ON PRECOMPACT MULTIPLICATION OPERATORS ON WEIGHTED FUNCTION SPACES

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Abstract. Let X be a completely regular Hausdorff space, E a Hausdorff topological vector space, $CL(E)$ the algebra of continuous operators on E, V a Nachbin family on X and $\mathcal{F} \subseteq CV_b(X,E)$ a topological vector space (for a given topology). If $\pi : X \to CL(E)$ is a mapping, consider the induced multiplication operator $M_\pi : \mathcal{F} \to \mathcal{F}$ given by

$$M_\pi(f)(x) := \pi(x)f(x), \quad f \in G, x \in coz(\mathcal{F}).$$

In this paper we give necessary and sufficient conditions for the induced linear mapping M_π to be (1) an equicontinuous operator, (2) a precompact operator and (3) a bounded operator on a subspace $\mathcal{F}$ of $CV_b(X,E)$ in the non-locally convex setting.

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1. Introduction

The fundamental work on weighted spaces of continuous scalar-valued functions has been done mainly by Nachbin [21, 22] in the 1960's. Since then it has been studied extensively for a variety of problems by Bierstedt [2, 3], Summers [35, 36], Prolla [25, 26], Ruess and Summers [27], Khan [10, 11], Singh and Summers [34], Nawrocki [23], Khan and Oubbi [12] and many others. The multiplication operators M_π on the Weighted spaces $CV_b(X,E)$ and $CV_o(X,E)$ were first considered by Singh and Manhas in [29] in the cases of $\pi : X \to \mathbb{C}$ and $\pi : X \to E$ and later in [30] in the case of $\pi : X \to CL(E)$, E a locally convex space. This class form a special class of the more general notion of weighted composition operators $W_{\pi,\varphi}$, where $\varphi : X \to X$ [8, 33, 34]. The extension of these results to non-locally convex setting have been given later in [13, 14, 20].

The compactness of weighted composition operators $W_{\pi,\varphi}$ and various other types of operators on spaces of continuous functions have also been studied extensively in recent years by many authors; see, e.g. [7, 8, 33, 6, 37, 38, 5, 16, 31, 32, 19, 17, 18]. We mention that it is not possible to get the behaviour of compact multiplication operators from the study of compact weighted composition operators. This is due to the reason that the conditions obtained earlier for a weighted composition operator to be compact are not satisfied by the identity map $\varphi : X \to X$. In [24], Oubbi considered multiplication operator M_π on a subspace $\mathcal{F}$ of $CV_b(X, E)$ and, in this case, gave necessary and sufficient conditions for M_π to be a (1) equicontinuous operator, (2) precompact operator, and (3) bounded operator. A characterization of compact operators on $CV_0(X, E)$ have also been considered in [20] in the case of E a general TVS.

In this paper, we consider characterization of precompact and related multiplication operators on a subspace $\mathcal{F}$ of $CV_b(X, E)$ in the general case. Our results extend and unify several well-known results.

2. Preliminaries

Throughout, we shall assume, unless stated otherwise, that X is a completely regular Hausdorff space and E is a non-trivial Hausdorff topological vector space (TVS) with a base $\mathcal{W}$ of closed balanced shrinkable neighbourhoods of 0. (A neighbourhood G of 0 in E is called *shrinkable* [15] if $r\overline{G} \subseteq int\ G$ for $0 \leq r < 1$. By ([**?**], Theorems 4 and 5), every Hausdorff TVS has a base of shrinkable neighbourhoods of 0 and also the Minkowski functional ρ_G of any such neighbourhood G is continuous and absolutely homogeneous (but not subadditive unless G is convex).

A *Nachbin family* V on X is a set of non-negative upper semicontinuous function on X, called *weights*, such that given $u, v \in V$ and $t \geq 0$, there exists $w \in V$ with $tu, tv \leq w$ (pointwise) and, for each $x \in X$, ther exists $v \in V$ with $v(x) > 0$; due to this later condition, we sometimes write $V > 0$. Let $C(X, E)$ be the vector space of all continuous E-valued functions on X, and let $C_b(X, E)$ (resp. $C_o(X, E)$) denote the subspace of $C(X, E)$ consisting of those functions which are bounded (resp. vanish at infinity, have compact support). Further, let

$$\begin{aligned} CV_b(X,E) &= \{f \in C(X,E) : (vf)(X) \text{ is bounded in } E \text{ for all } v \in V\}, \\ CV_o(X,E) &= \{f \in C(X,E) : vf \text{ vanishes at infinity on } X \text{ for all } v \in V\} \\ CV_{pc}(X,E) &= \{f \in C(X,E) : (vf)(X) \text{ is precompact in } E \text{ for all } v \in V\}. \end{aligned}$$

