ON CLOSE-TO-CONVEX FOR CERTAIN INTEGRAL OPERATORS

Aabed Mohammed, Maslina Darus and Daniel Breaz

Abstract. In this paper, we consider some sufficient conditions for two integral operators to be close-to-convex function defined in the open unit disk.

Keywords: Analytic functions, closed-to-convex, close-to-star, integral operators.

2000 Mathematics Subject Classification: 30C45

1. Introduction

Let $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk and $\mathcal{A}$ denotes the class of functions normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the open unit disk $\mathcal{U}$ and satisfy the condition $f(0) = f'(0) - 1 = 0$. We also denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are also univalent in $\mathcal{U}$.

A function $f \in \mathcal{A}$ is the convex function of order ρ, $0 \le \rho < 1$, if f satisfies the following inequality

$$\mathrm{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > \rho, \quad z \in \mathcal{U}$$

and we denote this class by $\mathcal{K}(\rho)$.

Similarly, if $f \in \mathcal{A}$ satisfies the following inequality

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \rho, \quad z \in \mathcal{U}$$

for some ρ, $0 \le \rho < 1$, then f is said to be starlike of order ρ and we denote this class by $\mathcal{S}^*(\rho)$. We note that $f \in \mathcal{K} \Leftrightarrow zf'(z) \in \mathcal{S}^*$, $z \in \mathcal{U}$. In particular case, the classes $\mathcal{K}(0) = \mathcal{K}$ and $\mathcal{S}^* = \mathcal{S}^*$ are familiar classes of starlike and convex functions in $\mathcal{U}$.

A function $f \in \mathcal{A}$ is called close-to-convex if there exist a convex function g such that

$$\mathrm{Re}\frac{f'(z)}{g'(z)} > 0, \quad z \in \mathcal{U}. \tag{2}$$

Since $f \in \mathcal{K} \Leftrightarrow zf'(z) \in \mathcal{S}^*$, $z \in \mathcal{U}$, we can replace (2) by the requirement that

$$\mathrm{Re}\frac{zf'(z)}{h(z)} > 0, \ z \in \mathcal{U},$$

where h is a starlike function on $\mathcal{U}$. Furthermore, f is closed-to-convex if and only if

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) d\theta > -\pi,$$

where $0 \leq \theta_1 < \theta_2 \leq 2\pi, \ z = re^{i\theta}$ and $r < 1$.

Let $\mathcal{C}$ denote the set of normalized close-to-convex functions on $\mathcal{U}$ it is clear that

$$\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{C} \subset \mathcal{S}.$$

A function $f \in \mathcal{A}$ is called close-to-star in $\mathcal{U}$ if there exist a starlike g such that

$$\mathrm{Re}\frac{f(z)}{g(z)} > 0, \ z \in \mathcal{U}.$$

Also f is close-to-star in $\mathcal{U}$ if and only if

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\frac{zf'(z)}{f(z)} d\theta > -\pi,$$

where $0 \leq \theta_1 < \theta_2 \leq 2\pi, \ z = re^{i\theta}$ and $r < 1$. Let $\mathcal{CS}^*$ denote the class of close-to-star functions in $\mathcal{U}$, it is known that the close-to-star functions are not necessarily univalent in $\mathcal{U}$.

Shukla and Kumar [5] introduced the following subclasses of $\mathcal{C}$ and $\mathcal{CS}^*$.

A function $f \in \mathcal{S}$ belongs to the class $\mathcal{C}(\beta, \rho)$ of close-to-convex functions of order β and type ρ if for some $g \in S^*(\rho)$,

$$\left| arg\left\{ \frac{zf'(z)}{g(z)} \right\} \right| < \frac{\beta\pi}{2},\ z \in \mathcal{U},$$

where $\beta \in [0,1]$.

A function $f \in \mathcal{S}$ belongs to the class $\mathcal{CS}^*(\beta,\rho)$ of close-to-star functions of order β and type ρ if for some $g \in \mathcal{S}^*(\rho)$,

$$\left| arg\left\{ \frac{f(z)}{g(z)} \right\} \right| < \frac{\beta\pi}{2},\ z \in \mathcal{U},$$

where $\beta \in [0,1]$.

It is clear that $\mathcal{C}(0,\rho) \equiv \mathcal{K}(\rho)$ and $\mathcal{CS}^*(0,\rho) \equiv \mathcal{S}^*(\rho)$.

Also $\mathcal{C}(\beta,\rho) \subset \mathcal{C}(1,0) \equiv \mathcal{C}$ and $\mathcal{CS}^*(\beta,\rho) \subset \mathcal{CS}^*(1,0) \equiv \mathcal{CS}^*$.

