbol{E} \subset \boldsymbol{L}$.

If M is slant then we have

iv) $F(gradf)_{|\Gamma NFTM} \subset \boldsymbol{E}$;

v) Let M^* be a totally geodesic hypersurface of submanifold M and $\vec{n}$ the unit vector field normal to the hypersurface M^*. The normal component $N\vec{n}$ of $F\vec{n}$ is a Legendre variation if and only if

$$g(\nabla^{\perp}_X N\vec{n}, NY) = g(\nabla^{\perp}_Y N\vec{n}, NX)$$

or equivalently

$$\eta(\nabla_X\vec{n})\eta(Y) - \eta(\nabla_Y\vec{n})\eta(X) = g(\vec{n}, [Y,X])$$

for all $X, Y \in \chi(M^*)$.

Proof. i) We consider $\vec{n} \in \Gamma(\tau(M))$ şi $X \in \chi(M)$. We have

$$g(\nabla^{\perp}_{Y}\vec{n}, NX) = -g(\vec{n}, Fh(Y,X)) + g(A_{\vec{n}}Y, TX) \tag{25}$$

and

$$g(\nabla^{\perp}_{X}\vec{n}, NY) = -g(\vec{n}, Fh(X,Y)) + g(A_{\vec{n}}X, TY) \tag{26}$$

for all $X, Y \in \chi(M), \vec{n} \in \Gamma(\tau(M))$. From (25) and (26) we obtain (24), that is i).

From (21), (22), (23) and the properties of the exterior derivative it results ii) and iii).

iv) We consider $f \in F(M)$. Then $gradf = \sum_{i,j=1}^{n} g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}$, where $(g^{ij})_{i,j=\overline{1,n}}$ is the inverse matrix of $(g_{ij})_{i,j=\overline{1,n}} = (g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^i}))_{i,j=\overline{1,n}}$. For $X \in D$ we obtain:

$$\begin{aligned} \alpha_{F(gradf)|\Gamma NFTM}(X) &= -\sum_{i,j=1}^{n} g^{ij} \frac{\partial f}{\partial x^i} g(\frac{\partial}{\partial x^j}, X) \\ &= -\sum_{i,k=1}^{n} \delta^i_k \frac{\partial f}{\partial x^i} X^k = -df(X) \end{aligned} \tag{27}$$

where X^k are the components of the vector field X with respect to the natural basis $\{\frac{\partial}{\partial x^i}\}_{i=\overline{1,n}}$. Because $\alpha_{F(gradf)|\Gamma NFTM}(\xi) = 0$ and using (27) we obtain iv).

v) Using (12), (6), (8), (9), for $X \in \chi(M^*)$ we obtain:

$$\widetilde{\nabla}_X(N\vec{n}) = -(\nabla_X T)\vec{n} + N\nabla_X\vec{n} - \eta(\vec{n})X. \tag{28}$$

Now, from Weingarten formula

$$\widetilde{\nabla}_X(N\vec{n}) = -A_{N\vec{n}}X + \nabla^{\perp}_X(N\vec{n}) \tag{29}$$

it follows:

$$A_{N\vec{n}}X = -(\nabla_X T)\vec{n} - \eta(\vec{n})X; \quad \nabla^{\perp}_X(N\vec{n}) = N\nabla_X\vec{n}. \tag{30}$$

But $N\vec{n}$ is a Legendre variation and then from Proposition 2.3 we obtain v). □

Let M be a slant (or a semi-slant) submanifold with slant angle θ of the Sasaki manifold $\widetilde{M}$ and $\vec{n} \in \chi^{\perp}(M)$. The first variation of the volume form of M, relative to the normal vector field $\vec{n}$ (that is the value at $t = 0$ of the first derivative of $V(\vec{n})$) can be expressed under the form [4]

$$V^{'}(\vec{n}) = -n \int_M g(\vec{n}, H)dv \tag{31}$$

where H is the mean curvature vector field of M. Then M is:
i) *l–minimal* if $V'(\vec{n}) = 0$, for all $\vec{n} \in \mathbf{L}$
ii) *e–minimal* if $V'(\vec{n}) = 0$, for all $\vec{n} \in \mathbf{E}$
iii) *h–minimal* if $V'(\vec{n}) = 0$, for all $\vec{n} \in \mathbf{H}$.
We also observe that:
a) If the slant (or semi-slant) submanifold M is minimal, then M is l, e and h–minimal.
b) If the slant (or semi-slant) submanifold M is e–minimal or h–minimal, then M is l–minimal.