Then $C_o(X, E) \subseteq C_b(X, E)$ and $CV_o(X, E) \subseteq CV_{pc}(X, E) \subseteq CV_b(X, E)$ (see [27], p.9). When $E = \mathbb{K}$ (= $\mathbb{R}$ or $\mathbb{C}$), the above spaces are denoted by $C(X)$, $C_b(X)$, $C_o(X)$, $C_{oo}(X)$, $CV_b(X)$ and $CV_o(X)$. We shall denote by $C(X) \otimes E$ the vector subspace of $C(X, E)$ spanned by the set of all functions of the form $\varphi \otimes a$, where $\varphi \in C(X)$, $a \in E$, and $(\varphi \otimes a) = \varphi(x)a$, $x \in X$.

Definition. *Given a Nachbin family V on X, the weighted topology ω_V on $CV_b(X,E)$ [21, 25, 10] is defined as the linear topology which has a base of neighbourhoods of 0 consisting of all sets of the form*

$$N(v,G) = \{f \in CV_b(X,E) : vf(X) \subseteq G\} = \{f \in CV_b(X,E) : \|f\|_{v,G} \leq 1\},$$

where $v \in V$, G is a closed shrinkable set in $\mathcal{W}$, and

$$\|f\|_{v,G} = \sup\{v(x)\rho_G(f(x)) : x \in X\}.$$

Some particular cases of the weighted toplogy are: the strict topology β on $C_b(X,E)$, the compact open topology k on $C(X,E)$ and the uniform topology u on $C_b(X,E)$ with $k \leq \beta \leq u$ on $C_b(X,E)$ [4, 9].

Let E and F be TVS, and let $CL(E,F)$ be the set of all continuous linear mappings $T : E \to F$. Then $CL(E,F)$ is a vector space with the usual poinwise operations. If $F = E$, $CL(E) = CL(E,E)$ is an algebra under composition.

Definition. *For any collection $\mathcal{A}$ of subsets of E, $CL_{\mathcal{A}}(E,F)$ denotes the subspace of $CL(E,F)$ consisting of those T which are bounded on the members of $\mathcal{A}$ together with the topology $t_{\mathcal{A}}$ of uniform convergence on the elements of $\mathcal{A}$. This topology has a base of neighbourhoods of 0 consisting of all sets of the form*

$$\begin{aligned} U(D,G) \quad : &= \{T \in CL_{\mathcal{A}}(E,F) : T(D) \subseteq G\} \\ &= \{T \in CL_{\mathcal{A}}(E,F) : \|T\|_{D,G} \leq 1\}, \end{aligned}$$

where $D \in \mathcal{A}$, G is a closed shrinkable neighbourhood of 0 in F, and

$$\|T\|_{D,G} = \sup\{\rho_G(T(a)) : a \in D\}.$$

If $\mathcal{A}$ consists of all bounded (resp. precompact, finite) subsets of E, then we will write $CL_u(E)$ (resp. $CL_{pc}(E), CL_p(E)$) for $CL_{\mathcal{A}}(E)$ and t_u (resp. t_{pc}, t_p) for $t_{\mathcal{A}}$. Clearly, $t_p \leq t_{pc} \leq t_u$.

For the general theory of topological vector spaces and continuous linear mappings, the reader is referred to to the book of Schaefer [28].

Definition. *(1) Let $\mathcal{F} \subseteq CV_b(X,E)$ be a topological vector space (for a given topology). Let $\pi : X \to CL(E)$ be a mapping and $F(X,E)$ a set of functions from X into E. For any $x \in X$, we denote $\pi(x) = \pi_x \in CL(E)$, and let $M_\pi : \mathcal{F} \to F(X,E)$ be the linear map defined by*

$$M_\pi(f)(x) := \pi(x)[f(x)] = \pi_x[f(x)], \quad f \in \mathcal{F}, \ x \in X.$$