Now, we consider the following integral operators

$$F_n(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdot ... \cdot \left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt, \tag{3}$$

and

$$F_{\alpha_1,...,\alpha_n}(z) = \int_0^z \left(f_1'(t)\right)^{\alpha_1} \cdot ... \cdot \left(f_n'(t)\right)^{\alpha_n} dt, \tag{4}$$

where $f_j \in \mathcal{A}$ and $\alpha_j > 0$, for all $j \in \{1,2,...,n\}$.

These operators are introduced by D.Breaz and N.Breaz [1] and studied by many authors (see [2], [3], [4]).

In the present paper, we obtain some sufficient conditions for the above integral operators to be in the class of close-to-convex function $\mathcal{C}$.

Before embarking on the proof of our results, we need the following Lemmas introduced by Shukla and Kumar [5].

Lemma 1. *If $f \in \mathcal{S}^*(\rho)$, then*

$$\rho(\theta_2-\theta_1) \le \int_{\theta_1}^{\theta_2} \mathrm{Re}\frac{zf'(z)}{f(z)} d\theta \le 2\pi(1-\rho) + \rho(\theta_2-\theta_1)$$

where $z = re^{i\theta}$ and $0 \le \theta_1 \le \theta_2 \le 2\pi$.

Lemma 2. *If $f \in \mathcal{C}(\beta,\rho)$ then*

$$-\beta\pi+\rho(\theta_2-\theta_1)\leq\int\limits_{\theta_1}^{\theta_2}\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)d\theta\leq\beta\pi+2\pi(1-\rho)+\rho(\theta_2-\theta_1)$$

where $z=re^{i\theta}$ *and* $0\leq\theta_1\leq\theta_2\leq2\pi$.

Lemma 3. *If* $f\in\mathcal{CS}^*(\beta,\rho)$ *then*

$$-\beta\pi+\rho(\theta_2-\theta_1)\leq\int\limits_{\theta_1}^{\theta_2}\mathrm{Re}\frac{zf'(z)}{f(z)}d\theta\leq\beta\pi+2\pi(1-\rho)+\rho(\theta_2-\theta_1)$$

where $z=re^{i\theta}$ *and* $0\leq\theta_1\leq\theta_2\leq2\pi$.

2.Main results

Theorem 1. *Let* $f_i\in\mathcal{S}^*(\rho_i)$, *for* $i\in\{1,2,...,n\}$. *If* $\sum\limits_{i=1}^{n}\alpha_i\leq1$, *then* $F_n(z)\in\mathcal{C}$, $z\in\mathcal{U}$, *where* F_n *is defined as in (3), and* $\mathcal{C}$ *is the class of close to convex functions.*

Proof. It is clear that $F_n'(z)\neq0$ for $z\in\mathcal{U}$. We calculate for F_n the derivatives of the first and second order. Since

$$F_n(z)=\int\limits_0^z\left(\frac{f_1(t)}{t}\right)^{\alpha_1}\cdot\ldots\cdot\left(\frac{f_n(t)}{t}\right)^{\alpha_n}dt,$$

then

$$F_n'(z)=\left(\frac{f_1(z)}{z}\right)^{\alpha_1}\ldots\left(\frac{f_n(z)}{z}\right)^{\alpha_n}.$$

Differentiating the above expression logarithmically, we have

$$\frac{F_n''(z)}{F_n'(z)}=\sum_{i=1}^{n}\alpha_i\left(\frac{f_i'(z)}{f_i(z)}-\frac{1}{z}\right).$$

By multiplying the above expression with z we obtain

$$\frac{zF_n''(z)}{F_n'(z)}=\sum_{i=1}^{n}\alpha_i\left(\frac{zf_i'(z)}{f_i(z)}-1\right).$$

That is equivalent to

$$1+\frac{zF_n''(z)}{F_n'(z)}=\sum_{i=1}^{n}\alpha_i\frac{zf_i'(z)}{f_i(z)}+1-\sum_{i=1}^{n}\alpha_i. \tag{5}$$

Taking real parts in (5) and integrating with respect to θ we get

$$\int_{\theta_1}^{\theta_2}\mathrm{Re}\left(1+\frac{zF_n''(z)}{F_n'(z)}\right)d\theta=\int_{\theta_1}^{\theta_2}\sum_{i=1}^{n}\alpha_i\mathrm{Re}\left(\frac{zf_i'(z)}{f_i(z)}\right)d\theta+\left(1-\sum_{i=1}^{n}\alpha_i\right)(\theta_2-\theta_1).$$

Since $f_i\in\mathcal{S}^*(\rho_i)$ then by applying Lemma 1, we obtain

$$\int_{\theta_1}^{\theta_2}\mathrm{Re}\left(1+\frac{zF_n''(z)}{F_n'(z)}\right)d\theta\geq\left(\sum_{i=1}^{n}\alpha_i\rho_i-\sum_{i=1}^{n}\alpha_i+1\right)(\theta_2-\theta_1).$$

Since $\left(\sum\limits_{i=1}^{n}\alpha_i\rho_i-\sum\limits_{i=1}^{n}\alpha_i+1\right)>0$ so, minimum is for $\theta_2=\theta_1$ we obtain that

$$\int_{\theta_1}^{\theta_2}\mathrm{Re}\left(1+\frac{zF_n''(z)}{F_n'(z)}\right)d\theta>-\pi,$$

then $F_n(z)\in\mathcal{C}$.