Theorem 3.3. *Let M be a compact slant (or semi-slant) submanifold of the Sasaki manifold $\widetilde{M}$. Then:*
i) M is l–minimal if and only if $H \in \Gamma NFTM$ and $\alpha_H = \sum_\mu f_\mu \Phi_\mu$, where $f_\mu \in F(M)$ and Φ_μ are co-exact 1-forms.
ii) M is e–minimal if and only if $H \in \Gamma NFTM$ and α_H is co-closed.
iii) M is h–minimal if and only if $H \in \Gamma NFTM$ and α_H is the sum of an exact 1–form and a co-exact 1–form.

Proof. i) " $\Rightarrow$ " We suppose that $\vec{n} \in \mathbf{L}$. Because M is l–minimal, then from (31) it results $g(H, \vec{n}) = 0$ for all $\vec{n} \in \mathbf{L}$. Taking into account Proposition 3.2 i) we obtain $g(H, \vec{n}) = 0$ for all $\vec{n} \in \Gamma(\tau(M))$, that is $H \in \Gamma NFTM$.
On the other hand, since α_H is a 1–form, it results that $*\alpha_H$ is a $(n-1)$–form and taking into account the definition of the $*$ operator and the properties of the exterior product, we obtain:

$$(\alpha_{\vec{n}} \wedge *\alpha_H)(X_1, ..., X_n) = g(\vec{n}, H) dv(X_1, ..., X_n)$$

for all X_1, X_2,...,X_n in $\chi(M)$, that is $\alpha_{\vec{n}} \wedge *\alpha_H = g(\vec{n}, H)dv = 0$.

From the definition of the scalar product $<,>$, defined on the real space of all 2–forms on M, we obtain:

$$< \alpha_{\vec{n}}, \alpha_H >= \int_M \alpha_{\vec{n}} \wedge *\alpha_H = 0.$$

Because $< \alpha_{\vec{n}}, \alpha_H >= 0$ and since M is compact, we have $\alpha_H = \sum_\mu f_\mu \Phi_\mu$, where $f_\mu \in F(M)$ and Φ_μ are co-exact 1–forms on M.
" $\Leftarrow$ " We suppose that $H \in \Gamma NFTM$ and $\alpha_H = \sum_\mu f_\mu \Phi_\mu$, Φ_μ are co-exact 1–forms. For $\vec{n} \in \mathbf{L}$ we have

$$< \alpha_{\vec{n}}, \alpha_H >=< \alpha_{\vec{n}}, \sum_\mu f_\mu \Phi_\mu >=< \alpha_{\vec{n}}, \sum_\mu f_\mu \delta\phi_\mu >= \sum_\mu < d\alpha_{\vec{n}}, f_\mu \phi_\mu >= 0$$

and then

$$V'(\vec{n}) = \int_M g(H, \vec{n}) dv = \int_M \alpha_{\vec{n}} \wedge *\alpha_H =< \alpha_{\vec{n}}, \alpha_H >= 0.$$

ii) " $\Rightarrow$ " We suppose that M is l–minimal. Then from i) we have $H \in \Gamma NFTM$. But $\vec{n} \in \mathbf{E}$ implies that $\alpha_{\vec{n}}$ is exact, that is there exists $f \in F(M)$ so that $\alpha_{\vec{n}} = df$. Thus $d\alpha_{\vec{n}} = d^2 f = 0$ and $< \alpha_{\vec{n}}, \delta\alpha_{\vec{H}} >=$
$< d\alpha_{\vec{n}}, \alpha_H >= 0$, that is α_H is a co-closed 1-f-orm.
" $\Leftarrow$ " We suppose that $H \in \Gamma NFTM$ and α_H co-closed. Now, we consider $\vec{n} \in \mathbf{E}$. Then it exists $f \in F(M)$ so that $\alpha_{\vec{n}} = df$. We obtain
$< \alpha_{\vec{n}}, \alpha_H >= < df, \alpha_H >=< f, \delta\alpha_H >= 0$ and then

$$V'(\vec{n}) = \int_M g(H, \vec{n}) dv = \int_M \alpha_{\vec{n}} \wedge *\alpha_H =< \alpha_{\vec{n}}, \alpha_H >= 0$$

that is M e–minimal.