Note that M_π is linear since each π_x is linear. Then M_π is said to be a multiplication operator on $\mathcal{F}$ if (i) $M_\pi(\mathcal{F}) \subseteq \mathcal{F}$ and (ii) $M_\pi : \mathcal{F} \to \mathcal{F}$ is continuous on $\mathcal{F}$. A self-map $\varphi : X \to X$ give rise to a linear mapping $W_{\pi,\varphi} : \mathcal{F} \to F(X,E)$ defined as

$$W_{\pi,\varphi}(f)(x) = \pi(x)(f(\varphi(x))) = \pi_x(f(\varphi(x))), \ f \in \mathcal{F}, \ x \in X.$$

Then $W_{\pi,\varphi}$ is said to be a weighted composition operator on $\mathcal{F}$ if (i) $W_{\pi,\varphi}(\mathcal{F}) \subseteq \mathcal{F}$ and (ii) $W_{\pi,\varphi} : \mathcal{F} \to \mathcal{F}$ is continuous on $\mathcal{F}$. Clearly, if $\varphi : X \to X$ is the identity map, then $W_{\pi,\varphi}$ is the multiplication operator M_π on $\mathcal{F}$

(2) We define the cozero set of $\mathcal{F} \subseteq C(X, E)$ by

$$coz(\mathcal{F}) := \{x \in X : f(x) \neq 0 \text{ for some } f \in \mathcal{F}\}.$$

If $coz(\mathcal{F}) = X$, i.e. if $\mathcal{F}$ does not vanish on X, then $\mathcal{F}$ is said to be essential. In general, $\mathcal{F} = CV_o(X, E)$ and $\mathcal{F} = CV_b(X, E)$ need not be essential. If $CV_o(X)$ is essential, then clearly $CV_b(X)$ is also essential. If $CV_o(X)$ is essential and E is a non-trivial TVS, then $CV_o(X) \otimes E$ and hence $CV_b(X) \otimes E$, $CV_b(X, E)$ and $CV_b(X, E)$ are also essential.

Definition. (cf. [24, 1]) *(a) A subspace $\mathcal{F}$ of $CV_b(X, E)$ is said to be E-solid if, for every $g \in C(X, E)$, $g \in \mathcal{F} \Leftrightarrow$ for any $G \in \mathcal{W}$, there exist $H \in \mathcal{W}$, $f \in \mathcal{F}$ such that*

$$\rho_G \circ g \leq \rho_H \circ f \quad \text{(pointwise) on } coz(\mathcal{F}). \tag{ES}$$

(b) A subspace $\mathcal{F}$ of $CV_b(X, E)$ is said to be EV-solid if, for every $g \in C(X, E)$, $g \in \mathcal{F} \Leftrightarrow$ for any $u \in \mathcal{V}, G \in \mathcal{W}$, there exist $u \in \mathcal{V}$, $H \in \mathcal{W}$, $f \in \mathcal{F}$ such that

$$v\rho_G \circ g \leq u\rho_H \circ f \quad \text{(pointwise) on } coz(\mathcal{F})). \tag{EVS}$$

(c) A subspace $\mathcal{F}$ of $CV_b(X, E)$ is said to have the property (M) if

$$(\rho_G \circ f) \otimes a) \in \mathcal{F} \text{ for all } G \in \mathcal{W}, a \in E \text{ and } f \in \mathcal{F}. \tag{M}$$

Note. (i) The classical solid spaces (such as $C_b(\mathbb{R})$ and $C_o(\mathbb{R})$) are nothing but the $\mathbb{K}$-solid ones.

(ii) Every EV-solid subspace of $CV_b(X, E)$ is E-solid.

(iii) Every E-solid subspace $\mathcal{F}$ of $CV_b(X, E)$ satisfies both conditions (a) $C_b(X)\mathcal{F} \subseteq \mathcal{F}$ and (b) (M).

Examples. (1) The spaces $CV_b(X, E)$, $CV_o(X, E)$ and $C_{oo}(X, E)$ are all EV-solid.

(2) $CV_b(X, E) \cap C_b(X, E)$, $CV_o(X, E) \cap C_b(X, E)$, $CV_b(X, E) \cap C_o(X, E)$ and $CV_o(X, E) \cap C_o(X, E)$ are E-solid but need not be EV-solid.

(3) $C_o(\mathbb{R}, \mathbb{C})$ and $C_b(\mathbb{R}, \mathbb{C})$ are not $\mathbb{C}V$-solid for $V = \{\lambda e^{-\frac{1}{n}}, n \in \mathbb{N}, \lambda > 0\}$.