Corollary 2. *Let $f_i\in\mathcal{S}^*(\rho)$, for $i\in\{1,2,...,n\}$. If $\sum\limits_{i=1}^{n}\alpha_i\leq 1$, then $F_n(z)\in\mathcal{C}$, $z\in\mathcal{U}$, where F_n is defined as in (3), and $\mathcal{C}$ is the class of close to convex functions.*

Proof. We consider in Theorem 1, $\rho_1=\rho_2=...=\rho_n$.

Theorem 3. *Let $f_i\in\mathcal{CS}^*$, $i=\{1,2,...,n\}$. If $\sum\limits_{i=1}^{n}\alpha_i\leq 1$, then $F_n(z)\in\mathcal{C}$, $z\in\mathcal{U}$, where F_n is defined as in (3), and $\mathcal{C}$ is the class of close to convex functions.*

Proof. Following the same steps as in Theorem 1, we obtain that

$$\int_{\theta_1}^{\theta_2}\mathrm{Re}\left(1+\frac{zF_n''(z)}{F_n'(z)}\right)d\theta=\int_{\theta_1}^{\theta_2}\sum_{i=1}^{n}\alpha_i\mathrm{Re}\left(\frac{zf_i'(z)}{f_i(z)}\right)d\theta+\left(1-\sum_{i=1}^{n}\alpha_i\right)(\theta_2-\theta_1).$$

Since $f_i\in\mathcal{CS}^*$, then

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left(1+\frac{zF_n''(z)}{F_n'(z)}\right)d\theta > -\pi\sum_{i=1}^{n}\alpha_i + \left(1-\sum_{i=1}^{n}\alpha_i\right)(\theta_2-\theta_1).$$

Since $1-\sum\limits_{i=1}^{n}\alpha_i > 0$ so, minimum is for $\theta_1=\theta_2$ we obtain that

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left(1+\frac{zF_n''(z)}{F_n'(z)}\right)d\theta > -\pi$$

Then $F_n \in \mathcal{C}$.

Theorem 4. *Let $f_i \in \mathcal{C}(\beta_i,\rho_i)$, $i=\{1,2,...,n\}$. If $\sum\limits_{i=1}^{n}\alpha_i\beta_i \le 1$, then $F_{\alpha_1,\alpha_2,...,\alpha_n}(z) \in \mathcal{C}$, $z\in\mathcal{U}$, where $F_{\alpha_1,\alpha_2,...,\alpha_n}$ is defined as in (4), and $\mathcal{C}$ is the class of close to convex functions.*

Proof. It is clear that $F'_{\alpha_1,...,\alpha_n}(z) \neq 0$ for $z\in\mathcal{U}$. Since

$$F_{\alpha_1,...,\alpha_n}(z) = \int_0^z \left(f_1'(t)\right)^{\alpha_1}\cdot ...\cdot\left(f_n'(t)\right)^{\alpha_n}dt.$$

Following the same steps as in Theorem 1, we obtain that

$$\frac{zF''_{\alpha_1,...,\alpha_n}(z)}{F'_{\alpha_1,...,\alpha_n}(z)} = \sum_{i=1}^{n}\alpha_i\frac{zf_i''(z)}{f_i'(z)}.$$

That is equivalent to

$$1+\frac{zF''_{\alpha_1,...,\alpha_n}(z)}{F'_{\alpha_1,...,\alpha_n}(z)} = \sum_{i=1}^{n}\alpha_i\left(\frac{zf_i''(z)}{f_i'(z)}+1\right)+1-\sum_{i=1}^{n}\alpha_i. \qquad (6)$$

Taking real parts in (6) and integrating with respect to θ we get

$$\int_{\theta_1}^{\theta_2}\mathrm{Re}\left(1+\frac{zF''_{\alpha_1,...,\alpha_n}(z)}{F'_{\alpha_1,...,\alpha_n}(z)}\right)d\theta = \int_{\theta_1}^{\theta_2}\sum_{i=1}^{n}\alpha_i\mathrm{Re}\left(\frac{zf_i''(z)}{f_i'(z)}+1\right)d\theta + \left(1-\sum_{i=1}^{n}\alpha_i\right)(\theta_2-\theta_1).$$