iii) " $\Rightarrow$ " We consider that M is h–minimal. From Proposition 3.2 iii), i) we have that M is l–minimal and $H \in \Gamma NFTM$. For $\vec{n} \in \mathbf{H}$ we have

$$< \alpha_{\vec{n}}, \Delta\alpha_H >=< \Delta\alpha_{\vec{n}}, \alpha_H >= 0$$

that is $\Delta\alpha_H = 0$ or there are an exact 1-form ω_1 and a co-exact 1–form ω_2 so that $\alpha_H = \omega_1 + \omega_2$.
" $\Leftarrow$ " We suppose that $H \in \Gamma NFTM$, $\alpha_H = \omega_1 + \omega_2$, $\omega_1 = df$, $\omega_2 = \delta\omega$, ω a 2–form. For $\vec{n} \in \mathbf{H}$, we have $\delta\alpha_{\vec{n}} = 0$ and

$$< \alpha_{\vec{n}}, \alpha_H > \; = \; < \alpha_{\vec{n}}, df + \delta\omega >=< \delta\alpha_{\vec{n}}, f > + < d\alpha_{\vec{n}}, \omega >= 0.$$

Thus

$$V'(\vec{n}) = \int_M g(H, \vec{n}) dv = \int_M \alpha_{\vec{n}} \wedge *\alpha_H =< \alpha_{\vec{n}}, \alpha_H >= 0$$

, that is M h–harmonic. □

Theorem 3.4. *Let M be a compact slant(or semi-slant) submanifold of the Sasaki manifold $\widetilde{M}$. If H is a Legendre variation of M, then:*
i) M is l-minimal iff M is minimal.
ii) M is e-minimal iff H is an harmonic variation.
iii) M is h-minimal iff H is an Hamiltonian variation.

Proof. Because the argument is the same for the proof of all afirmations we only prove i).

" $\Rightarrow$ " We suppose that H is a Legendre variation. Then $d\alpha_H = 0$. Because M is l-minimal, then from Proposition 3.2 we have $H \in \Gamma NFTM$ and

$$< \alpha_H, \alpha_H >=< \alpha_H, \sum_\mu f_\mu \Phi_\mu >=< \alpha_H, \sum_\mu \delta\phi_\mu >=< d\alpha_H, \sum_\mu f_\mu \phi_\mu >= 0$$

that is $\alpha_H = 0$ and then $H = 0$.
" $\Leftarrow$ " We consider $H = 0$. Then from (31) we have $V'(\vec{n}) = 0$ for all $\vec{n} \in \mathbf{L}$, that is M is l–minimal. □

Proposition 3.5. *Let M be a slant submanifold of the Sasaki manifold $\widetilde{M}$ such that $[h(X, TY) = h(Y, TX)$ for all $X, Y \in \chi(M)$ and the mean curvature vector H of M is parallel. Then:*
i) H is a Legendre variation;
ii) H is e-minimal iff

$$\sum_{i=1}^{n-1} g(A_H e_i, Te_i) = 0$$

Proof. i) Writing (24) for $\vec{n} = H$ and taking into account the fact that H is parallel, we have $d\alpha_H = 0$, that is H is a Legendre variation. Using (6) for $X = e_a$ and $Y = H$ we have $(\widetilde{\nabla} e_a F)H = 0$ for $e_a \in \{e_1, ..., e_{n-1}\}$. Moreover, taking into account the definition of the co-derivative operator of the 1–form α_H, (6) and (9), we deduce

$$\delta\alpha_H = -\sum_{i=1}^{n} g(A_H e_i, Te_i).$$

Now, our affirmation results because $T\xi = 0$. □

4. Second order deformations of slant submanifolds

Let $V''(\vec{n})$ be the second variation of the volum form of a n–dimensional slant submanifold M in the Sasaki manifold $\widetilde{M}$. By [4] this is given by

$$\begin{aligned} V"(\vec{n}) &= \int_M \left\{ \left\| \nabla^\perp \vec{n} \right\|^2 - \|A_{\vec{n}}\|^2 \right\} dv \\ &+ \int_M \left\{ n^2 g^2(H, \vec{n}) - ng(H, \widetilde{\nabla}_{\vec{n}} \vec{n}) - \sum_{a=1}^{n} \widetilde{R}(\vec{n}, e_a, \vec{n}, e_a) \right\} dv \end{aligned} \tag{32}$$

where $\vec{n} \in \chi^\perp(M)$ and $\widetilde{R}$ is the Riemann Christoffel tensor of the manifold $\widetilde{M}$. Then:
i) M is *stable* if $V"(\vec{n}) \geq 0$, for all $\vec{n} \in \chi^\perp(M)$;

ii) M is *l–stable* if $V"(\vec{n}) \geq 0$, for all $\vec{n} \in \mathbf{L}$;
iii) M is *e–stable* if $V"(\vec{n}) \geq 0$, for all $\vec{n} \in \mathbf{E}$;
iv) M is *h–stable* if $V"(\vec{n}) \geq 0$, for all $\vec{n} \in \mathbf{H}$.