3. Characterization of Precompact and Bounded Multiplication Operators

In this section, we characterize equicontinuous, precompact and bounded multiplication operators on $CV_b(X, E)$.

Recall that a linear map $T : \mathcal{F} \subseteq CV_b(X, E) \to \mathcal{F}$ is said to be *compact* if it maps some 0-neighbourhood into a compact subset of $\mathcal{F}$. More generally, a linear map $T : \mathcal{F} \subseteq CV_b(X, E) \to \mathcal{F}$ is said to be *precompact* (resp. *equicontinuous*, *bounded*) if it maps some 0-neighbourhood into a precompact (resp. equicontinuous, bounded) subset of $\mathcal{F}$.

We first consider two results on equicontinuity of precompact subsets of $CV_b(X, E)$. These are given in [12] without proof. We include the proof for reader's interest and later use. Earlier versions of these results are due to Bierstedt [2, 3], Ruess and Summers [27], and Oubbi [24] established in the locally convex setting. We begin by considering the evaluation maps δ_x and Δ. For any $x \in X$, let $\delta_x : CV_b(X, E) \to E$ denote the evaluation map given by

$$\delta_x(f) = f(x),\ f \in CV_b(X, E).$$

Clearly, $\delta_x \in CL(CV_b(X, E), E)$. Next, we define $\Delta : X \to CL(CV_b(X, E), E)$ as the evaluation map given by

$$\Delta(x) = \delta_x,\ x \in X.$$

Lemma 1. *The evaluation map $\Delta : X \to CL(CV_b(X, E), E)$ is continuous $\Leftrightarrow$ every precompact subset of $CV_b(X, E)$ is equicontinuous.*

Proof. **(⇒)** Suppose $\Delta : X \to CL_{pc}(CV_b(X, E), E)$ is continuous, and let P be a precompact subset of $CV_b(X, E)$. To show that P is equicontinuous, let $x_o \in X$ and $G \in \mathcal{W}$. Since Δ is continuous at x_o, there exists an open neighbourhood D of x_o in X such that $\Delta(D) \subseteq \Delta(x_o) + U(P, G)$; that is

$$\delta_x(f) - \delta_{x_o}(f) \in G \text{ for all } x \in D \text{ and } f \in P.$$

Hence $f(D) \subseteq f(x_o) + G$ for all $f \in P$, and so P is equicontinuous.

(⇐) Suppose every precompact subset of $CV_b(X, E)$ is equicontinuous. To show that $\Delta : X \to CL_{pc}(CV_b(X, E), E)$ is continuous, let $x_o \in X$ and let P be a precompact subset of $CV_b(X, E)$ and G a balanced set in $\mathcal{W}$. Since P is equicontinuous (by hypothesis), there exists an open neighbourhood D of x_o in X such that

$$f(D) \subseteq f(x_o) + G \text{ for all } f \in P;$$

that is $\delta_x - \delta_{x_o} \in U(P, G)$ for all $x \in D$. Hence $\Delta(D) \subseteq \Delta(x_o) + U(P, G)$, showing that Δ is continuous at $x_o \in X$. □

Theorem 2. *Let X be a $V_{\mathbb{R}}$-space. Then every precompact subset of $CV_b(X, E)$ is equicontinuous.*

Proof. In view of Lemma 1, it suffices to show that the evaluation map $\Delta : X \to CL_{pc}(CV_b(X,E),E)$ is continuous. Since X is a $V_{\mathbb{R}}$-space, we only need to show that Δ is continuous on each $S_{v,1} = \{x \in X : v(x) \geq 1\}$, $v \in V$. Let $v_o \in V$ and $x_o \in S_{v_o,1}$, and let P be a precompact subset of $CV_b(X,E)$ and $G \in \mathcal{W}$. Choose a balanced $H \in \mathcal{W}$ with $H + H \subseteq G$. Since P is precompact, there exist $h_1, ..., h_n \in A$ such that

$$P \subseteq \bigcup_{i=1}^{n} (h_i + N(v_o, H)). \tag{1}$$

Since each h_i is continuous, there exists a neighbourhood D_i of x in X such that

$$h_i(y) - h_i(x_o) \in H \text{ for all } y \in D_i \ (i = 1, ..., n). \tag{2}$$