Since $f_i \in \mathcal{C}(\beta_i,\rho_i)$ then by applying Lemma 2, we obtain

$$\int_{\theta_1}^{\theta_2} \text{Re}\left(1+\frac{zF''_{\alpha_1,...,\alpha_n}(z)}{F'_{\alpha_1,...,\alpha_n}(z)}\right)d\theta \geq \sum_{i=1}^{n}\alpha_i\left[-\beta_i\pi+\rho_i(\theta_2-\theta_1)\right]+\left(1-\sum_{i=1}^{n}\alpha_i\right)(\theta_2-\theta_1)=$$

$$=\left(\sum_{i=1}^{n}\alpha_i\rho_i-\sum_{i=1}^{n}\alpha_i+1\right)(\theta_2-\theta_1)-\pi\sum_{i=1}^{n}\alpha_i\beta_i.$$

Since $\sum\limits_{i=1}^{n}\alpha_i\rho_i-\sum\limits_{i=1}^{n}\alpha_i+1>0$ so, minimum is for $\theta_1=\theta_2$ we obtain that

$$\int_{\theta_1}^{\theta_2} \text{Re}\left(1+\frac{zF''_{\alpha_1,...,\alpha_n}(z)}{F'_{\alpha_1,...,\alpha_n}(z)}\right)d\theta > -\pi$$

then $F_{\alpha_1,...,\alpha_n}(z)\in\mathcal{C}$.

Corollary 5. *Let $f_i\in\mathcal{C}(\beta,\rho)$, $i=\{1,2,...,n\}$. If $\sum\limits_{i=1}^{n}\alpha_i\leq 1$, then $F_{\alpha_1,\alpha_2,...,\alpha_n}(z)\in\mathcal{C}$, $z\in\mathcal{U}$, where $F_{\alpha_1,\alpha_2,...,\alpha_n}$ is defined as in (4), and $\mathcal{C}$ is the class of close to convex functions.*

Proof. We considet in Theorem 4 $\beta_1=...=\beta_n$ and $\rho_1=...=\rho_n$.

Theorem 6. *If $f_i\in\mathcal{CS}^*(\beta_i,\rho_i)$, $i=\{1,2,...,n\}$ and $\sum\limits_{i=1}^{n}\alpha_i\beta_i\leq 1$, then $F_n\in\mathcal{C}$, where F_n is defined as in (3), and $\mathcal{C}$ is the class of close to convex functions.*

Proof. Since the proof is similar to the proof of theorems in (1), (3) and (4), it will be omitted.

Corollary 7. *If $f_i\in\mathcal{CS}^*(\beta,\rho)$, $i=\{1,2,...,n\}$ and $\sum\limits_{i=1}^{n}\alpha_i\leq 1$, then $F_n\in\mathcal{C}$, where F_n is defined as in (3), and $\mathcal{C}$ is the class of close to convex functions.*

Proof. We consider in Theorem 6 $\beta_1=...=\beta_n$ and $\rho_1=...=\rho_n$.

Acknowledgement: The work here is fully supported by UKM-GUP-TMK-07-02-107, UKM.

References

[1] D. Breaz and N. Breaz, *Two integral operators*, Studia Universitatis Babes-Bolyai, Mathematica, 47:3(2002), 13-19.

[2] D. Breaz, S. Owa and N. Breaz, *A new integral univalent operator*, Acta Universitatis Apulensis, No 16/2008, pp. 11-16.

[3] D. Breaz, *A convexity property for an integral operator on the class* $\mathcal{S}_p(\beta)$, Journal of Inequalities and Applications, vol. 2008, Article ID 143869.

[4] D. Breaz, *Certain Integral Operators On the Classes* $\mathcal{M}(\beta_i)$ *and* $\mathcal{N}(\beta_i)$, Journal of Inequalities and Applications, vol. 2008, Article ID 719354.

[5] S. L. Shukla and V. Kumar, *On The products of close-to-starlike and close-to-convex functions*, Indian J. Pure appl. Math. 16(3): (1985), 279-290.

Aabed Mohammed
School of Mathematical Sceinces,
Faculty of Science and Technology,
Universiti Kebangsaan Malaysia
43600 Bangi, Selangor D. Ehsan,
Malaysia
e-mail:*aabedukm@yahoo.com*

Maslina Darus
School of Mathematical Sceinces,
Faculty of Science and Technology,
Universiti Kebangsaan Malaysia
43600 Bangi, Selangor D. Ehsan,
Malaysia
e-mail:*maslina@ukm.my*

Daniel Breaz
“1 Decembrie 1918” University of Alba Iulia,
Faculty of Science,
Department of Mathematics-Informatics,
510009 Alba Iulia,
Romania
e-mail:*dbreaz@uab.ro*