Example: If we consider the minimal slant submanifold M of $\mathbb{R}^5$ with $0 \leq v \leq \frac{1}{2}$ and its harmonic variation $\vec{n}_1 = 2\frac{\partial}{\partial y^2}$, taken in the Section 3, then we have:

$$\|A_{\vec{n_1}}\|^2 = \frac{\cos^2\theta}{4}; \quad \left\|\nabla^{\perp}\vec{n_1}\right\|^2 = \frac{\sin^2\theta}{4}.$$

Using the properties of the Riemann Christoffel tensor $\widetilde{R}$, we obtain:

$$\widetilde{R}(\vec{n}_1, \vec{v}_1, \vec{n}_1, \vec{v}_1) = 0; \quad \widetilde{R}(\vec{n}_1, \xi, \vec{n}_1, \xi) = 1;$$
$$\widetilde{R}(\vec{n}_1, \vec{v}_2, \vec{n}_1, \vec{v}_2) = \cos^2\theta(32v^5 + 16v^4 - 16v^3 - 2v - 3).$$

From (32) and these last relations we have:

$$\begin{aligned} V"(\vec{n}_1) &= \frac{\cos^2\theta}{2}\int_M(-64v^5 - 32v^4 + 32v^3 + 4v + 5)dv \\ &= \frac{\cos^2\theta}{2}\int_M[4v\left\{v^2[9 - (4v+1)^2] + 1\right\} + 5]dv \end{aligned}$$

that is $V"(\vec{n_1}) \geq 0$.

From Proposition 3.2 ii), iii) we have:

Proposition 4.1. *Let M be a slant submanifold of the Sasaki manifold $\widetilde{M}$. Then:*
i) if M is l–stable then M is h–stable and e–stable.
ii) if M is h–unstable or e–unstable then M is l–unstable.

Proposition 4.2. *If M is a totally geodesic slant submanifold of the Sasaki manifold $\widetilde{M}$ with negative defined Riemann Christoffel tensor $\widetilde{R}$ then M is stable.*

Lemma 4.3. *If M is a slant submanifold with slant angle θ of the Sasaki manifold $\widetilde{M}$ then:*

i) For $\vec{n} \in \Gamma NFTM$ *we have:*

$$\begin{aligned}\left\|\nabla^{\perp}\vec{n}\right\|^2 &= \frac{1}{\sin^2\theta}\sum_{a=1}^{n}\sum_{b=1}^{n-1}[-g(\nabla_{e_a}(t\vec{n}), e_b) + g(h(e_a, e_b), n\vec{n}) \\ &+ g(h(e_a, Te_b), \vec{n})]^2 + \sum_{a=1}^{n}\sum_{j=2n}^{2m+1}[g(\nabla_{e_a}(t\vec{n}), te_j) + g(\nabla^{\perp}_{e_a}(n\vec{n}), ne_j) \\ &+ g(h(e_a, t\vec{n}), ne_j) - g(h(e_a, te_j), n\vec{n})]^2 \qquad (33)\end{aligned}$$

and

$$\begin{aligned}\|A_{\vec{n}}\|^2 &= \sum_{a,b=1}^{n} g^2(h(e_a, e_b), \vec{n}) \\ &= \sum_{a,b=1}^{n}[g(\nabla_{e_a}(Te_b), t\vec{n}) - g(h(e_a, t\vec{n}), Ne_b) \\ &+ g(\nabla^{\perp}_{e_a}Ne_b, n\vec{n}) - \eta(e_b)g(Ne_a, \vec{n})]^2 \qquad (34)\end{aligned}$$

ii) For $\vec{n} \in \Gamma(\tau(M))$ *we have*

$$\begin{aligned}\left\|\nabla^{\perp}\vec{n}\right\|^2 &= \frac{1}{\sin^2\theta}\sum_{a=1}^{n}\sum_{b=1}^{n-1}[g(h(e_a, e_b), n\vec{n}) + g(h(e_a, Te_b), \vec{n})]^2 \\ &+ \sum_{a=1}^{n}\sum_{j=2n}^{2m+1}[g(\nabla_{e_a}(t\vec{n}), te_j) + g(\nabla^{\perp}_{e_a}(n\vec{n}), ne_j) \\ &+ g(h(e_a, t\vec{n}), ne_j) - g(h(e_a, te_j), n\vec{n})]^2 \qquad (35)\end{aligned}$$