Let $D = \cup_{i=1}^{n} D_i$. Now, if $y \in D \cap S_{v_o,1}$ and $f \in P$, then by (1), $f = h_i + g$ for some $i \in \{1, ..., n\}$ and $g \in N(v_o, H)$. Hence, using (2),

$$\begin{aligned} \delta_y(f) - \delta_{x_o}(f) &= f(y) - f(x_o) = h_i(y) + g(y) - h_i(x_o) - g(x_o) \\ &= h_i(y) - h_i(x_o) + \frac{1}{v_o(y)} v_o(y) g(y) - \frac{1}{v_o(x_o)} v_o(x_o) g(y) \\ &\in H + \frac{1}{v_o(y)} H - \frac{1}{v_o(x)} H \subseteq H + H - H \subseteq G; \end{aligned}$$

that is, $\delta_y - \delta_{x_o} \in U(P,G)$ for all $y \in D \cap S_{v_o,1}$. Hence $\Delta(D \cap S_{v_o,1}) \subseteq \Delta(x) + U(P,G)$, showing that Δ is continuous on each $S_{v_o,1}$. □

Remark. The above result was proved in [27] for the subspace $CV_{pc}(X,E)$ with E a locally convex space.

Next, we shall see that the precompact (and then the compact) multiplication operators are often trivial. Recall that if $A \subseteq X$, then a point $x \in A$ is called an *isolated point* of A if x is not a limit point of A.

Theorem 3. *Let $\mathcal{F} \subseteq CV_b(X,E)$ be a $C_b(X)$-module and $\pi : X \to CL(E)$ a map such that $M_\pi(\mathcal{F}) \subseteq C(X,E)$. If X has no isolated points and M_π is equicontinuous on $\mathcal{F}$, then $M_\pi = 0$.*

*Proof.*Suppose M_π is equicontinuous on $\mathcal{F}$ but $M_\pi(f_o) \neq 0$ for some $f_o \in \mathcal{F}$. Then there exists $x_o \in coz(\mathcal{F})$ with $\pi_{x_o}(f_o(x_o)) \neq 0$. Since M_π is equicontinuous, there exist some $v \in V$ and $G \in \mathcal{W}$ such that $M_\pi(N(v,G) \cap \mathcal{F})$ is equicontinuous on X and in particular at x_o. We may assume that $f_o \in N(v,G)$ (since $N(v,G)$ is absorbing). Hence, for every balanced $H \in \mathcal{W}$, there exists a neighbourhood D of x_o in X such that

$$\pi_y(f(y)) - \pi_{x_o}(f(x_o)) \in H \text{ for all } y \in D \text{ and } f \in N(v,G) \cap \mathcal{F}.$$

Since x_o is not isolated, there exists some $y \in D \cap coz(\mathcal{F})$ with $y \neq x_o$. Choose then $g_y \in C_b(X)$ with $0 \leq g_y \leq 1$, $g_y(y) = 0$, and $g_y(x_o) = 1$. Then $g_y f_o \in N(v,G) \cap \mathcal{F}$

and so

$$\pi_y(g_y(y)f_o(y)) - \pi_{x_o}(g_y(x_o)f_o(x_o)) \in H;$$

that is, $\pi_{x_o}(f_o(x_o)) \in H$. Since $H \in \mathcal{W}$ is arbitrary and E is Hausdorff (i.e. $\bigcap_{H\in\mathcal{W}} H = \{0\}$), we have $\pi_{x_o}(f_o(x_o)) = 0$. This is the desired contradiction. □

Corollary 4. *Let $\mathcal{F} \subseteq CV_b(X,E)$ be a $C_b(X)$-module and $\pi : X \to CL(E)$ a map such that $M_\pi(\mathcal{F}) \subseteq C(X,E)$. If X is a $V_{\mathbb{R}}$-space without isolated points, then M_π is precompact $\Leftrightarrow M_\pi = 0$.*

Proof. Suppose M_π is precompact. Then, by Theorem 1, M_π is equicontinuous. Hence, by Theorem 2, $M_\pi = 0$. The converse is trivial. □

Next, we consider characterization for bounded operators. To do this, we first need to obtain the following generalization of ([24], Lemma 10).

Lemma 5. *Let $\mathcal{F} \subseteq CV_b(X,E)$ with $C_b(X)\mathcal{F} \subseteq \mathcal{F}$ and $\mathcal{F}$ satisfies (M). Then, for any $v \in V$, $G \in \mathcal{W}$ and $x \in coz(\mathcal{F})$,*

$$\begin{aligned} \frac{1}{v(x)} &= \sup\{\rho_G(f(x)) : f \in N(v,G)\cap\mathcal{F}\} \\ &= \sup\{\rho_G(f(x)) : f \in \mathcal{F} \text{ with } \|f\|_{v,G} \le 1\}. \end{aligned}$$

Proof. Let $x \in coz(\mathcal{F})$, $v \in V$, $G \in \mathcal{W}$. There exist $f \in \mathcal{F}$ and $H \in \mathcal{W}$ such that $(\rho_H \circ f)(x) = 1$. Choose $a \in E$ with $\rho_G(a) = 1$.

Case I. Suppose $v(x) = 0$. For each $n \ge 1$, set

$$U_n := \{y \in X : v(y) < \frac{1}{n} \text{ and } 1 - \frac{1}{n} < \rho_H(f(y)) < 1 + \frac{1}{n}\};$$

and consider $h_n \in C_b(X)$ with

$$0 \le h_n \le n, \; h_n(x) = n, \text{ and supp } h_n \subseteq U_n.$$

By (M), the function $g_n := \frac{n}{n+1}h_n \; \rho_H \circ f \otimes a \in \mathcal{F}$. Further,

$$\begin{aligned} \|g_n\|_{v,G} &= \sup\{v(y)\frac{n}{n+1}h_n(y)\rho_H(f(y))\rho_G(a) : y \in X\} \\ &= \sup\{v(y)\frac{n}{n+1}h_n(y)\rho_H(f(y))\rho_G(a) : y \in U_n\} \\ &< \frac{1}{n}\cdot\frac{n}{n+1}\cdot n\cdot(1+\frac{1}{n})\cdot 1 = 1; \end{aligned}$$

hence,

$$\begin{aligned} \sup\{\rho_G(f(x)) &: f \in \mathcal{F}, \|f\|_{v,G} \le 1\} \\ &\ge \sup\{\rho_G(g_n(x)) : n \in \mathbb{N}\} \\ &= \sup\{\frac{n}{n+1}h_n(x)\rho_H(f(x))\rho_G(a) : n \in \mathbb{N}\} \\ &= \sup\{\frac{n}{n+1}\cdot n\cdot 1\cdot 1 : n \in \mathbb{N}\} = \infty = \frac{1}{v(x)}. \end{aligned}$$

Case II. Suppose $v(x) \neq 0$. For $n > \frac{1}{v(x)}$, let

$$\begin{aligned} F_n &: = \{y \in X : \frac{v(x)}{v(x)+\frac{1}{2n}} \leq \rho_H(f(y)) \leq \frac{v(x)}{v(x)-\frac{1}{2n}}\}, \\ U_n &: = \{y \in X : \frac{v(x)}{v(x)+\frac{1}{n}} < \rho_H(f(y)) < \frac{v(x)}{v(x)-\frac{1}{n}}\}. \end{aligned}$$

Then choose $h_n, k_n \in C_b(X)$ with

$$\begin{aligned} 0 &\leq h_n \leq \frac{1}{v(x)+\frac{1}{n}},\ h_n(x) = \frac{1}{v(x)+\frac{1}{n}} \text{ on } F_n, \text{ and supp } h_n \subseteq U_n, \\ 0 &\leq k_n \leq 1,\ k_n(x) = 1, \text{ and supp } k_n \subseteq J_n := F_n \cap \{y \in X : v(y) < v(x) + \frac{1}{n}\}. \end{aligned}$$

By (M), the function $g_n := \frac{v(x)-\frac{1}{2n}}{v(x)} h_n k_n \rho_H \circ f \otimes a \in \mathcal{F}$. Further,

$$\begin{aligned} \|g_n\|_{v,G} &= \sup\{v(y)\rho_G(g_n(y)) : y \in X\} \\ &= \sup\{v(y)\frac{v(x)-\frac{1}{2n}}{v(x)} h_n(y)k_n(y)\rho_H(f(y))\rho_G(a) : y \in X\} \\ &= \sup\{v(y)\frac{v(x)-\frac{1}{2n}}{v(x)} h_n(y)k_n(y)\rho_H(f(y))\rho_G(a) : y \in J_n\} \\ &< (v(x)+\frac{1}{2n})\frac{v(x)-\frac{1}{2n}}{v(x)} \cdot \frac{1}{v(x)+\frac{1}{2n}} \cdot 1\frac{v(x)}{v(x)-\frac{1}{2n}} = 1; \end{aligned}$$

hence

$$\begin{aligned} \sup\{\rho_G(f(x)) &: f \in \mathcal{F}, \|f\|_{v,G} \leq 1\} \\ &\geq \sup\{\rho_G(g_n(x)) : n \in \mathbb{N}\} \\ &= \sup\{\frac{v(x)-\frac{1}{2n}}{v(x)} \cdot \frac{1}{v(x)-\frac{1}{2n}} \cdot 1 \cdot 1 \cdot 1 : n \in \mathbb{N}\} = \frac{1}{v(x)}.1 \qquad (1) \end{aligned}$$