and

$$\begin{aligned}\|A_{\vec{n}}\|^2 &= \sum_{a,b=1}^{n}[g(\nabla_{e_a}Te_b, t\vec{n}) + g(h(e_a, Te_b), n\vec{n}) \\ &- g(h(e_a, t\vec{n}), Ne_b) + g(\nabla^{\perp}_{e_a}Ne_b, n\vec{n})]^2. \qquad (36)\end{aligned}$$

Proof. i) We have

$$\begin{aligned}\left\|\nabla^{\perp}\vec{n}\right\|^{2} &= \sum_{a=1}^{n}\left\|\nabla^{\perp}_{e_a}\vec{n}\right\|^{2}=\sum_{a=1}^{n}g(\nabla^{\perp}_{e_a}\vec{n},\nabla^{\perp}_{e_a}\vec{n})\\ &= \sum_{a=1}^{n}[\sum_{b=1}^{n-1}g^{2}(\nabla^{\perp}_{e_a}\vec{n},\frac{Ne_b}{\sin\theta})+\sum_{j=2n}^{2m+1}g^{2}(\nabla^{\perp}_{e_a}\vec{n},e_j)]\\ &= \frac{1}{\sin^{2}\theta}[\sum_{a=1}^{n}\sum_{b=1}^{n-1}g^{2}(\nabla^{\perp}_{e_a},Ne_b)]+\sum_{a=1}^{n}\sum_{j=2n}^{2m+1}g^{2}(\nabla^{\perp}_{e_a}\vec{n},e_j) \qquad (37)\end{aligned}$$

From (6), (8) ,(9) and (12) we obtain:

$$g(\nabla^{\perp}_{e_a}\vec{n},Ne_b)=-g(\nabla_{e_a}(t\vec{n}),e_a)+g(h(e_a,e_b),\vec{n})+g(h(e_a,Te_b),n\vec{n})$$

For $e_j\in\Gamma(\tau(M))$, $j=\overline{2n,2m+1}$, using (6), (8), (9), (12) it results

$$\begin{aligned}g(\nabla^{\perp}_{e_a}\vec{n},e_j) &= g(\nabla_{e_a}(t\vec{n}),te_j)+g(h(e_a,t\vec{n}),ne_j)\\ &- g(h(e_a,te_j),n\vec{n})+g(\nabla^{\perp}_{e_a}n\vec{n},ne_j)\end{aligned}$$

From these last two relations we obtain (33).

Also, by a similar argument, we obtain (34) and for $\vec{n}\in\Gamma(\tau(M))$ we deduce (35) and (36). □

Proposition 4.4. *Let M be a slant umbilical submanifold of the Sasaki manifold $\widetilde{M}$, so that the mean curvature vector of M is parallel with respect to the Levi-Civita connection $\widetilde{\nabla}$. If M is l–minimal and the Riemann Christoffel tensor of $\widetilde{M}$ is negatively defined, then*

$$V''(\vec{n})\geq 0. \qquad (38)$$

for all $\vec{n}\in\Gamma(\tau(M))$.

Proof. Because M is umbilical, we have:

$$\|A_{\vec{n}}\|^{2}=ng^{2}(H,\vec{n}). \qquad (39)$$

From Teorema 3.3 i) and taking into account the properties of the Levi-Civita connection, we obtain:

$$\begin{aligned}V"(\vec{n}) &= \int_{M}\left\|\nabla^{\perp}\vec{n}\right\|^{2}-ng^{2}(H,\vec{n})+n^{2}g^{2}(H,\vec{n})\\ &- n[\vec{n}(g(H,\vec{n}))-g(\tilde{\nabla}_{\vec{n}}H,\vec{n})]-\sum_{a=1}^{n}\widetilde{R}(\vec{n},e_a,\vec{n},e_a)dv\\ &= \int_{M}[\left\|\nabla^{\perp}\vec{n}\right\|^{2}+ng(\widetilde{\nabla}_{\vec{n}}H,\vec{n})-\sum_{a=1}^{n}\widetilde{R}(\vec{n},e_a,\vec{n},e_a)]dv.\end{aligned}$$

and then (38). □

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