On the other hand,

$$\begin{aligned} \sup\{\rho_G(f(x)) &: f \in \mathcal{F}, \|f\|_{v,G} \leq 1\} \\ &= \frac{1}{v(x)} \sup\{v(x)\rho_G(f(x)) : f \in \mathcal{F}, \|f\|_{v,G} \leq 1\} \\ &\leq \frac{1}{v(x)} \sup\{\sup_{y\in X} v(y)\rho_G(f(y)) : f \in \mathcal{F}, \|f\|_{v,G} \leq 1\} \\ &= \frac{1}{v(x)} \sup\{\|f\|_{v,G} : f \in \mathcal{F}, \|f\|_{v,G} \leq 1\} \leq \frac{1}{v(x)} \cdot 1 = \frac{1}{v(x)}. \square \end{aligned}$$

Theorem 6. *Let $F \subseteq CV_b(X, E)$ and $\pi : X \to CL(E)$ be such that $M_\pi(F) \subseteq C(X, E)$ and F satisfies (M). If M_π is a bounded multiplication operator, then there exist $v \in V$ and $G \in \mathcal{W}$ such that for any $u \in V$ and $H \in \mathcal{W}$, there exists $\lambda > 0$ such that*

$$\lambda u(x)a \in G \text{ implies } u(x)\pi_x(a) \in H,\ x \in coz(\mathcal{F}),\ a \in E,$$

or equivalently

$$u(x)\rho_H(\pi_x(a)) \leq \lambda v(x)\rho_G(a),\ x \in coz(\mathcal{F}),\ a \in E. \tag{2}$$

In, in addition, F is EV-solid, then the converse also holds.

Proof. $(\Rightarrow)$ Suppose π_x is bounded. Then it is bounded on $N(v, G) \cap \mathcal{F}$ for some $v \in V$ and $G \in \mathcal{W}$. Then, for any $u \in V$ and closed balanced $H \in \mathcal{W}$, there exists $\lambda > 0$ such that

$$M_\pi(N(v, G) \cap \mathcal{F}) \subseteq \lambda N(u, H) \cap \mathcal{F}.$$

In particular,

$$u(x)M_\pi(f(x)) \in \lambda H \ \text{ for all } f \in N(v, G) \cap \mathcal{F} \text{ and } x \in X.$$

Now, for any $f \in N(v, G) \cap \mathcal{F}$ and $a \in G$, the function $\rho_G \circ f \otimes a \in N(v, G)$ and, by (M), to $\mathcal{F}$. Hence

$$u(x)\rho_G \circ f(x)\rho_H(\pi_x(a)) \leq \lambda \ \text{ for all } f \in \mathcal{F}, \|f\|_{v,G} \leq 1, \text{ and } x \in X.$$

Taking supremum over $f \in \mathcal{F}, \|f\|_{v,G} \leq 1$, and using Lemma 2, we have

$$u(x)\rho_H(\pi_x(a)) \leq \lambda v(x) \text{ for all } x \in X. \tag{$*$}$$

Now, let $a \in E$ be arbitrary. If $\rho_G(a) \neq 0$, then $\frac{a}{\rho_G(a)} \in G$ and so replacing $\frac{a}{\rho_G(a)}$ in $(*)$, we get

$$u(x)\rho_H(\pi_x(a)) \leq \lambda v(x)\rho_G(a).$$

If $\rho_G(a) = 0$, then $\rho_G(na) = 0$ for all $n \in \mathbb{N}$ and so replacing a by na in $(*)$,

$$u(x)\rho_H(\pi_x(a)) \leq \frac{1}{n}\lambda v(x);$$

hence $u(x)\rho_H(\pi_x(a)) = 0$. Thus (2) holds for all $a \in E$.

$(\Leftarrow)$ Suppose $\mathcal{F}$ is EV-solid and that (2) holds. We need to show that $M_\pi(\mathcal{F}) \subseteq \mathcal{F}$ and that $M_\pi(\mathcal{F}) \subseteq \mathcal{F}$ and that M_π is bounded on $\mathcal{F}$. Let $v \in V$ and $G \in \mathcal{W}$. We claim that $M_\pi(N(v,G)\cap\mathcal{F})$ is contained and bounded in $\mathcal{F}$. Indeed, for any $u\in V$ and $H\in\mathcal{W}$, by (2), ther exists $\lambda>0$ such that

$$\lambda v(x)a\in G \text{ implies } \pi_x(a)\in H \text{ for all } x\in coz(\mathcal{F}),\ a\in E.$$

In particular,

$$v(x)(\lambda f)(x)\in G \text{ implies } u(x)\pi_x(f(x))\in H \text{ for all } f\in\mathcal{F},\ x\in coz(\mathcal{F}). \qquad (**)$$

Now, for any $f\in\mathcal{F}$, $\lambda f\in\mathcal{F}$ and so, since $\mathcal{F}$ is EV-solid, $(**)$ implies that $\pi f\in\mathcal{F}$; hence $M_\pi(\mathcal{F})\subseteq\mathcal{F}$. Further, $(**)$ can be expressed as

$$M_\pi(N(v,G)\cap\mathcal{F})\subseteq\lambda N(u,H)\cap\mathcal{F},$$

which shows that $M_\pi$is bounded on $\mathcal{F}$. □

Finally, we examine the cases $\omega_v=k$ and $\omega_v=\beta$.

Theorem 7. *Let $\pi: X\to CL(E)$ be a map and $\mathcal{F}$ a subspace of $CV_b(X,E)$ satisfying (M) with V such that $\omega_v\in\{k,\beta\}$.*

(1) If $M_\pi$is a bounded multiplication operator on $(\mathcal{F},k)$, then the support of π is contained in $K\cap z(\mathcal{F})$ for some compact $K\subseteq X$. Here $z(\mathcal{F})=X\backslash coz(\mathcal{F})$.

(2) If $M_\pi$is a bounded multiplication operator on $(\mathcal{F},\beta)$, then π vanishes at infinity, when $CL(E)$ is endowed with the topology t_p.

Proof. (1) Suppose $M_\pi$is a bounded multiplication operator on $(\mathcal{F},k)$. Then, by Theorem 3, there exist compact set $K\subseteq X$ and $G\in\mathcal{W}$ such that, for every compact set $J\subseteq X$ and $H\in\mathcal{W}$, there exists $\lambda>0$ such that

$$\lambda\chi_K(x)a\in G \text{ implies } \chi_J(x)\pi_x(a)\in H \text{ for all } a\in E,\ x\in coz(\mathcal{F}). \qquad (*)$$

Let $y\notin z(\mathcal{F})\cap K$. Taking a compact set J containing y and not intersecting (e.g. $J=\{y\}$), we have by $(*)$

$$\chi_J(y)\rho_H(a))\leq\lambda\chi_K(y)\rho_G(a)=0 \text{ for all } a\in E.$$

Hence $\pi_x=0$. Thus supp $\pi\subseteq K\cap z(\mathcal{F})$.

(2) By Theorem 3, there exist $v\in S_o^+(X)$ and $G\in\mathcal{W}$ such that, for $u(x)=\sqrt{v(x)}$ and any $H\in W$, there exists $\lambda>0$ such that

$$\begin{aligned}\sqrt{v(x)}\rho_H(\pi_x(a)) &\leq \lambda v(x)\rho_G(a) \text{ for all } a\in E,\ x\in coz(\mathcal{F}),\\ \text{or}\quad ||\pi_x||_{\{a\},H} &= \rho_H(\pi_x(a))\leq\lambda\rho_G(a)\sqrt{v(x)}.\end{aligned}$$

Since $\sqrt{v}$ vanishes at infinity, so does $||\pi(.)||_{\{a\},H}$. (Here $||\pi(.)||_{\{a\},H}(x):=||\pi_x||_{\{a\},H}$, $x\in coz(\mathcal{F})$.) Hence $\pi: X\to CL_p(E)$ vanishes at infinity. □

Remark. We mention that the above results are obtained for the subset $coz(\mathcal{F})$ of X. However, by assuming that the spaces $\mathcal{F} = CV_b(X, E)$ and $\mathcal{F} = CV_o(X, E)$ be essential, we can replace $coz(\mathcal{F})$ by X in the above proofs.

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